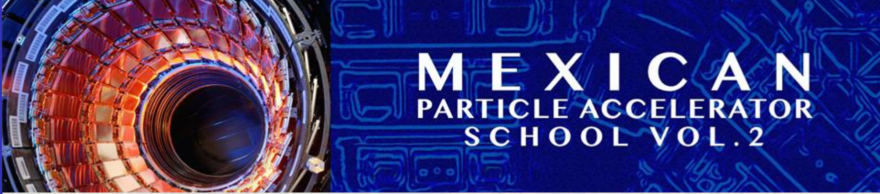

Beam Dynamics: Lectures I - II

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División de Ciencias e Ingenierías

Guanajuato, GTO, Mexico | *November, 2015*

Syllabus

- Six lectures:
  - Two on Friday 13th.
  - Two on Saturday 14th.
  - Two on Tuesday 17th.
- We will focus on circular machines.
- Homework:
  - Assigned each lecture.
  - Collected at the start of the following day.
  - Work in groups.
  - Assigned time at the end of the day for doing homework.
- Final exam: similar to homework. Notes allowed?
- MePAS grade: 60% homework, 40% final exam.
- Our emails:
  a.castilla@cern.ch
  lmedinam@cern.ch

Begging for caffeine and eating churros at the Ferney market...

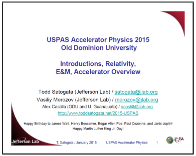
Bibliography

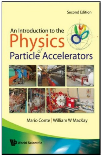
This course is mostly based on the USPAS courses by **Todd Satogata** (JLAB), available at `http://toddsatogata.net/`, which in turn follows

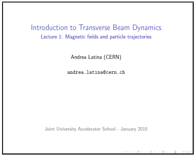
M. Conte, W. W. MacKay, *An Introduction to the Physics of Particle Accelerators*. Second Edition. World Scientific. Singapore, 2008.

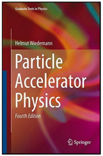
Some extracts has been taken from the JUAS 2015 courses on Transverse Dynamics **Andrea Latina** (CERN) and on Longitudinal Dynamics by **Elias Metral** (CERN). Both of them are available at `https://indico.cern.ch/event/356897/`

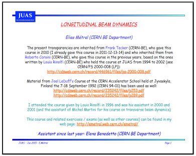
Of course, **Wiedemann**'s book and the **Chao and Tigner**'s Handbook are the canonical bibliography on Beam Dynamics...

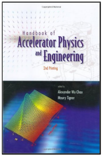












Also... **Free lectures**[1] on accelerators and more (on demand)! Visit `https://www.cockcroft.ac.uk/lectures`

[1] Please send an email to `graeme.burt@cockcroft.ac.uk` and tell Graeme you are using them (this is only for internal statistics).

A. Castilla
L. Medina

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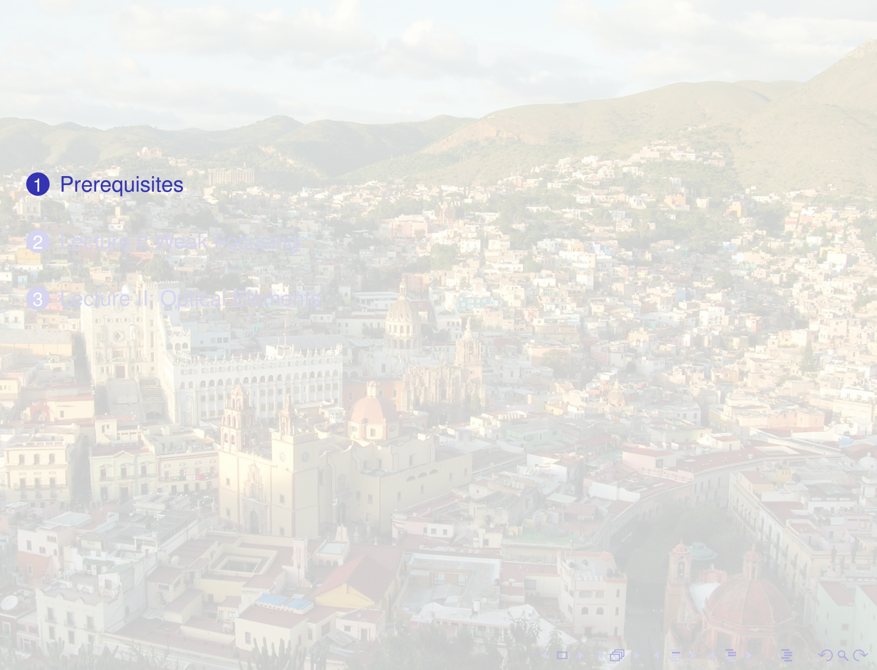


Relativity review

Particle accelerators: applied electromagnetism and **special relativity**.

It will come in handy writing down some of the useful formulas,

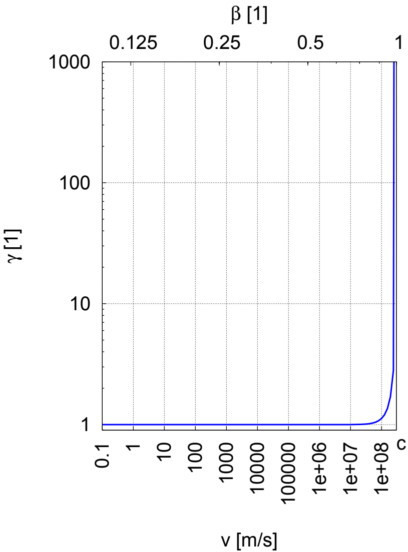
Lorentz factors

$$\beta_r \equiv \frac{v}{c}, \qquad \gamma_r \equiv \frac{1}{\sqrt{1-\beta_r^2}} \tag{1.1}$$

where v is the speed of the object and c is the speed of light,

$$\begin{aligned} c &= 299,792,458 \ \frac{\text{m}}{\text{s}} \\ &\approx 3 \times 10^8 \ \frac{\text{m}}{\text{s}} \end{aligned} \tag{1.2}$$

Note that β_r and γ_r are dimensionless.



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Example: *Lorentz factors*

For a car traveling at 150 km/h,

$$v = 150\ \frac{\mathrm{km}}{\mathrm{h}}\cdot\frac{1000\ \mathrm{m}}{\mathrm{km}}\cdot\frac{3600\ \mathrm{s}}{\mathrm{h}} = 41.7\ \frac{\mathrm{m}}{\mathrm{s}}$$

$$\beta_r = \frac{41.7\ \mathrm{m/s}}{3\times10^8\ \mathrm{m/s}} = 0.00000014$$

$$\gamma = \frac{1}{\sqrt{1-(1.4\times10^{-7})^2}} \approx 1$$

The Helios-2 probe (the fastest man- made object), travels at 70.2 km/s, This corresponds to the Lorentz factors

$$\beta_r = 0.00023, \quad \gamma_r = 1.00000003$$

Compare both results to

$$\beta_r = 0.99999999, \quad \gamma_r = 7453.56$$

corresponding to a proton in the LHC.

Relativity review (cont.)

Rest energy

$$E_0 = mc^2 \tag{1.3}$$

Total energy

$$E = \gamma_r mc^2 \tag{1.4}$$

Kinetic energy

$$E_K = E - E_0 = (\gamma_r - 1)mc^2 \tag{1.5}$$

Momentum

$$p = \gamma_r m(\beta_r c) = \beta_r \frac{E}{c} \tag{1.6}$$

for an object with mass m.

The unit of energy in the International System is the joule (J). A more suitable unit in particle Physics is the

Electron-volt (eV)

$$\begin{aligned} 1\ \text{eV} &= (1.602 \times 10^{-19}\ \text{C})(1\ \text{V}) \\ &= 1.602 \times 10^{-19}\ \text{J} \end{aligned} \tag{1.7}$$

It corresponds to the amount of **energy gained/lost** by a particle with a charge e (the **elementary charge**), when it is moved across an electric potential difference of **one volt**.

Relativity review (cont.)

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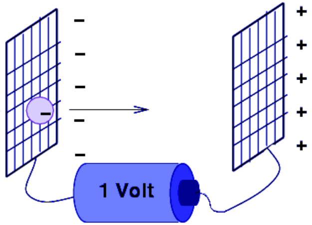


Energy units

Prefix	Sym.	Value
tera-	T	10^{12}
giga-	G	10^{9}
mega-	M	10^{6}
kilo-	k	10^{3}
mili-	m	10^{-3}
micro-	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}

Table 1: SI prefixes.

	Unit
Energy	eV
Mass	eV/c^2
Momentum	eV/c

Table 2: Units of energy, mass and momentum in terms of eV. **Note: it is often set $c = 1$.**

Example: *Energies in eV*

The mass of an electron is 1.673×10^{-27} kg. According to (7), its rest energy is

$$\begin{aligned} E_0 &= (1.673 \times 10^{-27}\ \mathrm{kg})(3 \times 10^{8}\ \mathrm{m/s})^2 \\ &= 8.19 \times 10^{-14}\ \mathrm{J} \cdot \frac{1\ \mathrm{eV}}{1.602 \times 10^{-19}\ \mathrm{J}} \\ &\approx 511,000\ \mathrm{eV} \end{aligned}$$

Since $E_0 = mc^2$, the mass can also be written in terms of eV (instead of kg). So, for an electron,

$$m = \frac{E_0}{c^2} = \mathbf{0.511\ MeV/c^2}$$

If it is traveling at 10% of the speed of light, then

$$v = 0.1c, \qquad \beta_r = 0.1, \qquad \gamma_r = 1.005$$

Thus, the total energy of the electron is

$$E = 1.005 \cdot 0.511\ \frac{\mathrm{MeV}}{\mathrm{c}^2} \cdot c^2 = 0.513\ \mathrm{MeV}$$

Electromagnetism review

The

Newton's second law

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt} \qquad (1.8)$$

describes the **motion** of a particle of mass m due to an **external force** $\vec{F}$.

The

Lorentz force

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \qquad (1.9)$$

defines the **force** experienced by a charge q under and **electric and magnetic fields**.

Example: *Electric vs. magnetic forces*

Typical values for the strength of electric and magnetic fields are

$$|\vec{E}| \approx 1\ \frac{\mathrm{MV}}{\mathrm{m}}, \qquad |\vec{B}| \approx 1\ T = 1\ \frac{\mathrm{Vs}}{\mathrm{m}^2}$$

Suppose we have a particle with the elementary charge and velocity equal to $v = \beta_r c$. Then, the ratio between the magnetic and electric forces is

$$\begin{aligned} \frac{F_m}{F_e} &= \frac{q|\vec{v}||\vec{B}|}{q|\vec{E}|} = \frac{e(\beta_r c)(1\ \frac{\mathrm{Vs}}{\mathrm{m}^2})}{e(1\ \frac{\mathrm{MV}}{\mathrm{m}})} \\ &= \frac{\beta_r\left(3\times 10^8\ \frac{\mathrm{m}}{\mathrm{s}}\right)\frac{\mathrm{Vs}}{\mathrm{m}^2}}{1\times 10^6\ \frac{\mathrm{V}}{\mathrm{m}}} = 300\beta_r \end{aligned}$$

What can we conclude from this?

For charged particles with speeds close to c, $\beta_r \approx 1$. Then, if we want to exert a force to change its motion, we better use magnetic forces (they're $300\times$ stronger!).

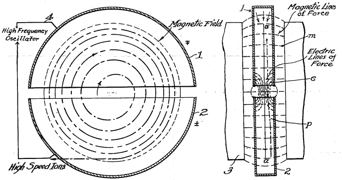
The cyclotron I

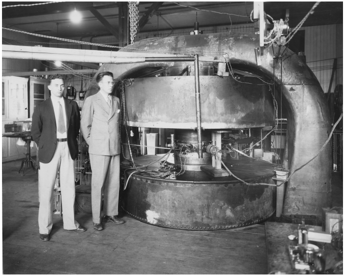
Equating (1.8) and (1.9) in the absence of electric field ($\vec{E} = 0$),

$$\begin{aligned} q\vec{v} \times \vec{B} &= \frac{d\vec{p}}{dt} = \frac{d(\gamma_r m \vec{v})}{dt} \\ &= m\left(\gamma_r \frac{d\vec{v}}{dt} + \frac{d\gamma_r}{dt}\vec{v}\right) \\ &= \gamma_r m \frac{d\vec{v}}{dt} \end{aligned}$$

since $\beta_r = |\vec{\beta}_r|$ is constant, which implies $d\gamma_r/dt = 0$. Now, with the aid of the **angular velocity** $\vec{\omega}$, defined by $\vec{v} \equiv \vec{\omega} \times \vec{\rho}$,

$$\begin{aligned} q\vec{v} \times \vec{B} &= \gamma_r m \frac{d(\vec{\omega} \times \vec{\rho})}{dt} \\ &= \gamma_r m \left(\vec{\omega} \times \frac{d\vec{\rho}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{\rho}\right) \end{aligned}$$





The cyclotron II

or

$$q\vec{v} \times \vec{B} = \gamma_r m\vec{\omega} \times \frac{d\vec{\rho}}{dt}$$

since ω is constant for a central force of constant magnitude. Now, the **cyclotron (or bending) radius** ρ is just the radius of the particle's orbit, then,

$$q\vec{v} \times \vec{B} = \gamma_r m\vec{\omega} \times \vec{v}$$

In the particular case when $\vec{B}$ and $\vec{v}$ are perpendicular,

$$qvB = \gamma_r m\omega v = \frac{\gamma_r mv^2}{\rho} \qquad (1.10)$$

with $\omega = v/\rho$. Arranging this equation,

$$qB = \frac{\gamma_r mv}{\rho} = \frac{p}{\rho} \qquad (1.11)$$

we get the

Rigidity

$$(B\rho) = \frac{p}{q} \qquad (1.12)$$

and its units are Tm.

The **rigidity** give us an idea on *how hard/easy* is a particle to deflect, Note how relates machines properties *(left)* with beam properties *(right)*.

When working with particles with the **elementary charge**, we can rewrite the

Rigidity (in practical units)

$$p\ [\mathrm{GeV/c}] \approx 0.3\ B\ [\mathrm{T}]\ \rho\ [\mathrm{m}] \qquad (1.13)$$

The cyclotron III

Example: *Rigidity*

Let us consider an electron ring with radius $R = 200$ m. If only 50% of the circumference $C = 2\pi R$ is occupied by bending magnets, this length has to correspond to a circumference given by $2\pi\rho$. In other words,

$$0.5C = 0.5 \cdot 2\pi R = 2\pi\rho$$

or

$$\rho = 0.5R = 100 \text{ m}$$

If the momentum of the electrons is 12 GeV/c, the rigidity is

$$B\rho \approx \frac{p[\text{GeV/c}]}{0.3} = 40 \text{ Tm}$$

and therefore $B = 0.4$ T.

Rearranging (1.10) in a different way, we obtain the

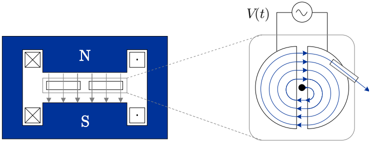
Cyclotron (angular) frequency

$$\omega = \frac{qB}{\gamma_r m}, \qquad f = \frac{\omega}{2\pi} \qquad (1.14)$$

which gives us the **number of turns** a particle can perform in the cyclotron, **per unit of time**.

In order to accelerate the particles, an RF voltage has to be provided, and its frequency has to match the revolution frequency,

$$f_{rf} = f = \frac{\omega}{2\pi} \qquad (1.15)$$



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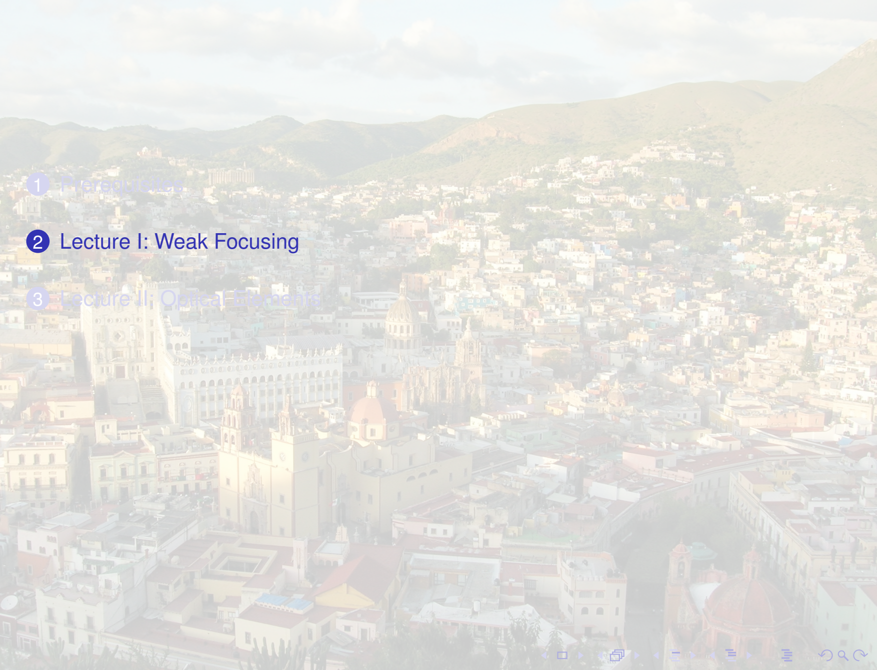


Parametrization and approximations

The **ideal particle** defines a trajectory, the **design orbit**.

To describe the motion of a given particle, we use a **local coordinate system** $(\hat{x}, \hat{y}, \hat{s})$ that moves (rotates) with the ideal particle, the so-called Frenet-Serret frame.

From the figure,

$$R = \rho + x \tag{2.1}$$

where

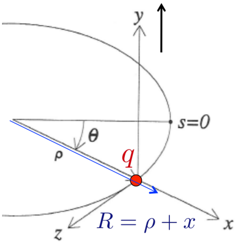
$$\theta = \frac{s}{R} = \frac{(\beta_r c)t}{R} \tag{2.2}$$

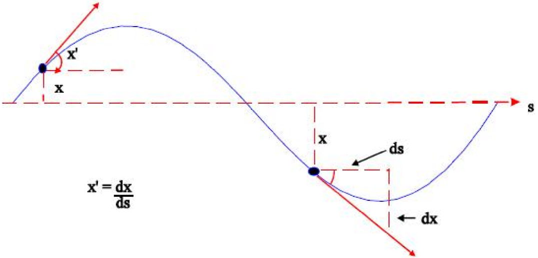
The slope

$$x' \equiv \frac{dx}{ds} = \frac{1}{R}\frac{dx}{d\theta} \tag{2.3}$$

is the local trajectory angle. Also note

$$x' = \frac{v_x}{v_z} = \frac{p_x}{p_s} \approx \frac{p_y}{p} \tag{2.4}$$





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Approximations

1. No local currents (near-vacuum).
2. Paraxial approximation:
$$x', y' \ll 1, \text{ or } p_x, p_y \ll p_s \tag{2.5}$$
3. Perturbative coordinates:
$$x, y \ll \rho \tag{2.6}$$
4. Transverse linear $\vec{B}$ field:
$$\begin{aligned} \vec{B} &= B_x\hat{x} + B_y\hat{y} \\ &= B_0\hat{y} + (x\hat{y} + y\hat{x})\frac{\partial B_y}{\partial x} \end{aligned} \tag{2.7}$$
where $B_0 \neq 0$.
5. Negligible $\vec{E}$ field: $\gamma_r \approx$ constant.

Equations of motion I

We begin with the Lorentz force equation of motion,

$$\vec{F} = q\vec{v} \times \vec{B} = \frac{d(\gamma_r m\vec{v})}{dt} \qquad (2.8)$$

Given the **position** vector,

$$\vec{r} = R\hat{x} + y\hat{y} \qquad (2.9)$$

we need to calculate the corresponding **velocity** and **acceleration** as follows,

$$\vec{v} = \dot{\vec{r}} = \dot{R}\hat{x} + R\dot{\hat{x}} + \dot{y}\hat{y} = \dot{R}\hat{x} + R\dot{\theta}\hat{s} + \dot{y}\hat{y} \qquad (2.10)$$

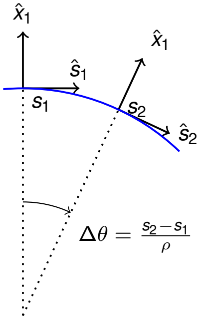
$$\vec{a} = \dot{\vec{v}} = \ddot{R}\hat{x} + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{s} + R\dot{\theta}\dot{\hat{s}} + \ddot{y}\hat{y} \qquad (2.11)$$

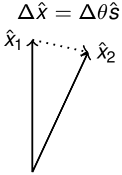
If we calculate $\dot{\hat{s}}$,

$$\dot{\hat{s}} = -\dot{\theta}\hat{x} = -\frac{v}{R}\hat{x} \qquad (2.12)$$

and insert it in (2.11), we obtain

$$\begin{aligned} \vec{a} &= (\ddot{R} - R\dot{\theta}^2)\hat{x} + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{s} + \ddot{y}\hat{y} \\ &= \left(\ddot{x} - \frac{v^2}{R}\right)\hat{x} + \frac{2\dot{x}v}{R}\hat{s} + \ddot{y}\hat{y} \qquad (2.13) \end{aligned}$$





Equations of motion II

We study **each component** separately. For the vertical motion,

$$F_y = q\beta_r c B_x = \gamma_r m \ddot{y} \tag{2.14}$$

Solving for $\ddot{y}$,

$$\ddot{y} - \frac{q\beta_r c B_x}{\gamma_r m} = 0 \tag{2.15}$$

We can change the derivative w.r.t. (*with respect to*) **time**, to a derivative w.r.t. the **angle** θ:

$$t = \frac{R}{\beta_r c}\theta \quad \Rightarrow \quad \frac{d}{dt} = \frac{\beta_r c}{R}\frac{d}{d\theta} \tag{2.16}$$

By doing so,

$$\left(\frac{\beta_r c}{R}\right)^2 \frac{d^2 y}{d\theta^2} - \frac{q\beta_r c B_x}{\gamma_r m} = 0 \tag{2.17}$$

After dropping the common term $\beta_r c$, and multiplying the equation by R^2,

$$\frac{d^2 y}{d\theta^2} - \frac{qB_x}{\gamma_r m \beta_r c}R^2 = 0 \tag{2.18}$$

Following a similar procedure in the horizontal plane, we get

$$\begin{aligned} F_x &= -q\beta_r c B_y \\ &= \gamma_r m\left(\ddot{x} - \frac{v^2}{R}\right) \end{aligned} \tag{2.19}$$

or

$$\frac{d^2 x}{d\theta^2} + \left(\frac{qB_y}{p}R - 1\right)R = 0 \tag{2.20}$$

We know from the **rigidity** that

$$p = qB_0\rho \tag{2.21}$$

and we can also replace R by (2.1) with the **paraxial aproximation**,

$$R = \rho\left(1 + \frac{x}{\rho}\right) \tag{2.22}$$

Equations of motion III

to get

$$\frac{d^2x}{d\theta^2} + \left[\frac{B_y}{B_0}\left(1+\frac{x}{\rho}\right) - 1\right] R = 0 \qquad (2.23)$$

Since the approximation on the **linearization** of the magnetic field is $\vec{B} = B_x\hat{x} + B_y\hat{y}$, where

$$B_x = \left(\frac{\partial B_y}{\partial x}\right) y, \qquad B_y = B_0 + \left(\frac{\partial B_y}{\partial x}\right) x \qquad (2.24)$$

then, in the case of the horizontal equation of motion, we obtain

$$\frac{d^2x}{d\theta^2} + \left[\left(1+\frac{1}{B_0}\frac{\partial B_y}{\partial x}x\right)\left(1+\frac{x}{\rho}\right) - 1\right]\rho\left(1+\frac{x}{\rho}\right) = 0$$

After performing the multiplications, we will ignore terms of second order on x and higher. On the vertical equation, we apply of course the same series of approximations. The resulting equations gives us what is known as the

Betatron motion

$$\frac{d^2x}{d\theta^2} + (1-n)x = 0 \qquad (2.25)$$

$$\frac{d^2y}{d\theta^2} + ny = 0 \qquad (2.26)$$

Stability condition

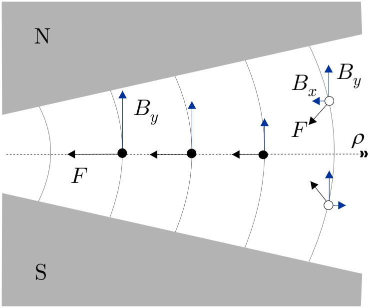
where n is the

Field index

$$n \equiv -\frac{\rho}{B_0}\left(\frac{\partial B_y}{\partial x}\right) \quad (2.27)$$

Note that in order to have a **stable motion** in both planes, the following conditions must be satisfied:

$$0 < n < 1 \quad (2.28)$$



Equations (2.25) and (2.26) describe the **weak focusing**. Note that they are **simple harmonic oscillators**, therefore their solutions are well known. In particular, for the horizontal plane,

$$x(\theta) = A\cos(\theta\sqrt{1-n}) \quad + B\sin(\theta\sqrt{1-n}) \quad (2.29)$$

with derivative

$$\frac{dx}{d\theta} = \sqrt{1-n}[-A\sin(\theta\sqrt{1-n}) + B\cos(\theta\sqrt{1-n})] \quad (2.30)$$

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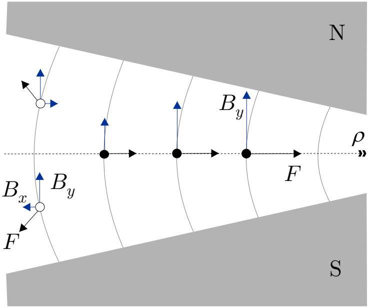


Equations (2.25) and (2.26) describe the **weak focusing**. Note that they are **simple harmonic oscillators**, therefore their solutions are well known. In particular, for the horizontal plane,

$$x(\theta) = A\cos(\theta\sqrt{1-n}) \quad + B\sin(\theta\sqrt{1-n}) \qquad (2.29)$$

with derivative

$$\frac{dx}{d\theta} = \sqrt{1-n}[-A\sin(\theta\sqrt{1-n}) + B\cos(\theta\sqrt{1-n})] \qquad (2.30)$$

Betatron motion

Applying the **initial conditions**

$$x_0 = x(\theta = 0) = A, \qquad x_0' = \frac{1}{\rho}\left(\frac{dx}{d\theta}\right)_{\theta=0} = \frac{\sqrt{1-n}}{\rho}B \tag{2.31}$$

the constants are given in terms of them,

$$A = x_0, \qquad B = \frac{\rho}{\sqrt{1-n}}x_0' \tag{2.32}$$

Substituting the last equations in the equation of motion,

$$x(\theta) = \cos(\theta\sqrt{1-n})x_0 + \frac{\rho}{\sqrt{1-n}}\sin(\theta\sqrt{1-n})x_0' \tag{2.33}$$

$$x'(\theta) = \frac{1}{\rho}\frac{dx}{d\theta}\frac{1}{\rho} = -\frac{\sqrt{1-n}}{\rho}\sin(\theta\sqrt{1-n})x_0 + \cos(\theta\sqrt{1-n})x_0' \tag{2.34}$$

we can write the solution in **matrix form** (for the vertical plane follow the same steps),

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos\phi_x & \frac{\rho}{\sqrt{1-n}}\sin\phi_x \\ -\frac{\sqrt{1-n}}{\rho}\sin\phi_x & \cos\phi_x \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = M_H\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \tag{2.35}$$

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi_y & \frac{\rho}{\sqrt{n}}\sin\phi_y \\ -\frac{\sqrt{n}}{\rho}\sin\phi_y & \cos\phi_y \end{pmatrix}\begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = M_V\begin{pmatrix} y_0 \\ y_0' \end{pmatrix} \tag{2.36}$$

Betatron tune

where the **phase advances** are given by

$$\phi_x(s) \equiv \theta\sqrt{1-n} = \frac{s}{\rho}\sqrt{1-n} \quad (2.37)$$

$$\phi_y(s) \equiv \theta\sqrt{n} = \frac{s}{\rho}\sqrt{n} \quad (2.38)$$

Particles move in transverse **betatron oscillations** around the design trajectory.

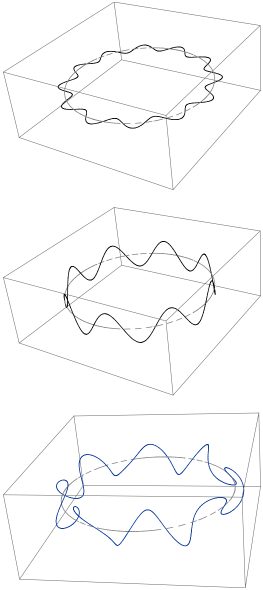
The **number of oscillations** performed by a particle in a particular plane (horizontal or vertical), is accounted by the

Betatron tune

$$Q_{x,y} = \frac{\phi_{x,y}(2\pi)}{2\pi} \quad (2.39)$$

Note that the weak focusing condition can be written as

$$Q_x^2 + Q_y^2 = 1 \quad (2.40)$$



A. Castilla
L. Medina

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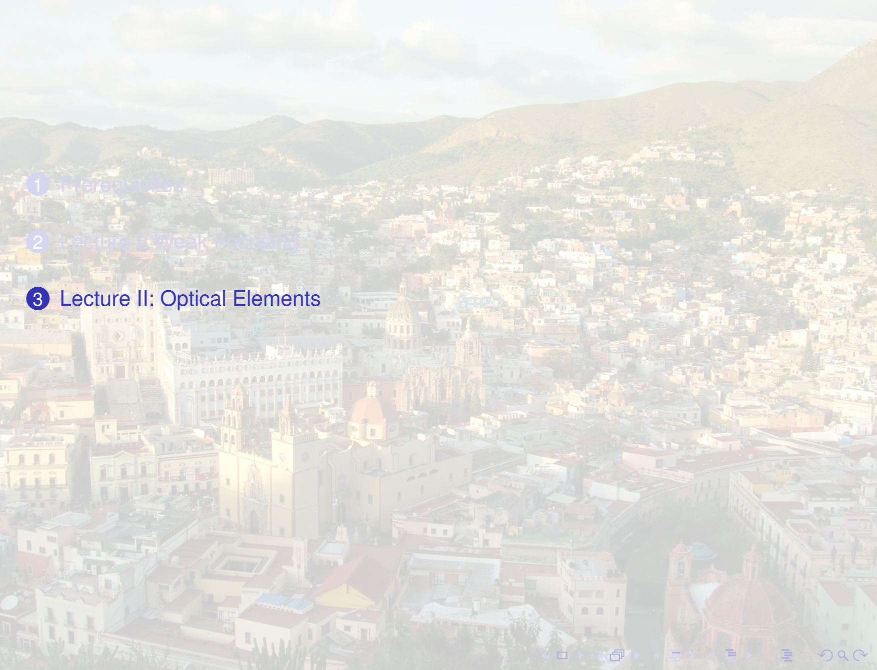


Field expansion

The **transverse magnetic field** is

$$\vec{B} = B_x\hat{x} + B_y\hat{y} \tag{3.1}$$

Making a **Taylor expansion** of the field components we get

$$B_x = B_{0x} + \frac{\partial B_x}{\partial y}y + \frac{1}{2!}\frac{\partial^2 B_x}{\partial y^2}y^2 + \ldots \tag{3.2}$$

$$B_y = B_{0y} + \frac{\partial B_y}{\partial x}x + \frac{1}{2!}\frac{\partial^2 B_y}{\partial x^2}x^2 + \ldots \tag{3.3}$$

The first terms in both expansions correspond to **dipole** terms, the second terms to **quadrupoles**, then **sextupoles**, and so on.

Due to the **Maxwell's equation** for $\nabla \times \vec{H}$, in the absence of electric fields and currents,

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} \tag{3.4}$$

Most of the time we are not interested in vertical bending dipoles, then $B_{0x} = 0$, and we label $B_{0y} = B_0$.

We can write the linear field as

$$\begin{aligned}\vec{B} &= \left(\frac{\partial B_y}{\partial x}y\right)\hat{x} + \left(B_0 + \frac{\partial B_y}{\partial x}x\right)\hat{y} \\ &= B_0\hat{y} + (x\hat{y} + y\hat{x})\left(\frac{\partial B_y}{\partial x}\right)\end{aligned} \tag{3.5}$$

Another way to make the expansion, via **multipoles** (in complex notation):

$$B_x + iB_y = B_0\sum_{n=0}^{\infty}(a_n + ib_n)\left(\frac{x+iy}{a}\right)^n \tag{3.6}$$

where $n = 0$ corresponds to dipoles, $n = 1$ to quadrupoles, etc. The coefficients b_n and a_n are called **normal** and **skew**, respectively.

Unfortunately, this is not the only convention!

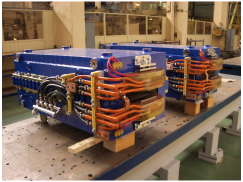
Linear optical elements

We can have two types of optical elements (magnets) with a linear magnetic field:

Dipoles

- Long magnets that **bend** the design trajectory.
- They may or may not include focusing (combined function).
- Special case: **drifts** (no field).

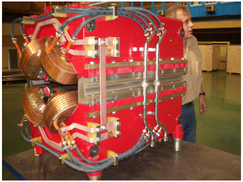
$$\vec{B} = B_0\hat{y} + (x\hat{y} + y\hat{x})\left(\frac{\partial B_y}{\partial x}\right) \tag{3.7}$$



Quadrupoles

- Design trajectory is **straight**.
- **Focus** particles moving out of the design orbit.
- Special case: **thin lens approximation**.

$$\vec{B} = (x\hat{y} + y\hat{x})\left(\frac{\partial B_y}{\partial x}\right) \tag{3.8}$$



The solution of the equation of motion of a particle that passes along each of them can be described with their corresponding **transport matrix**. We have already found the one for the dipole.

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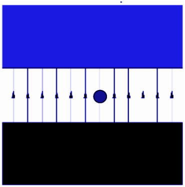

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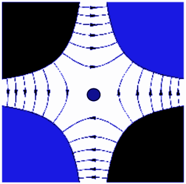


The solution of the equation of motion of a particle that passes along each of them can be described with their corresponding **transport matrix**. We have already found the one for the dipole.

Transport matrices

Therefore, we can build the accelerator optics out of *Lego* transport matrices.

Let $\vec{w}_0$ be the set of initial **transverse coordinates**. They are transformed to $\vec{w}$ after crossing a system represented by the matrix M according to

$$\vec{w} = M\vec{w}_0 \tag{3.9}$$

A **lattice** is usually made of substructures or **cells**, an array of magnets repeating along the machine.
To describe the motion of a particle along the accelerator, we can simply **multiply** the piecewise solutions given by the transport matrices.

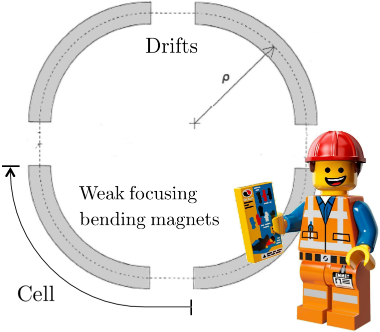
Take for example the machine on the right. Each cell is made of a drift and a dipole (bending),

$$M_{cell} = M_B M_D$$

The order in which we have to perform the matrices is from right to left, since it is the order in which the particle encounters the elements. For the machine composed of 4 cells,

$$M = (M_{cell})^4 = (M_B M_D)^4$$

gives the solution after one turn.



Dipoles

We know the general transport matrix for a **dipole without focusing**,

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi_x & \frac{\rho}{\sqrt{1-n}}\sin\phi_x & 0 & 0 \\ -\frac{\sqrt{1-n}}{\rho}\sin\phi_x & \cos\phi_x & 0 & 0 \\ 0 & 0 & \cos\phi_y & \frac{\rho}{\sqrt{n}}\sin\phi_y \\ 0 & 0 & -\frac{\sqrt{n}}{\rho}\sin\phi_y & \cos\phi_y \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix} \quad (3.10)$$

Remember that the phase advances are given by

$$\phi_x(s) = \frac{s}{\rho}\sqrt{1-n}, \qquad \phi_y(s) = \frac{s}{\rho}\sqrt{n}$$

Then, taking the field index to zero ($n \to 0$), we have the transport of a

Dipole (no focusing)

$$\begin{pmatrix} x(\theta) \\ x'(\theta) \\ y(\theta) \\ y'(\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 \\ -\frac{1}{\rho}\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & \rho\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix} \quad (3.11)$$

with bending angle θ and **no focusing**.

Note: In a circular machine composed of N dipoles of length L_B, the **integrated dipole field over one turn** is

$$\oint B\, ds \approx NBL_B = 2\pi \frac{p}{q} \quad (3.12)$$

Careful with units!

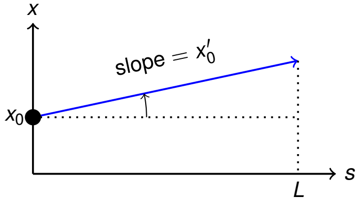
Drifts

The sub-matrix for the vertical plane represents a field-free **drift** where no bending is performed. This is valid in general for x or y when there is no field:

Drift

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \end{pmatrix} \tag{3.13}$$

where s is the length of the drift.



Interlude:

Note that we have switched to **4** $\times$ **4** matrices. In fact, a more complete study requires 6 coordinates and thus, 6×6 matrices): x, x', y, y', s, δ (as we will see later).

Nevertheless, when horizontal and vertical motions are **uncoupled** (that is, they are independent of each other), we can split our 4×4 matrices,

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} \multicolumn{2}{c}{M_x} & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{M_y} \\ 0 & 0 & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \end{pmatrix} \tag{3.14}$$

in pure horizontal/vertical motion, with our usual 2×2 matrices:

$$\vec{X} = M_x \vec{X}_0, \quad \vec{Y} = M_y \vec{Y}_0 \tag{3.15}$$

Quadrupoles: Quadrupole gradient

For the quadrupoles, the components of the magnetic field (3.8) are commonly written as

$$B_x = Gy, \qquad B_y = Gx \tag{3.16}$$

where G, measured in T/m, is the

Quadrupole gradient

$$G = \frac{B_{pole}}{a} = \frac{2\mu_0 NI}{a^2} \approx \frac{\partial B_y}{\partial x} \tag{3.17}$$

B_{pole} is the field at the pole tips, a is the inner radius of the quadrupole, μ_0 is the vacuum magnetic permeability,

$$\mu_0 = 4\pi \times 10^{-7}\ \frac{\text{Tm}}{\text{A}} \tag{3.18}$$

and N is the number of turns of current I around the poles.

We simply follow a similar procedure (and approximations) to the derivation of the equations of betatron motion.

The corresponding horizontal and vertical forces

$$F_x = -q\beta_r cGx \tag{3.19}$$

and

$$F_y = q\beta_r cGy \tag{3.20}$$

lead to the equations

$$x'' + \frac{G}{B_0\rho}x = 0 \tag{3.21}$$

and

$$y'' - \frac{G}{B_0\rho}y = 0 \tag{3.22}$$

Quadrupoles: Quadrupole strength

where we have replaced the derivatives w.r.t. θ to derivatives w.r.t. s,

$$\frac{d}{d\theta} = \frac{1}{R}\frac{d}{ds} \tag{3.23}$$

By using the

Quadrupole strength[2]

$$K \equiv \frac{1}{(B_0\rho)}\left(\frac{\partial B_y}{\partial x}\right) = \frac{G}{(B\rho)} \equiv k^2 \tag{3.24}$$

with units of m^{-2}, the equations of motion are

$$x'' + Kx = 0, \qquad y'' - Ky = 0 \tag{3.25}$$

where $''$ denotes derivation w.r.t. s.

The horizontal equation of motion in (3.25) is again a harmonic oscillator and its solution is given in terms of **sine and cosine**. In the vertical plane, the solution is given by **hyperbolic sine and cosine**.

We can then express the general transport matrix of a

[2] Two remarks: 1) Careful with K vs. k! 2) Some authors define K with a negative sign!

Quadrupoles: Focusing and defocusing

Quadrupole (horizontal **focusing** / vertical defocusing)

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} \cos(ks) & \frac{1}{k}\sin(ks) & 0 & 0 \\ -k\sin(ks) & \cos(ks) & 0 & 0 \\ 0 & 0 & \cosh(ks) & \frac{1}{k}\sinh(ks) \\ 0 & 0 & k\sinh(ks) & \cosh(ks) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \end{pmatrix} \tag{3.26}$$

This represent a **horizontal focusing** / **vertical defocusing quadrupole**.

To compensate this, quadrupoles are commonly placed in pairs, being the second quadrupole **horizontal defocusing** / **vertical focusing**.

Quadrupole (horizontal **defocusing** / vertical focusing)

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} \cosh(ks) & \frac{1}{k}\sinh(ks) & 0 & 0 \\ k\sinh(ks) & \cosh(ks) & 0 & 0 \\ 0 & 0 & \cos(ks) & \frac{1}{k}\sin(ks) \\ 0 & 0 & -k\sin(ks) & \cos(ks) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \end{pmatrix} \tag{3.27}$$

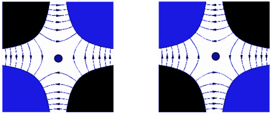
Another important remark: $k = \sqrt{K}$, and K is always taken positive, so for defocusing quads corresponding to negative K, the sign is explicitly written: $-K$.

Quadrupoles: Thin Lenses

Note that this correspond to change K (defined positive) to $-K$ since

$$\begin{aligned} \sinh u &= -i \sin(iu) \\ \cosh u &= \cos(iu) \end{aligned} \tag{3.28}$$

Physically, this represents a 90° rotation of the quadrupole, so the north and south poles end up interchanged.



In the **thin lens approximation**, we make $s = L \to 0$ while keeping KL constant. In addition, the **focal length** of the quadrupole is given by

$$f \equiv \frac{1}{KL} = \frac{1}{k^2 L} \tag{3.29}$$

The corresponding matrices are then

Thin quadrupole (horizontal focusing / vertical defocusing)

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix} \tag{3.30}$$

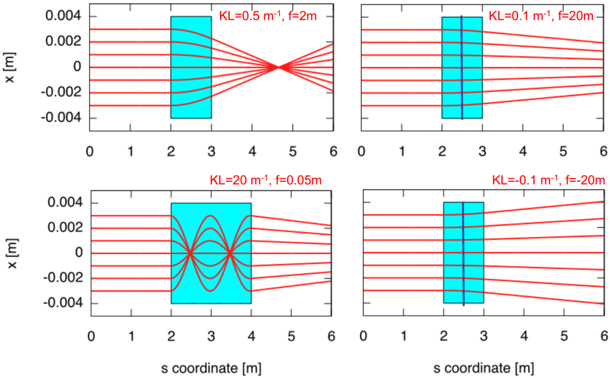
and

Thin quadrupole (horizontal defocusing / vertical focusing)

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix} \tag{3.31}$$

when we change $f \to -f$.

Quadrupoles: Focal Length

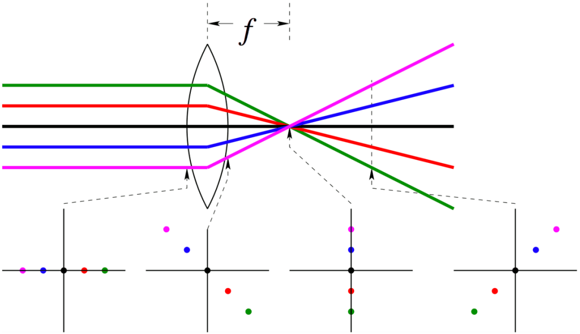


In general, a quadrupole can be treated as a thin quadrupole when

$$|f| \gg L \tag{3.32}$$

and then we can use the simpler transport matrices (3.30)-(3.31) instead of (3.26)-(3.27) where the trigonometric functions make the oscillatory motion evident (as in the thick quadrupole in the bottom left corner).

Quadrupoles: Focal Length



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Quadrupoles: Doublet I

Example: *Doublet*

A **doublet** is a system formed by a pair of quadrupoles, separated by a drift. Consider only the horizontal plane. If the first quadrupole is focusing and the second is defocusing, the transfer matrix of the system is

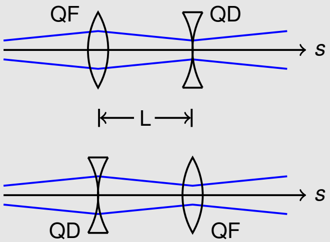
$$M_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_F} & L \\ \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} & 1 + \frac{L}{f_D} \end{pmatrix} \tag{3.33}$$

The element m_{12} gives us the inverse of the focal length of this system,

$$\frac{1}{f_{\text{doublet}}} \equiv \frac{1}{f_D} - \frac{1}{f_F} - \frac{L}{f_F f_D} \tag{3.34}$$

In the particular case that $f_D = f_F = f$,

$$\frac{1}{f_{\text{doublet}}} = -\frac{L}{f^2} \tag{3.35}$$



An **alternating gradient** system as this one provides **net focusing**, a fundamental feature for accelerator **strong focusing**.

Quadrupoles: Doublet II

Example: *Doublet (cont.)*

Consider an incoming paraxial ray $(x_0, 0)$. Then, after passing the doublet,

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M_{\text{doublet}} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{L}{f}\right) x_0 \\ -\frac{L}{f^2} x_0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f} \\ -\frac{L}{f^2} \end{pmatrix} x_0 \tag{3.36}$$

In order this to be focusing, x and x' must have **opposite signs**. Since both L and f are positive, we can analyse two cases:

If	$x_0 > 0,$	$x_0 < 0,$
then the slope	$x' = -\frac{L}{f^2} x_0 < 0,$	$x' = -\frac{L}{f^2} x_0 > 0,$
which demands	$x = \left(1 - \frac{L}{f}\right) x_0 > 0.$	$x = \left(1 - \frac{L}{f}\right) x_0 < 0\ .$
Now, since (again)	$x_0 > 0,$	$x_0 < 0,$
then	$1 - \frac{L}{f} > 0$	$1 - \frac{L}{f} > 0$
or	$f > L.$	$f > L.$

That is, an equal-strength doublet is **net focusing** under condition that the focal length of each lens is greater than the distance between them.

In the reversed case (defocusing quad first, followed by the focusing one), the same condition is reached after following a same procedure. Then, alternating quadrupoles continuously produces a system that is **overall net focusing and stable**.

Dispersion I

So far we have assumed that the design trajectory particle and our particle have the **same momentum**.

What happen when this is not the case? Let us suppose that

$$p = p_0 + \Delta p = p_0(1+\delta) \qquad (3.37)$$

where

Momentum deviation

$$\delta \equiv \frac{\Delta p}{p_0} \ll 1 \qquad (3.38)$$

Then, the equation of motion for the dipole in the **horizontal** plane is

$$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p_0(1+\delta)}R - 1\right)R = 0 \qquad (3.39)$$

Compare this equation with (2.20). Using the approximation

$$\frac{1}{1+\epsilon} \approx 1 - \epsilon \qquad (3.40)$$

we get

$$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p_0}(1-\delta)R - 1\right)R = 0 \qquad (3.41)$$

Rearranging,

$$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{p_0}R - 1\right)R = \frac{qB_y}{p_0}R^2\delta \qquad (3.42)$$

we obtain

$$\frac{d^2x}{d\theta^2} + (1-n)x = \rho\delta \qquad (3.43)$$

where n is the **field index** defined in (2.27) and δ is **constant**.

The momentum effect is called **dispersion**.

Dispersion II

We can see that the only change is the addition of an **inhomogeneous** term to the ordinary differential equation. Therefore, the solution of is given by the sum of the **homogeneous solution** x_h (already derived, a harmonic oscillator) and the **particular solution** x_p,

$$x(\theta) = x_h(\theta) + x_p(\theta) \tag{3.44}$$

Based on the form of the differential equation, we propose $x_p = C$, with C a constant, as particular solution. Substituting into the equation,

$$\frac{d^2C}{d\theta^2} + (1-n)C = 0 + (1-n)C = \rho\delta \tag{3.45}$$

where we can solve for C,

$$x_p = C = \frac{\rho\delta}{1-n} \tag{3.46}$$

The complete solution is then

$$x(\theta) = A\cos(\theta\sqrt{1-n}) + B\sin(\theta\sqrt{1-n}) + \frac{\rho}{1-n}\delta \tag{3.47}$$

We now get its derivative,

$$\frac{dx}{d\theta} = \sqrt{1-n}[-A\sin(\theta\sqrt{1-n}) + B\cos(\theta\sqrt{1-n})] \tag{3.48}$$

Dispersion III

in order to get A and B in terms of the initial conditions (x_0, x_0'). By doing so, we obtain the constants

$$A = x_0 - \frac{\rho}{1-n}\delta, \qquad B = \frac{\rho}{\sqrt{1-n}}x_0' \tag{3.49}$$

The solution can be written with a 3×3-matrix:

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \begin{pmatrix} \cos\phi_x & \frac{\rho}{\sqrt{1-n}}\sin\phi_x & \frac{\rho}{1-n}(1-\cos\phi_x) \\ -\frac{\sqrt{1-n}}{\rho}\sin\phi_x & \cos\phi_x & \frac{1}{1-n}\sin\phi_x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix} \tag{3.50}$$

where $\phi_x = \theta\sqrt{1-n}$. Note that δ has become a **"coordinate"**. If we take $n \to 0$, we obtain the transport matrix for a

Dipole (no focusing, with dispersion)

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix} \tag{3.51}$$

The momentum deviation **does not** gives rise to vertical dispersion in the presence of horizontal dipoles, as can be verified by carrying similar calculations out for the **vertical** plane.

Dispersion IV

Example: 180° *spectrometer magnet*

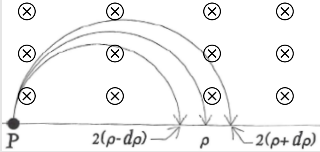
A mass **spectrometer** separates particles according to their energy.

Suppose a 180° bending magnet (π rad); its corresponding transport matrix is then

$$M = \begin{pmatrix} \cos\pi & \rho\sin\pi & \rho(1-\cos\pi) \\ -\frac{1}{\rho}\sin\pi & \cos\pi & \sin\pi \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & \rho[1-(-1)] \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In this magnet, a particle with initial coordinates $(0, 0, \pm\delta)$, will experience deflection and, at the exit of the dipole its final coordinates will be

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = M \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2\rho \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \pm\delta \end{pmatrix} = \begin{pmatrix} \pm 2\rho\delta \\ 0 \\ \pm\delta \end{pmatrix}$$



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Consider a 0.1 T magnet, used in a 20 MeV/c beam. Then, particles at the reference position, but with 1% deviation of momentum, will be displaced a distance

$$x = 2\rho\delta = 2 \cdot \frac{p[\mathrm{GeV/c}]}{0.3B[\mathrm{T}]} \cdot \delta = 2 \cdot \frac{0.02\ \mathrm{GeV/c}}{0.3(0.1\ \mathrm{T})} \cdot 0.01 = 2(0.67\ \mathrm{m})(0.01) \approx 1.33\ \mathrm{cm}$$

from the ideal orbit at the exit of the spectrometer.