Lecture 2

Motivations and Preliminaries:

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The Basics: Special Relativity

Energy & Mass units

- To describe the energy of individual particles, we use the eV, the energy that a unit charge

$$e = 1.6 \times 10^{-19} \text{ Coulomb}$$

gains when it falls through a potential, $\Delta\Phi = 1$ volt.

$$1 \ eV = 1.6 \times 10^{-19} \text{ Joule}$$

- We can use Einstein's relation to convert rest mass to energy units

$$E_o = mc^2$$

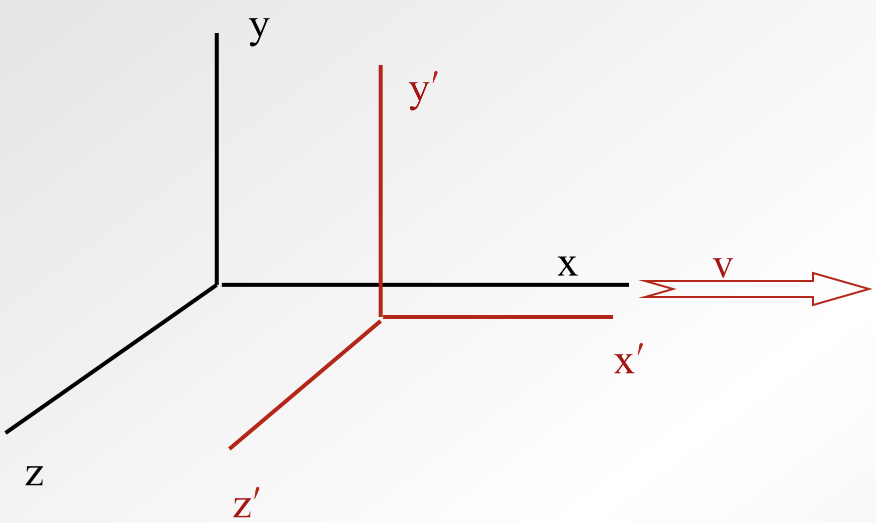
- For electrons,

$$E_{o,e} = 9.1\text{x}10^{-31} \text{ kg x } (3\text{x}10^{8} \text{ m/sec})^2 = 81.9\text{x}10^{-15} \text{ J } = 0.512 \text{ MeV}$$

- For protons,

$$E_{o,p} = 938 \text{ MeV}$$

Relativity: transformation of physical laws between inertial frames



What is an inertial frame?

How can you tell?

In an inertial frame free bodies have no acceleration

Postulate of Galilean relativity

Under the Galilean transformation

$$
\begin{aligned}
x' &= x - V_x t \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
\quad \Rightarrow \quad v'_x = v_x - V_x
$$

the laws of physics remain invariant in all inertial frames.

$$F = m \frac{d^2}{dt^2} x \quad \text{in ALL frames of reference}$$

Not true for electrodynamics !

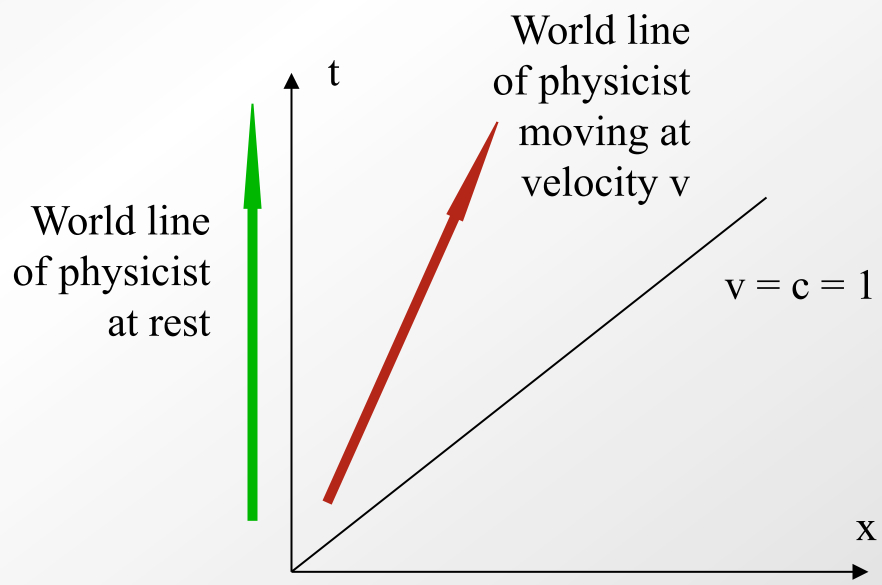
For example, the propagation of light

Observational basis of special relativity

*Observation 1: Light **never** overtakes light in empty space*
==> Velocity of light is the same for all observers

For this discussion let c = 1

Space-time diagrams



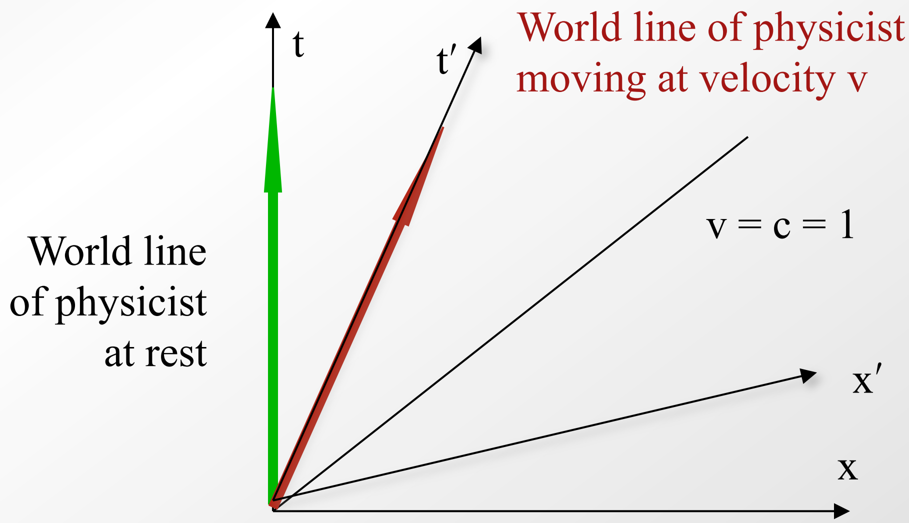
Relativistic invariance

Observation 2:

All the laws of physics are the same in all inertial frames

❖ This requires the invariance of the space-time interval

$$(ct')^2 - x'^2 - y'^2 - z'^2 = (ct)^2 - x^2 - y^2 - z^2$$



Lorentz boost replaces Galilean transformation

$$ct' = \gamma(ct - \beta z)$$

$$x' = x$$

$$y' = y$$

$$z' = \gamma(z - \beta ct)$$

where Einstein's relativistic factors are

$$\beta = \frac{|\mathbf{v}|}{c} \quad \text{and} \quad \gamma = 1 \Big/ \sqrt{1 - \beta^2}$$

Thus we have the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad , \quad t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}$$

$$y' = y \quad , \quad z' = z$$

Or in matrix form

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Show that the Lorentz transformation preserves 4-interval

Proper time & proper length

- Define proper time, τ, as duration *measured in the rest frame*

- The length of an object in its rest frame is L_o

- As seen by an observer moving at v, the duration, $\mathscr{T}$, is

$$\mathscr{T} = \frac{\tau}{\sqrt{1 - v^2/c^2}} \equiv \gamma\tau > \tau$$

And the length, L, is

$$L = L_o/\gamma$$

Lorentz scalar invariants & four-vectors

- To describe physical quantities in space-time, we introduce quantities with well defined transformations between different inertial frames.
- *Lorentz scalars* are quantities described by a single number that has the same value in all inertial frames.
  - That is, a scalar is a Lorentz invariant
  - Example: the charge on an electron is a scalar
  - What is another example?
- *Lorentz four-vectors*, w^α, have 1 time-like & 3 space-like components ($\alpha = 0, 1, 2, 3$)
  - $x^\alpha = (ct, x, y, z) = (ct, \mathbf{x})$ [Also, $x_\alpha = (ct, -x, -y, -z)$
  - Note Latin indices $i = 1, 2, 3$

Four-vectors & scalar products

- Norm of w^α is a *Lorentz scalar* ***(invariant in all frames)***

$$|w| = (w^\alpha w_\alpha)^{1/2} = (w_o^2 - w_1^2 - w_2^2 - w_3^2)^{1/2}$$

$$|w|^2 = g_{\mu\nu} w^\mu w^\nu$$ where the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- For two 4-vectors u^α and v^α their scalar (dot) product is

$$u \bullet v = (u^\alpha v_\alpha) = (u_0 v_0 - u_1 v_1 - u_2 v_2 - u_3 v_3)$$

Velocity, energy and momentum

- For a particle with 3-velocity $\boldsymbol{v}$, the *4-velocity* is

$$u^\alpha = (\gamma c, \gamma \mathbf{v}) = \frac{dx^\alpha}{d\tau}$$

- *Total energy, E*, of a particle equals its rest mass, m_o, plus kinetic energy, T

$$\boxed{E = m_o c^2 + T = \gamma m_o c^2}$$

Note that the energy is not a Lorentz invariant

$$\boxed{(E)^2 = p^2 c^2 + \left(mc^2\right)^2 = \left(\gamma m_o c^2\right)^2}$$

Tutorial exercise: 10 minutes

- The Fermilab Alvarez Linac accelerates protons to a *kinetic energy* of 400 MeV

  - a) Calculate the total energy of the protons in units of MeV
  - b) Calculate the momentum of the protons in units of MeV/c
  - c) Calculate the relativistic gamma factor
  - d) Calculate the proton velocity in units of the velocity of light.

Motivation: Discovery science

How do we understand
the underlying structure of things?

Motivations: How it all began

Paradigm 1: Fixed target experiments

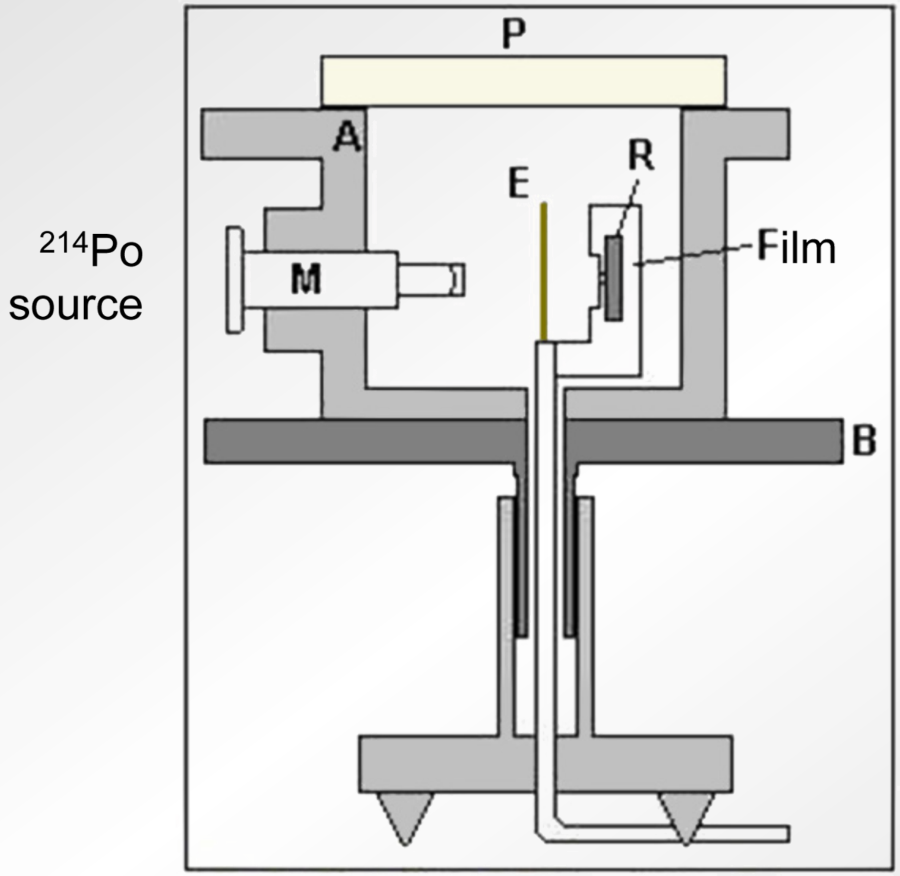


Fig1. Marsden-Geiger experiment.

Rutherford explains scattering of *alpha particles* on *gold* discovering the nucleus & urges … *On to higher energy probes!*

Rutherford articulated Figure of Merit 1

Particle energy on target

Why we use energetic beams for research?

- Resolution of "Matter" Microscopes
  - Wavelength of Particles (γ, e, p, ...) (de Broglie, 1923)

$$\boxed{\lambda = h/p} = 1.2 \;\; \mathrm{fm}/p\;[\;\mathrm{GeV/c}]$$

  - Higher momentum => shorter wavelength => better resolution

- Energy to Matter
  - Higher energy produces heavier particles

The advantage of the fixed target physics:
Figure of Merit 2

$$\frac{Events}{\text{second}} = \sigma_{process} \circ \underbrace{Flux \circ Target\ Number\ Density \circ Path\ Length}_{Luminosity}$$

Typical values:

Flux (particle current) ~ 10^{12} - 10^{14} s^{-1}
Number density of the target ~ $\rho N_A (Z/A)$ ~ $5 \times 6 \times 10^{23} / 2$
Path length through the target ~ 10 cm

Luminosity ~ $15 \times 10^{23} \times 10^{14}$ ~ $10^{36} – 10^{38}$ $cm^{-2}s^{-1}$

Ideal for precision & rare process physics,
BUT how much energy is available for new physics

Momentum & available energy

- The 4-momentum, p^μ, is

$$p^\mu = m_o u^\mu = (c\gamma m_0, \gamma m_0 \mathbf{v})$$

- Recalling that

$$\boxed{E = m_o c^2 + T = \gamma m_o c^2}$$

we have

$$p^\mu = m_o u^\mu = (c\gamma m_0, \gamma m_0 \mathbf{v}) = (\frac{E}{c}, \gamma m_0 \mathbf{v})$$

$$p^2 = (m^2 c^2 \gamma^2 - \gamma^2 m^2 v^2) = \left[m^2 c^2 \gamma^2 - \gamma^2 m^2 c^2 (1 - \frac{1}{\gamma^2}) \right]$$

$$= (m^2 c^2 \gamma^2 - \gamma^2 m^2 c^2 + m^2 c^2) = m^2 c^2$$

Scalars are Lorentz invariant:
Particle collisions

❖ Two particles have equal rest mass m_0.

Laboratory Frame (LF): one particle at rest, total energy is E.

projectile

target

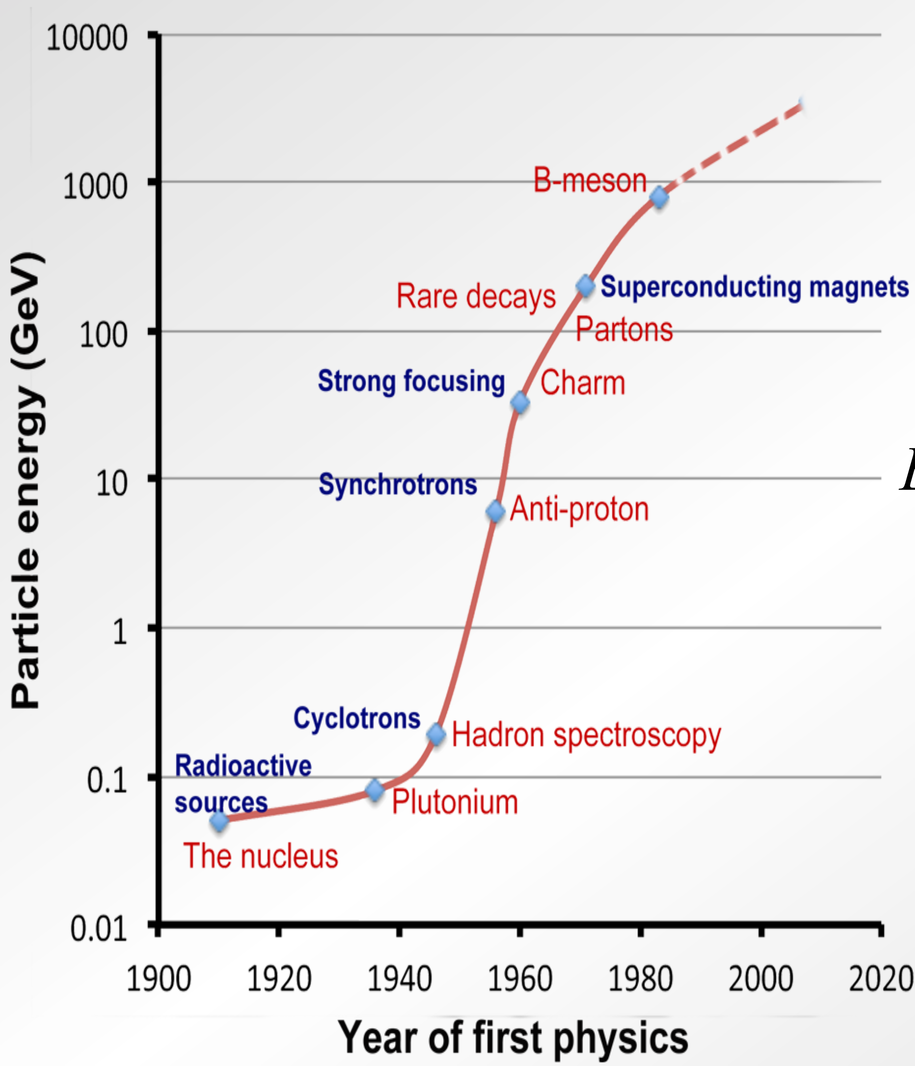
$$\mathbf{P_1} = (E_1/c, \mathbf{p_1}) \qquad \mathbf{P_2} = (m_0 c, \mathbf{0})$$

Centre of Momentum Frame (CMF): Velocities are equal & opposite, total energy is E_{cm}.

$$\mathbf{P_1} = (E_{\mathrm{cm}}/(2c), \mathbf{p}) \qquad \mathbf{P_2} = (E_{\mathrm{cm}}/(2c), -\mathbf{p})$$

Exercise: Relate E to E_{cm}

The fixed target paradigm has its limits



Invariance of

$$(p_1 + p_2)^{\mu} \cdot (p_1 + p_2)_{\mu}$$

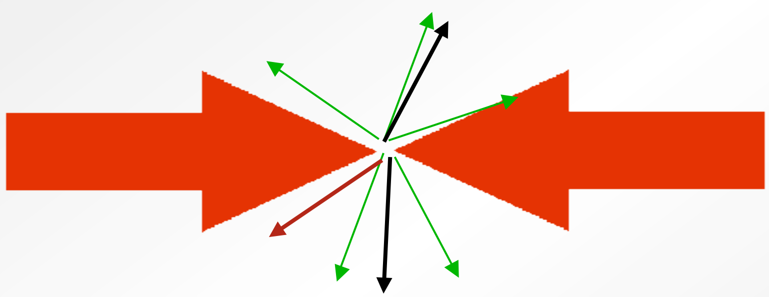
in Lorentz frames implies that

$$E_{cm} = \sqrt{m_1^2 + m_2^2 + 2m_2c^2E_{beam}}$$

$$\approx \sqrt{2mc^2E_{beam}} \quad \textit{for equal masses}$$

A great invention comes to the rescue

Collide beams !



If $m_1 = m_2$ and if $E_1 = E_2 = E$

$$E_{cm} = 2\ E$$

The full kinetic energy of both particles is now available to physical processes

ADA - The first storage ring collider (e^+e^-)
by B. Touschek at Frascati (1960)

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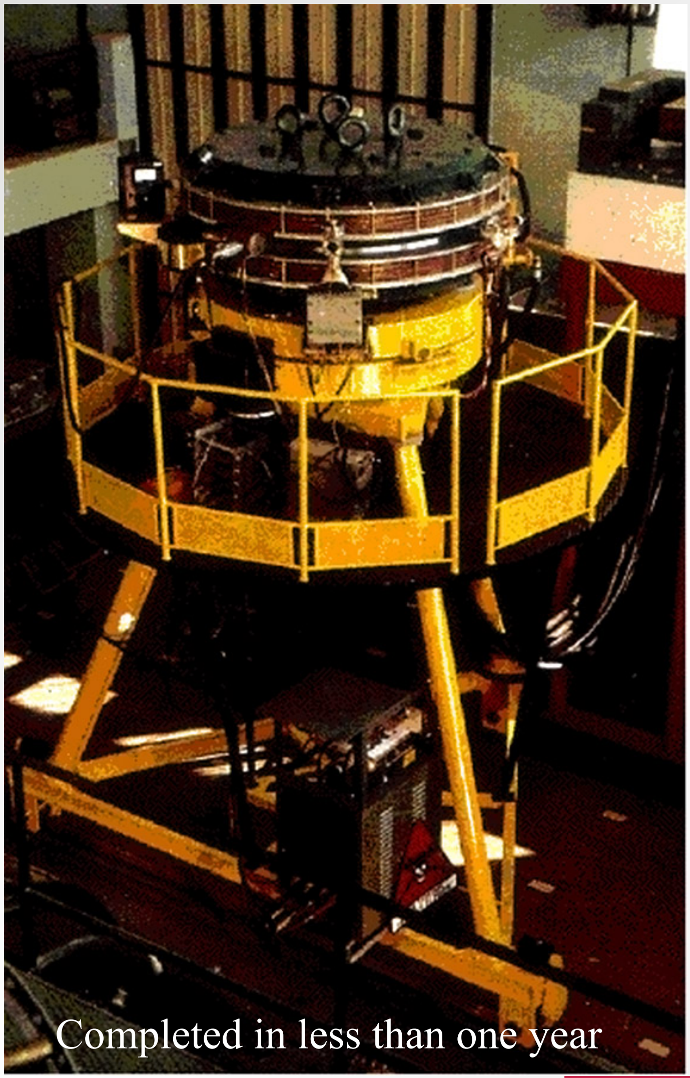
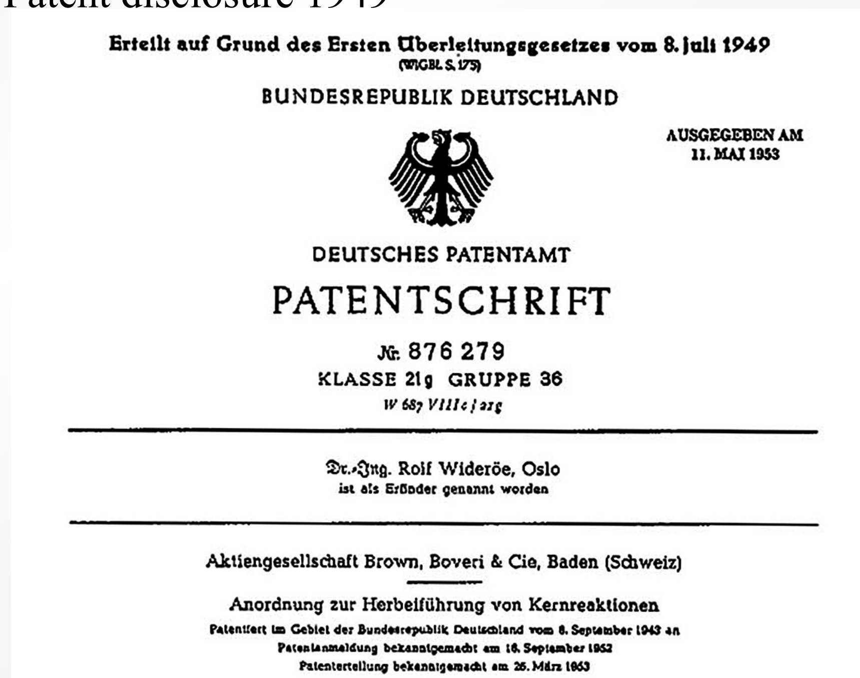
FACULTY OF ECONOMICS



Completed in less than one year

The storage ring collider idea was invented by Rolf Wiederoe in 1943!

– Collaboration with Bruno Touschek
– Patent disclosure 1949



Erteilt auf Grund des Ersten Überleitungsgesetzes vom 8. Juli 1949
(WiGBl. S. 175)

BUNDESREPUBLIK DEUTSCHLAND

AUSGEGEBEN AM
11. MAI 1953

DEUTSCHES PATENTAMT

PATENTSCHRIFT

Nr. 876 279

KLASSE 21g GRUPPE 36

W 687 VIIIc / 21g

Dr.-Ing. Rolf Wideröe, Oslo
ist als Erfinder genannt worden

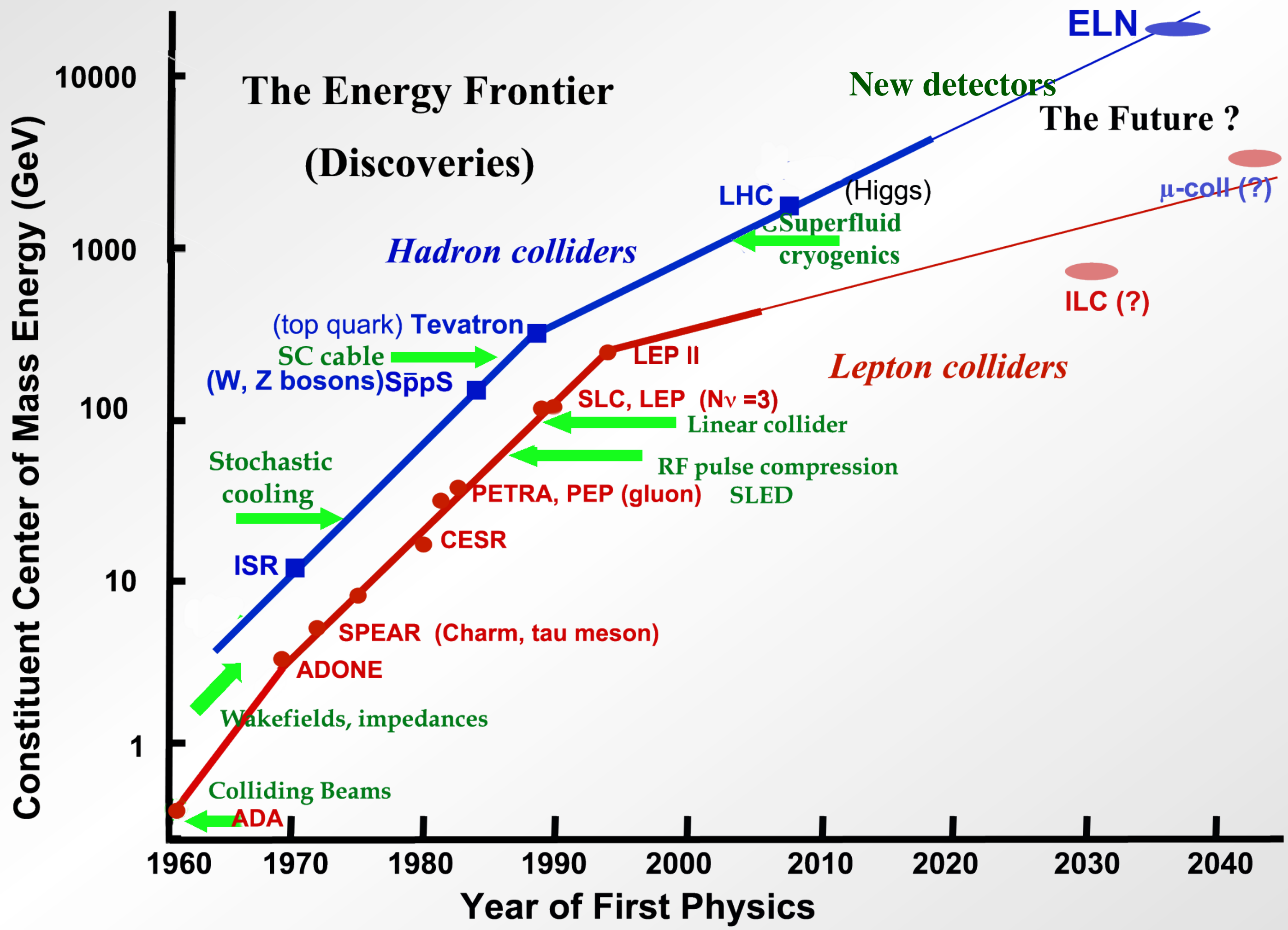
Aktiengesellschaft Brown, Boveri & Cie, Baden (Schweiz)

Anordnung zur Herbeiführung von Kernreaktionen

Patentiert im Gebiet der Bundesrepublik Deutschland vom 8. September 1943 an
Patentanmeldung bekanntgemacht am 18. September 1952
Patenterteilung bekanntgemacht am 26. März 1953

$$\boxed{E_{cm} = 2E_{beam}}$$

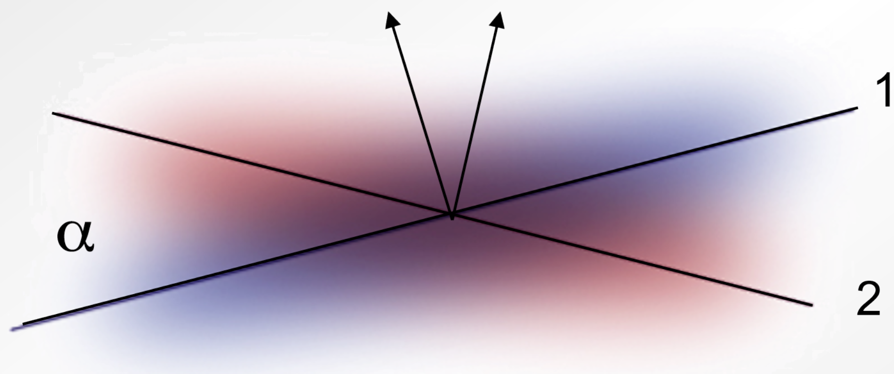
Discovery science requires discovery technology



This looks too good to be true!
What about the luminosity?

$$Events = Cross\text{-}section \times \langle Collision\ Rate \rangle \times Time$$

Beam energy: sets scale of physics accessible

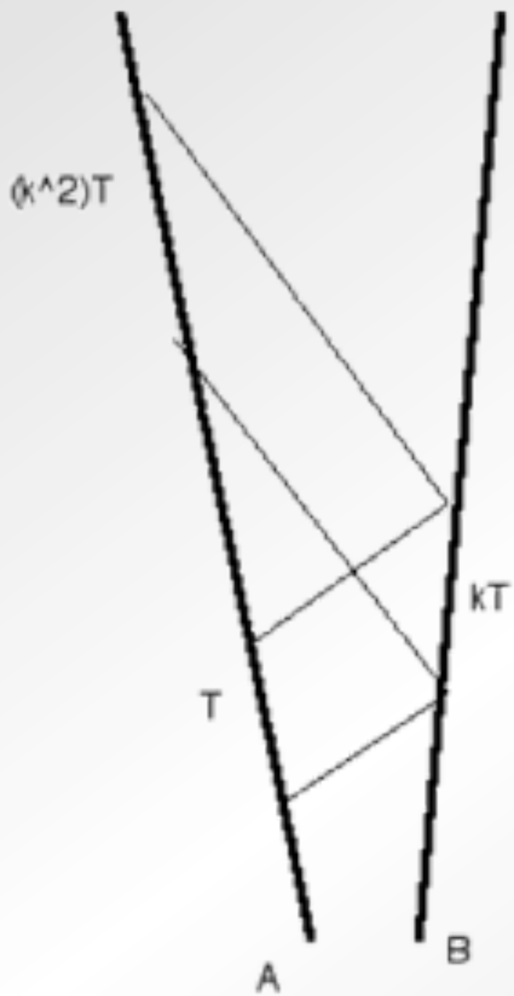


$$\text{Luminosity} = \frac{N_1 \times N_2 \times frequency}{Overlap\ Area} = \frac{N_1 \times N_2 \times f}{4\pi\sigma_x\sigma_y} \times Correction\ factors$$

We want large charge/bunch, high collision frequency & small spot size

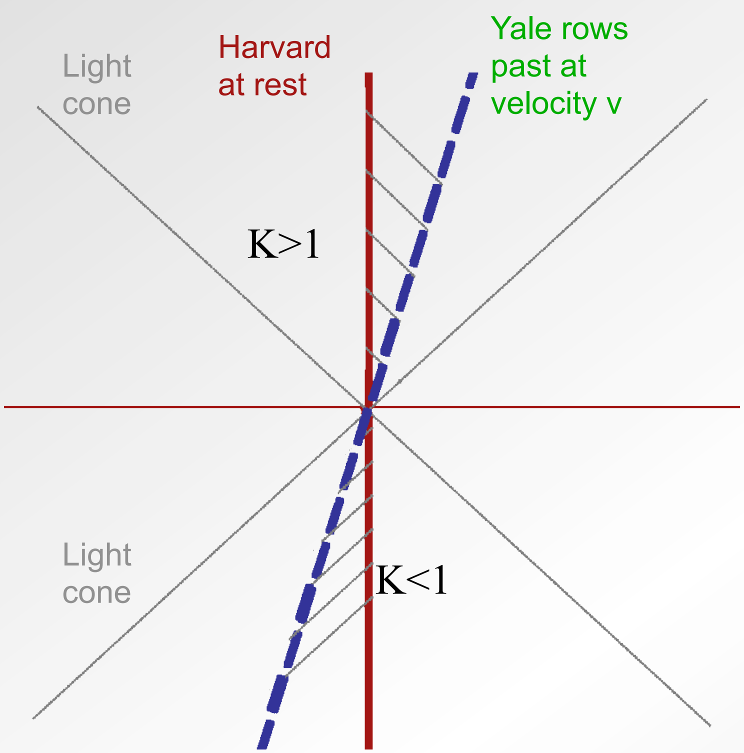
$$Luminosity \sim 10^{31} - 10^{34}\ cm^{-2}s^{-1}$$

Bondi's k-factor



- $k_{A\ to\ B} = k_{B\ to\ A}$
- k is known as the relativistic Doppler shift
- The diagram shows A & B moving apart; the Doppler shift decreases frequencies
- Measurements:
  - Time – how do we do it?
  - Distance– how do we do it?

Doppler shift of frequency
Harvard v. Yale crew teams



Distinguish between coordinate transformations & observations

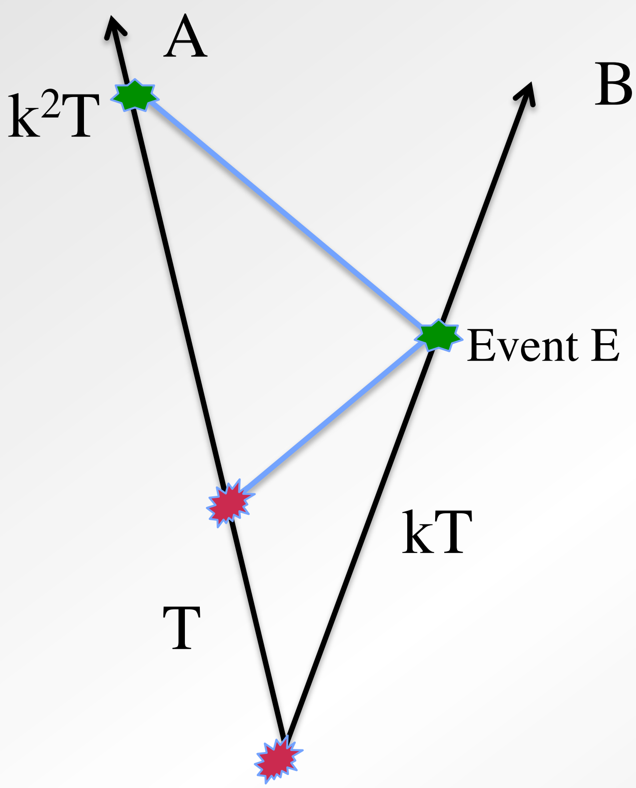
- Yale sets his signal to flash at a constant interval, $\Delta t'$
- Harvard sees the interval foreshortened by K(v) as Yale approaches
- Harvard see the interval stretched by K(-v) as Yale moves away

Homework: Using the world line diagram

Show $K(v) = K^{-1}(-v)$

For γ large find $K(\gamma)$

Computing the Doppler shift

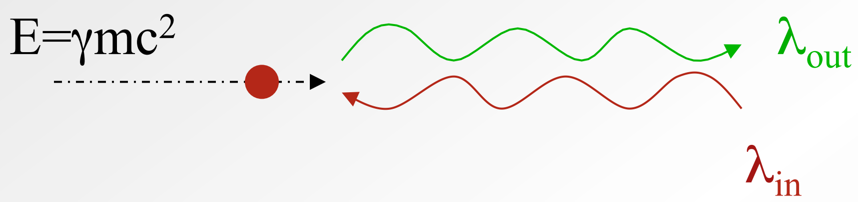


Observer A says that E happens at a position x and a time t

Then $v_{AB} = x / t$

Write x and t in terms of k and T

Head-on Compton scattering
by an ultra-relativistic electron

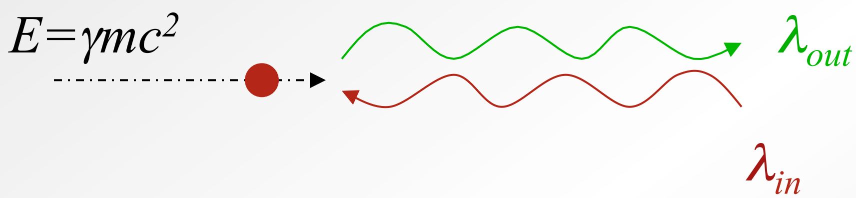


- What wavelength is the photon that is scattered by 180°?

Write your answer in terms of $K(\gamma)$

Source of beam loss in electron synchrotrons

- Remember the Compton scattering of photons up shifts the energy by 4 γ^2

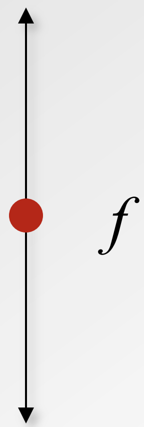


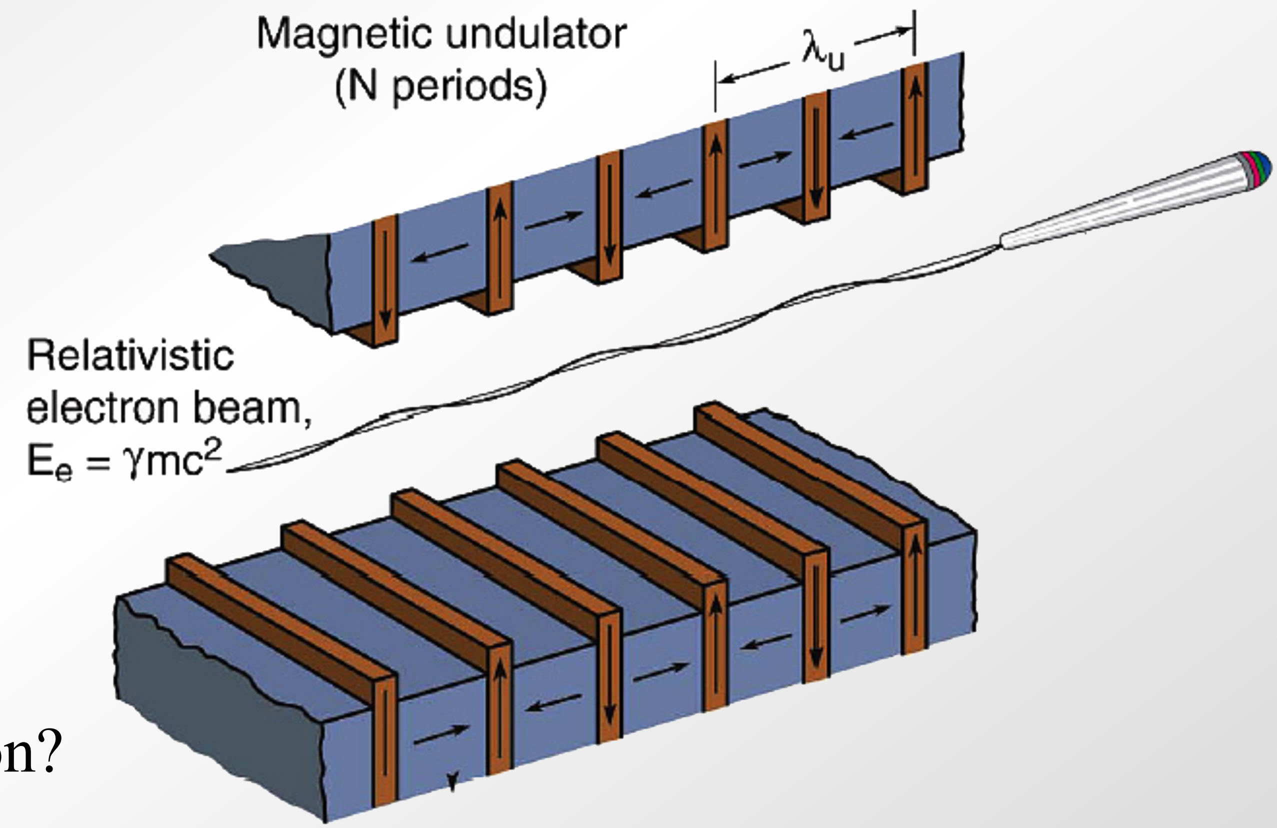
- Where are the photons?
  - The beam tube is filled with thermal photons (25 meV)

- In LEP-3 these photons can be up-shifted as much as 2.4 GeV
  - 2% of beam energy cannot be contained easily
  - We need to put in the Compton cross-section and photon density to find out how rapidly beam is lost

Undulator radiation: What is λ_{rad}?

An electron in the lab oscillating at frequency, f, emits dipole radiation of frequency f





What about the relativistic electron?

The Basics - Mechanics

Newton's law

- We all know

$$\mathbf{F} = \frac{d}{dt}\mathbf{p}$$

- The 4-vector form is

$$F^{\mu} = \left(\gamma c \frac{dm}{dt}, \gamma \frac{d\mathbf{p}}{dt} \right) = \frac{dp^{\mu}}{d\tau}$$

- Differentiate $p^2 = m_o^{\ 2} c^2$ with respect to τ

$$p_{\mu} \frac{dp^{\mu}}{d\tau} = p_{\mu} F^{\mu} = \frac{d(mc^2)}{dt} - \mathbf{F} \cdot \mathbf{v} = 0$$

- The work is the rate of changing mc^2

Harmonic oscillators & pendula

- Motion in the presence of a linear restoring force

$$F = -kx$$

$$\ddot{x} + \frac{k}{m} x = 0$$

$$x = A \sin \omega_o t \quad \text{where} \quad \omega_o = \sqrt{k/m}$$

- It is worth noting that the simple harmonic oscillator is a linearized example of the pendulum equation

$$\ddot{x} + \omega_o^2 \sin(x) \approx \ddot{x} + \omega_o^2 (x - x^3/6) = 0$$

that governs the free electron laser instability

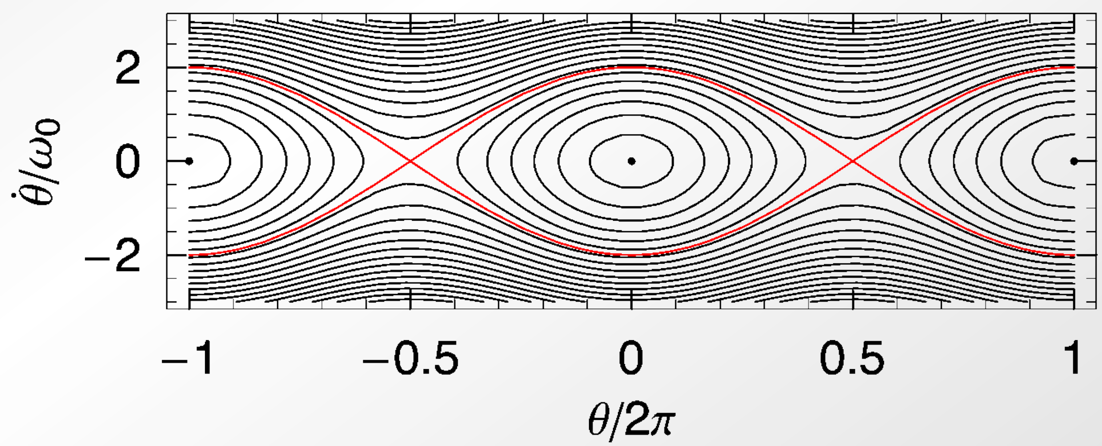
Solution to the pendulum equation

- Use energy conservation to solve the equation exactly
- Multiply $\ddot{x} + \omega_o^2 \sin(x) = 0$ by $\dot{x}$ to get

$$\frac{1}{2}\frac{d}{dt}\dot{x}^2 - \omega_o^2 \frac{d}{dt}\cos x = 0$$

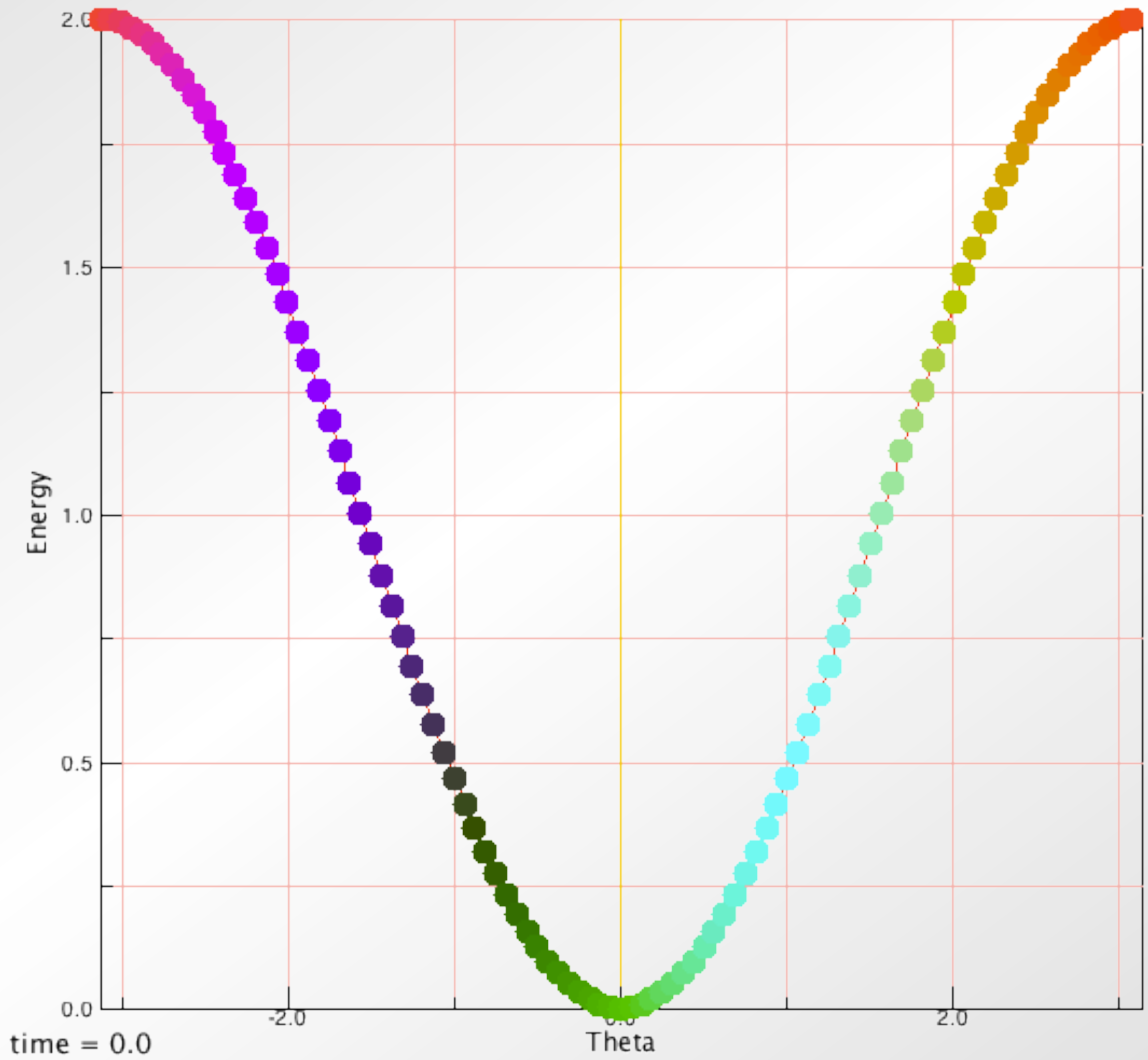
- Integrating we find that the energy is conserved

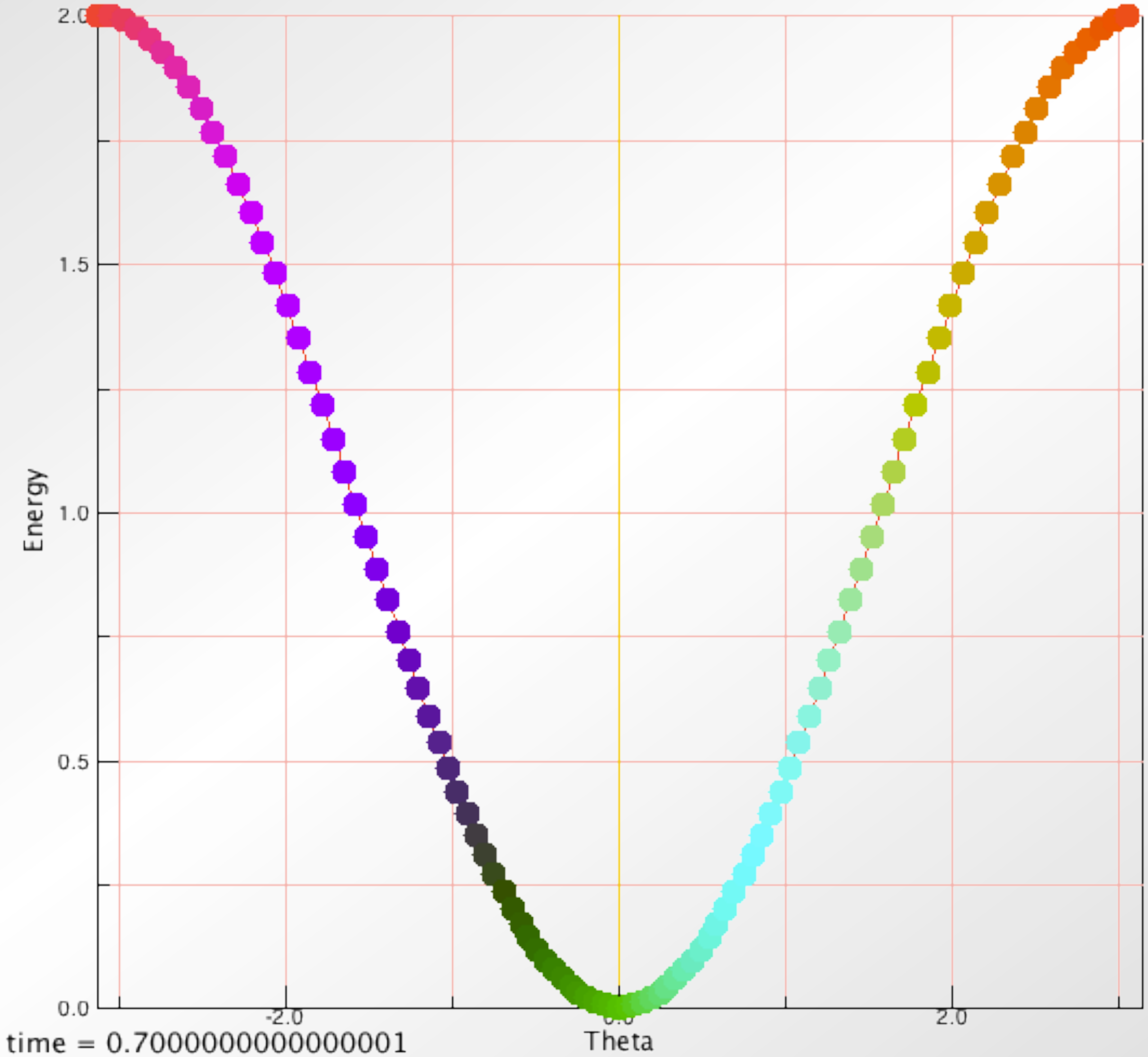
$$\frac{1}{2\omega_o^2}\dot{x}^2 - \cos x = \text{constant} = \text{ energy of the system} = E$$



With $x = \theta$

Simulation with 50 equally distributed pendula

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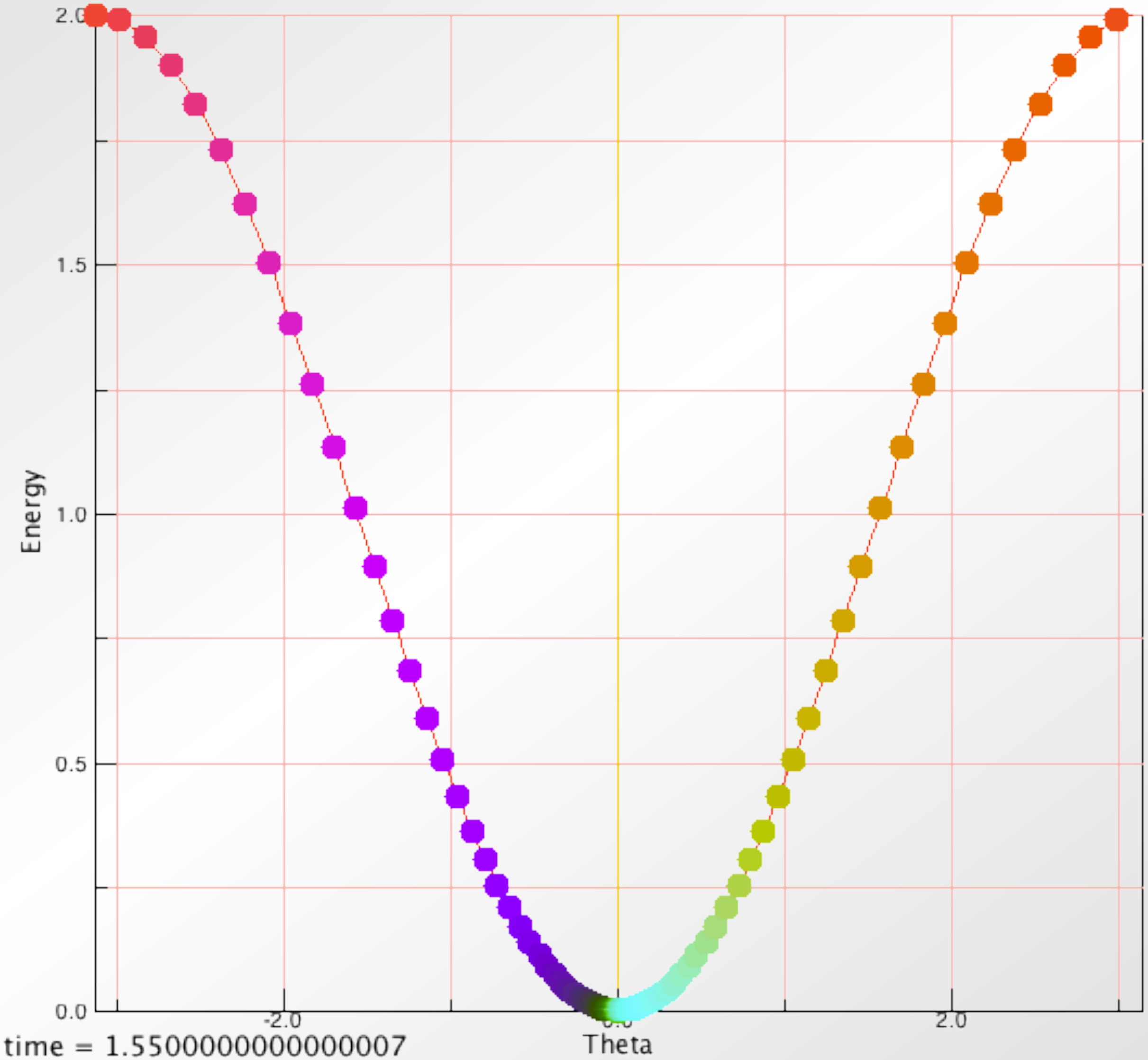
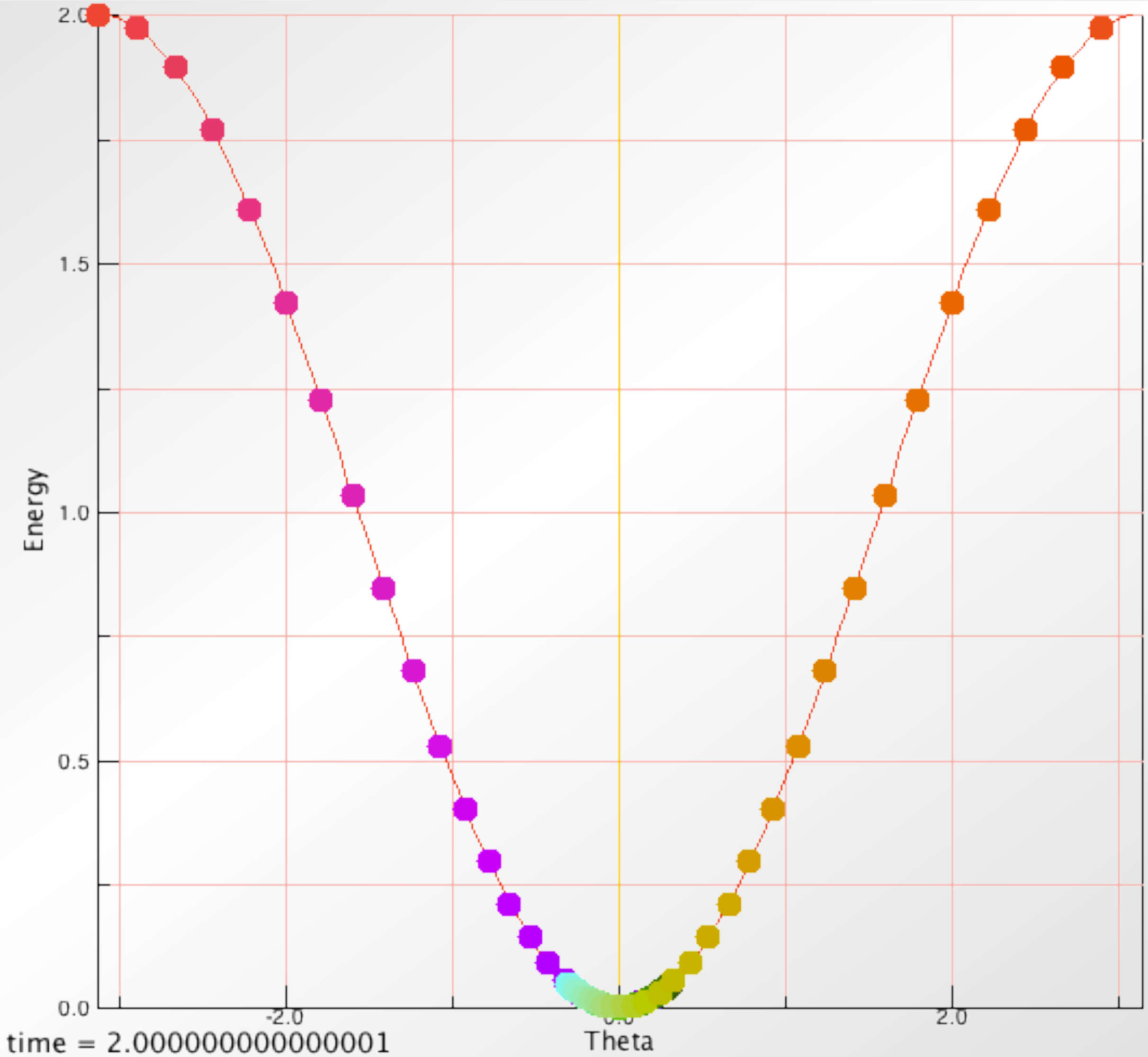
1.5
1.0
0.5
0.0
Energy
-2.0
0.0
2.0
Theta
time = 0.7000000000000001


2.0
1.5
1.0
0.5
0.0
Energy
-2.0
0.0
2.0
Theta
time = 1.5500000000000007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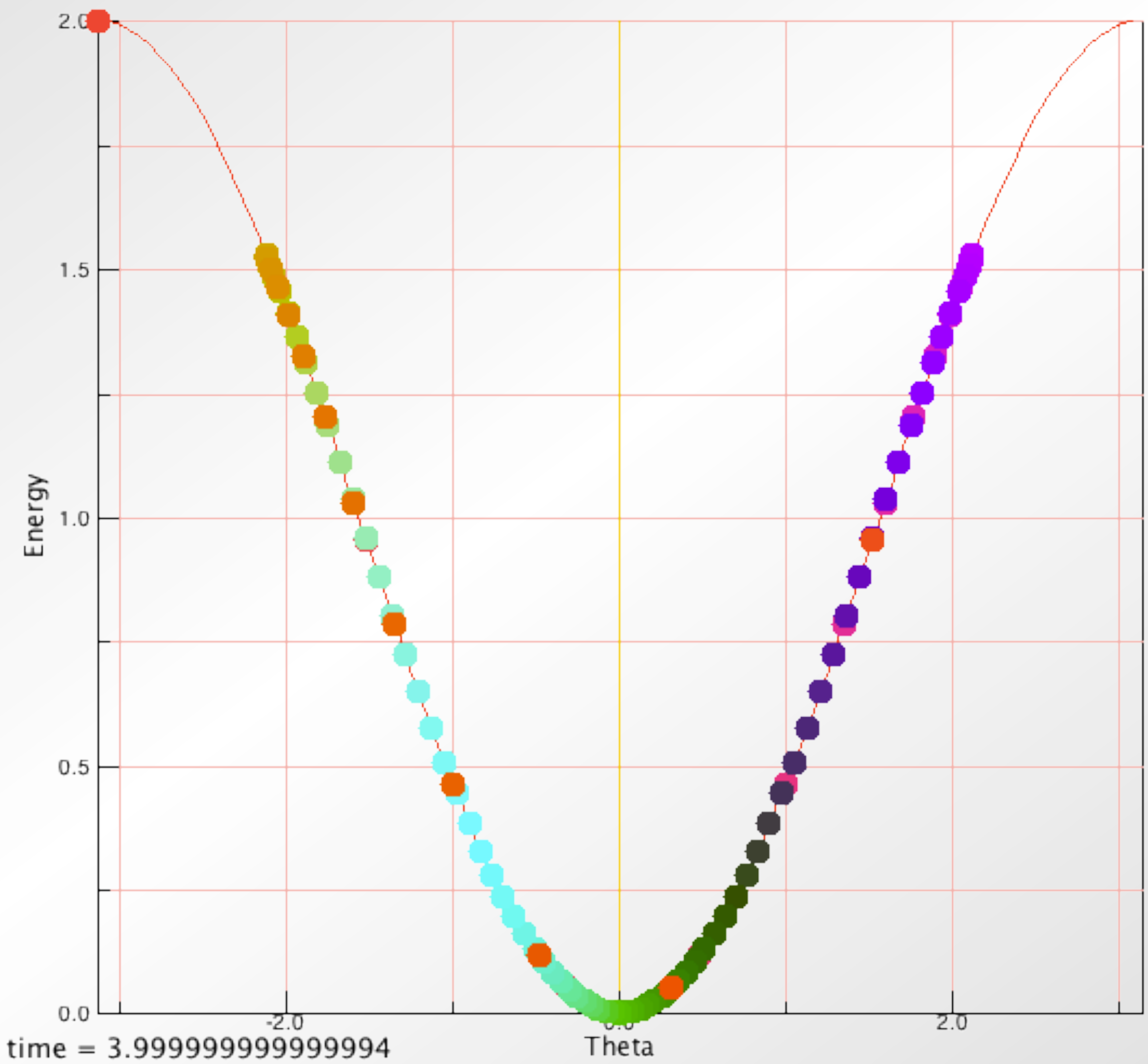
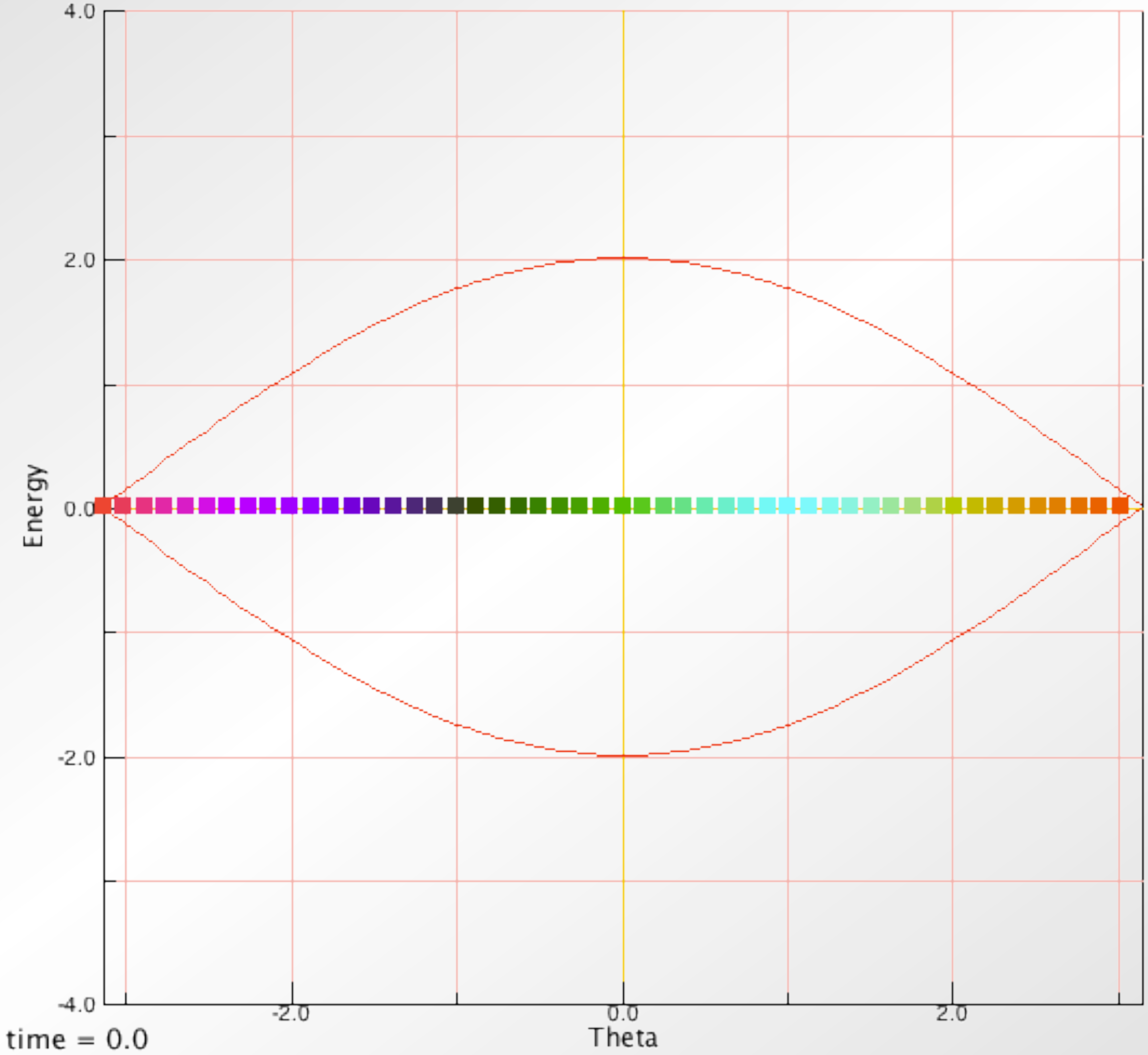
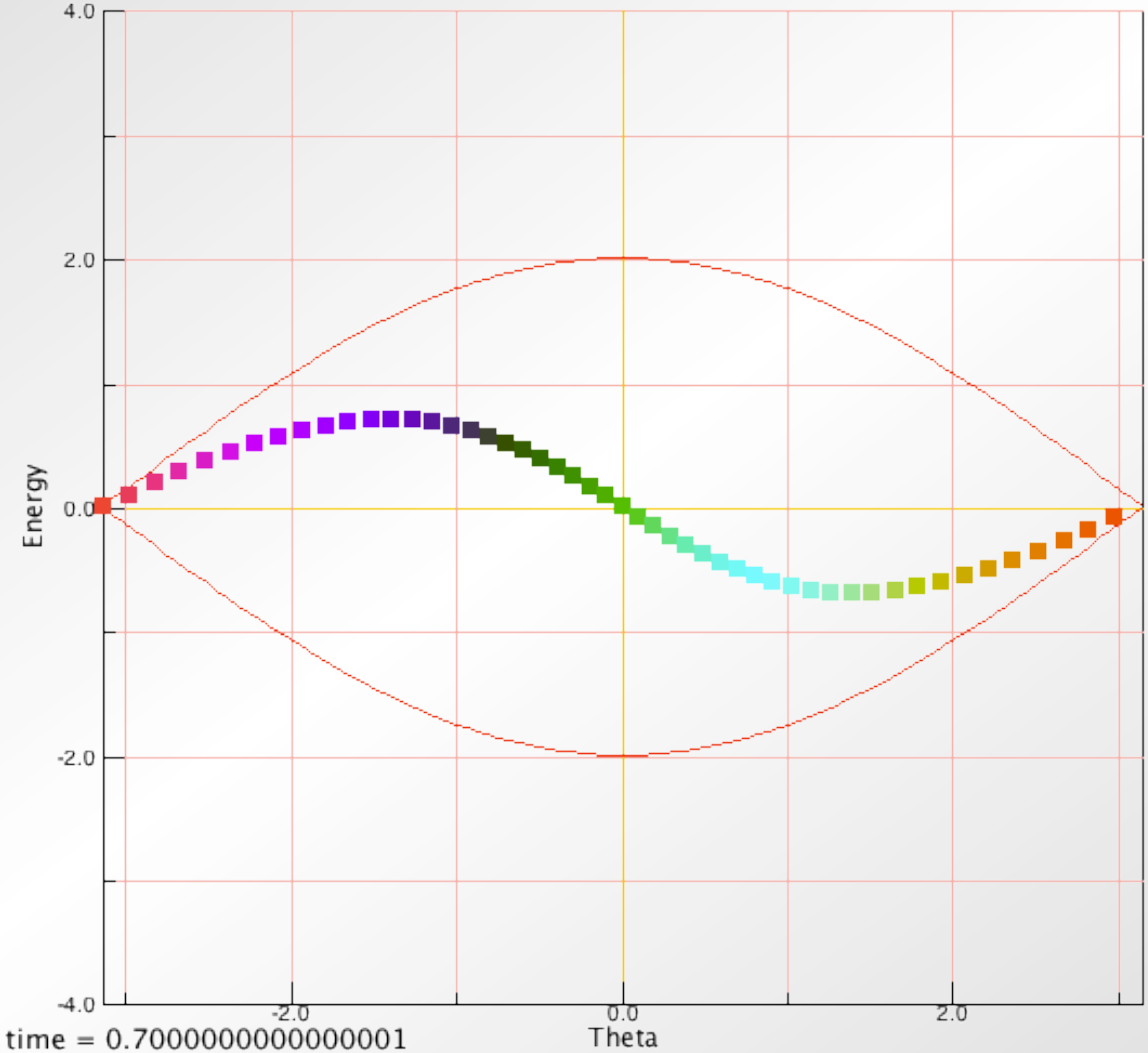
1.5
1.0
0.5
0.0
Energy
-2.0
0.0
2.0
Theta
time = 2.000000000000001

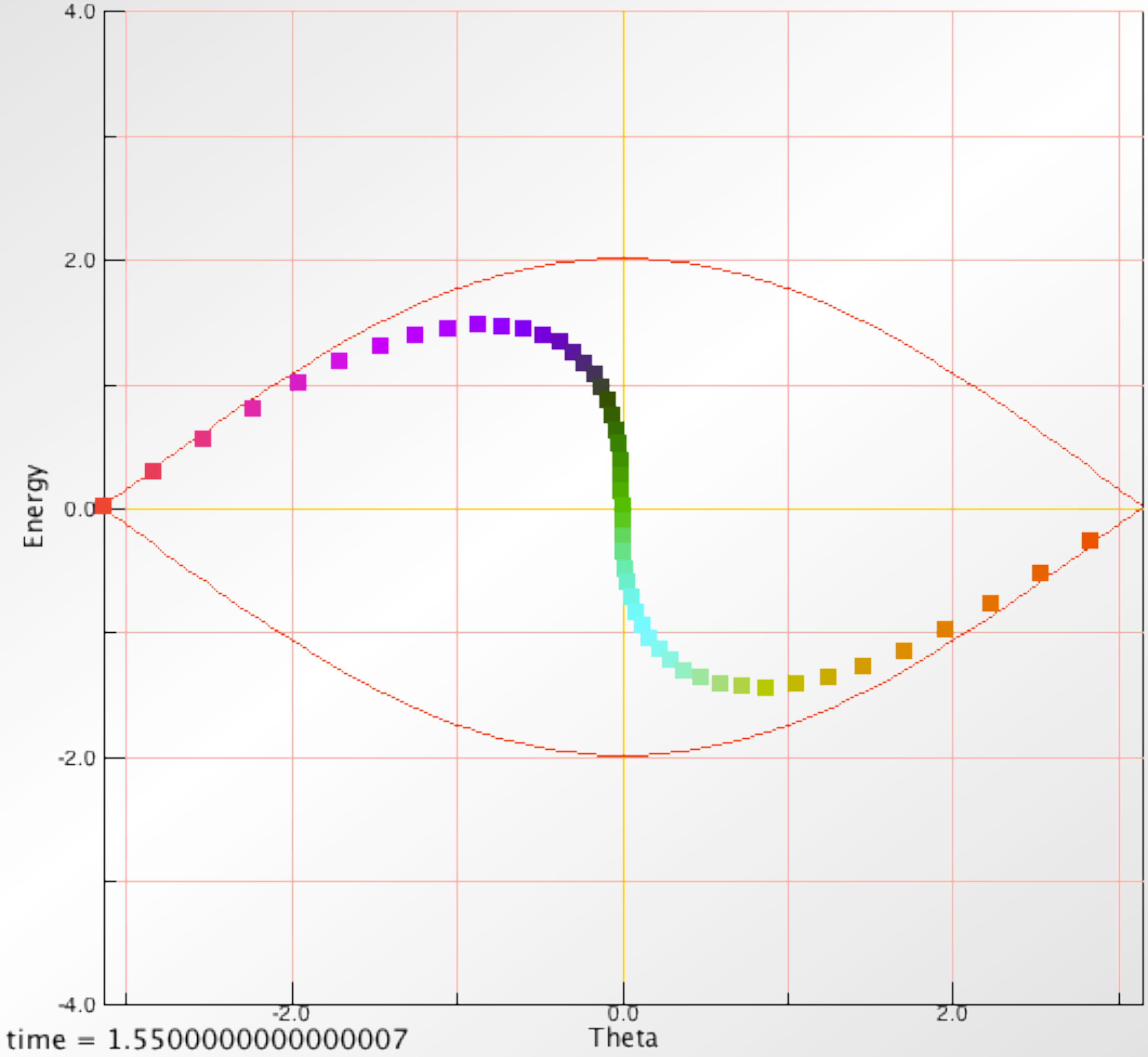


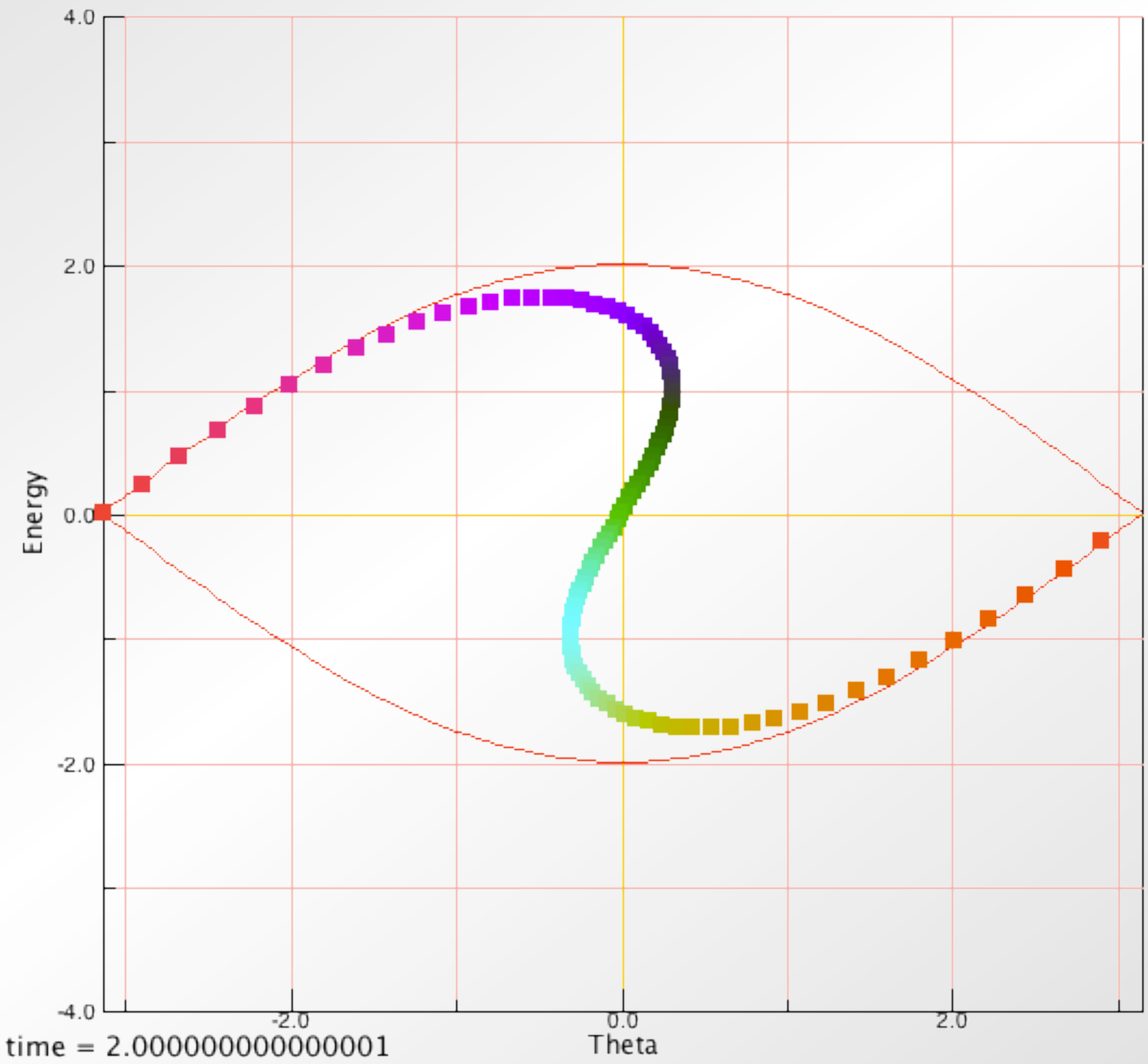
2.0
1.5
1.0
0.5
0.0
Energy
-2.0
0.0
2.0
Theta
time = 3.9999999999999994

Let's look at the phase space:
Simulation with 50 equally distributed pendula

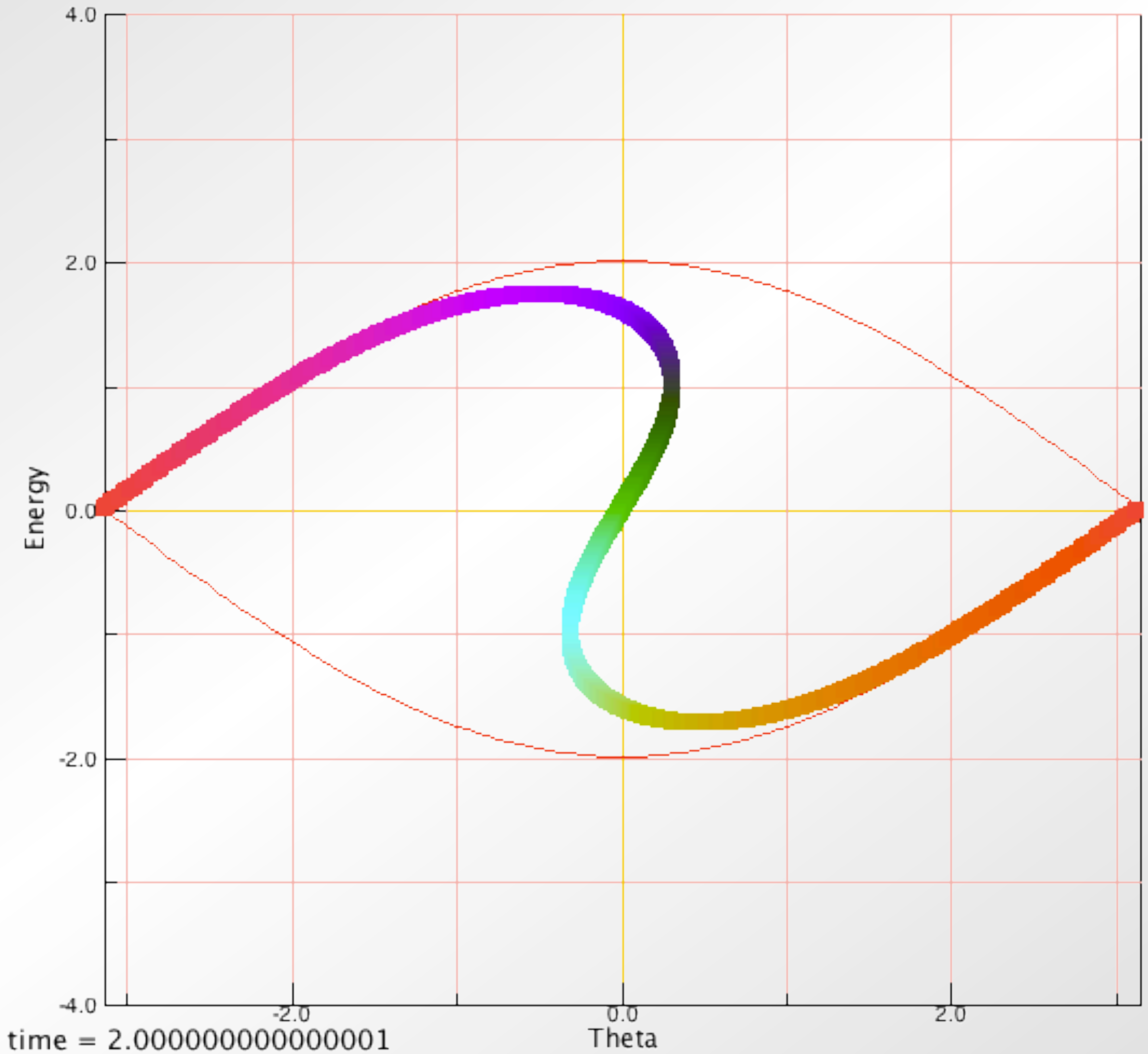


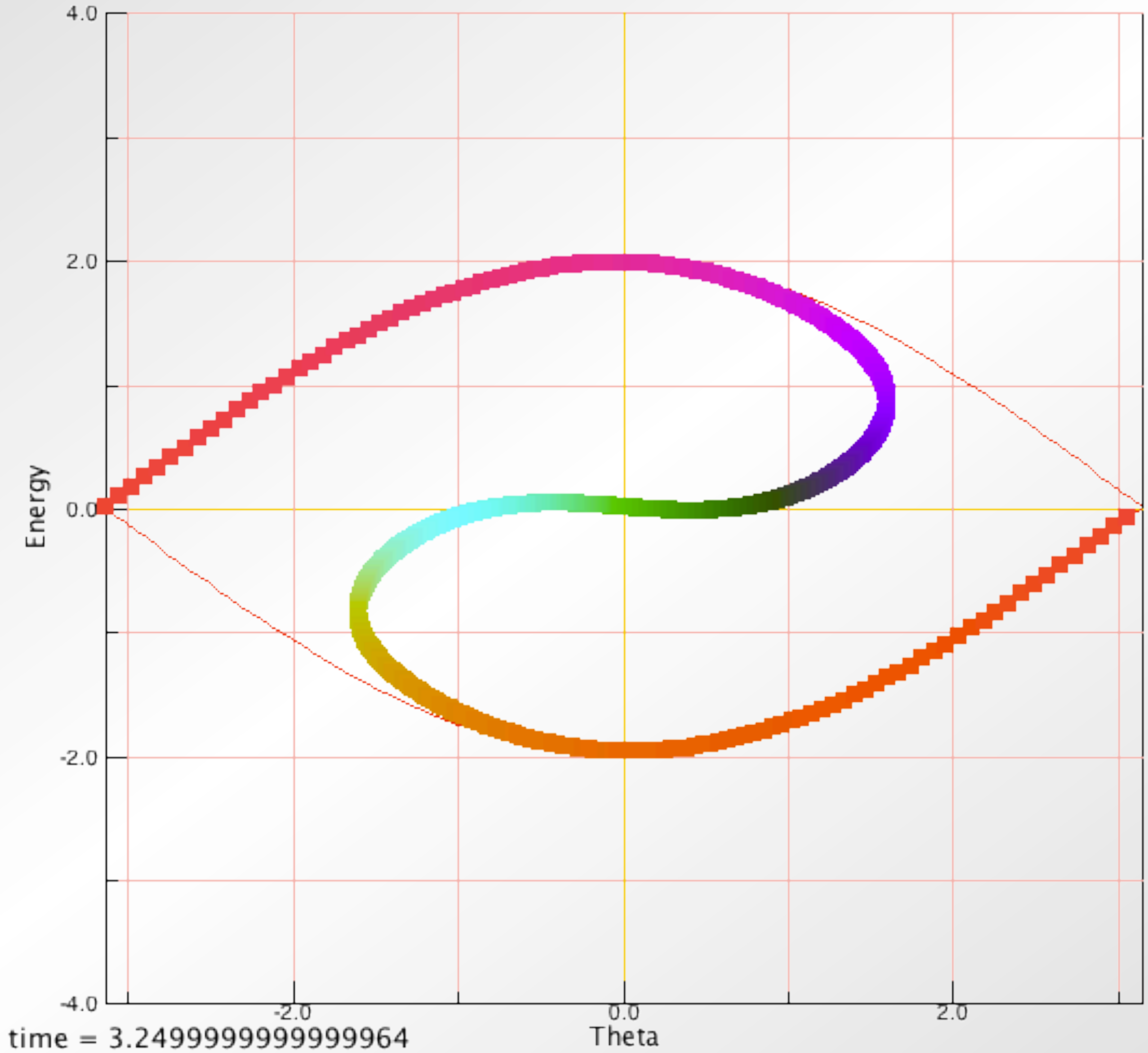

4.0
2.0
0.0
-2.0
-4.0
Energy
-2.0
0.0
2.0
Theta
time = 0.7000000000000001

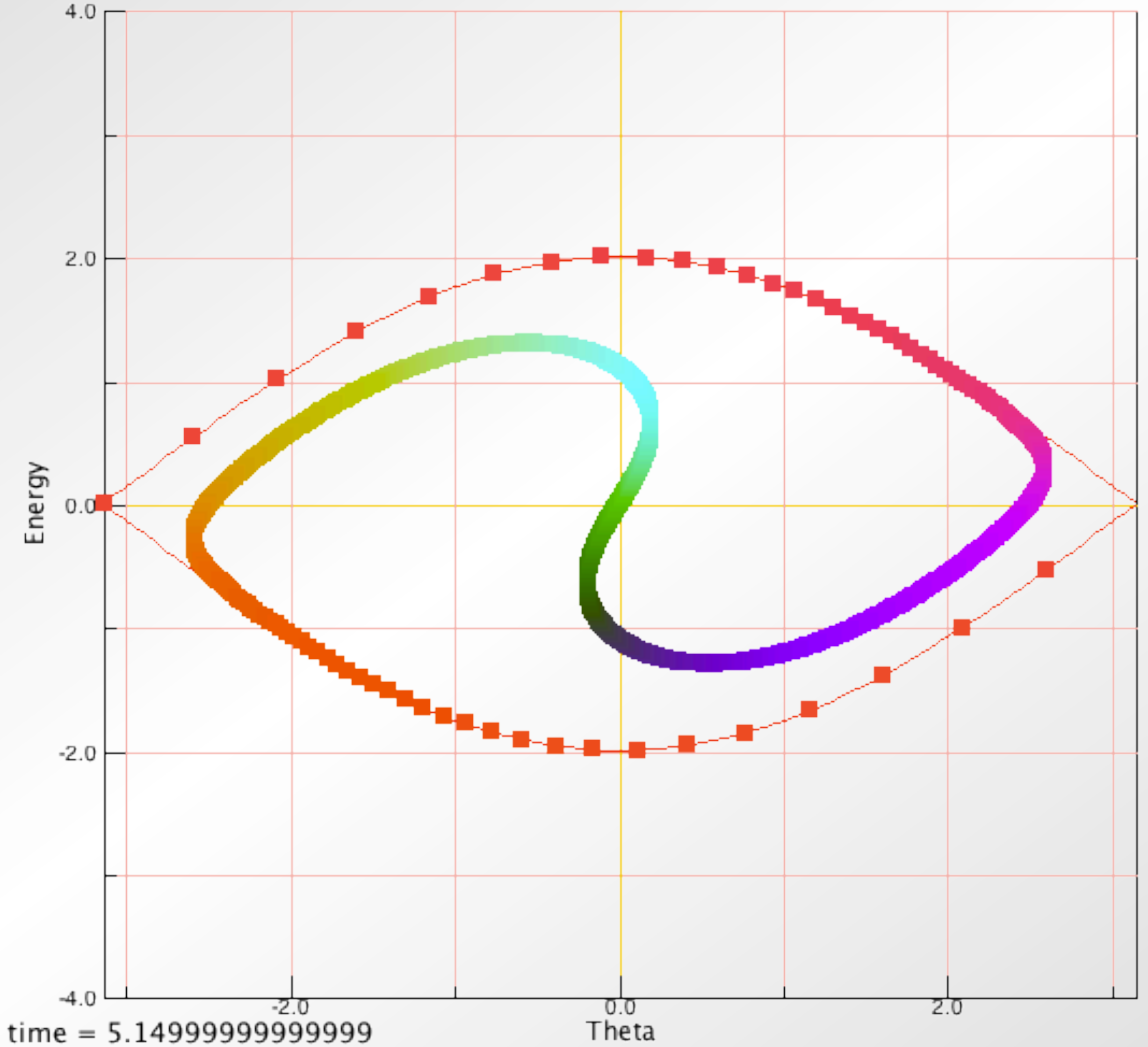

4.0
2.0
0.0
-2.0
-4.0
Energy
-2.0
0.0
2.0
Theta
time = 1.5500000000000007


4.0
2.0
0.0
-2.0
-4.0
Energy
-2.0
0.0
2.0
Theta
time = 2.000000000000001

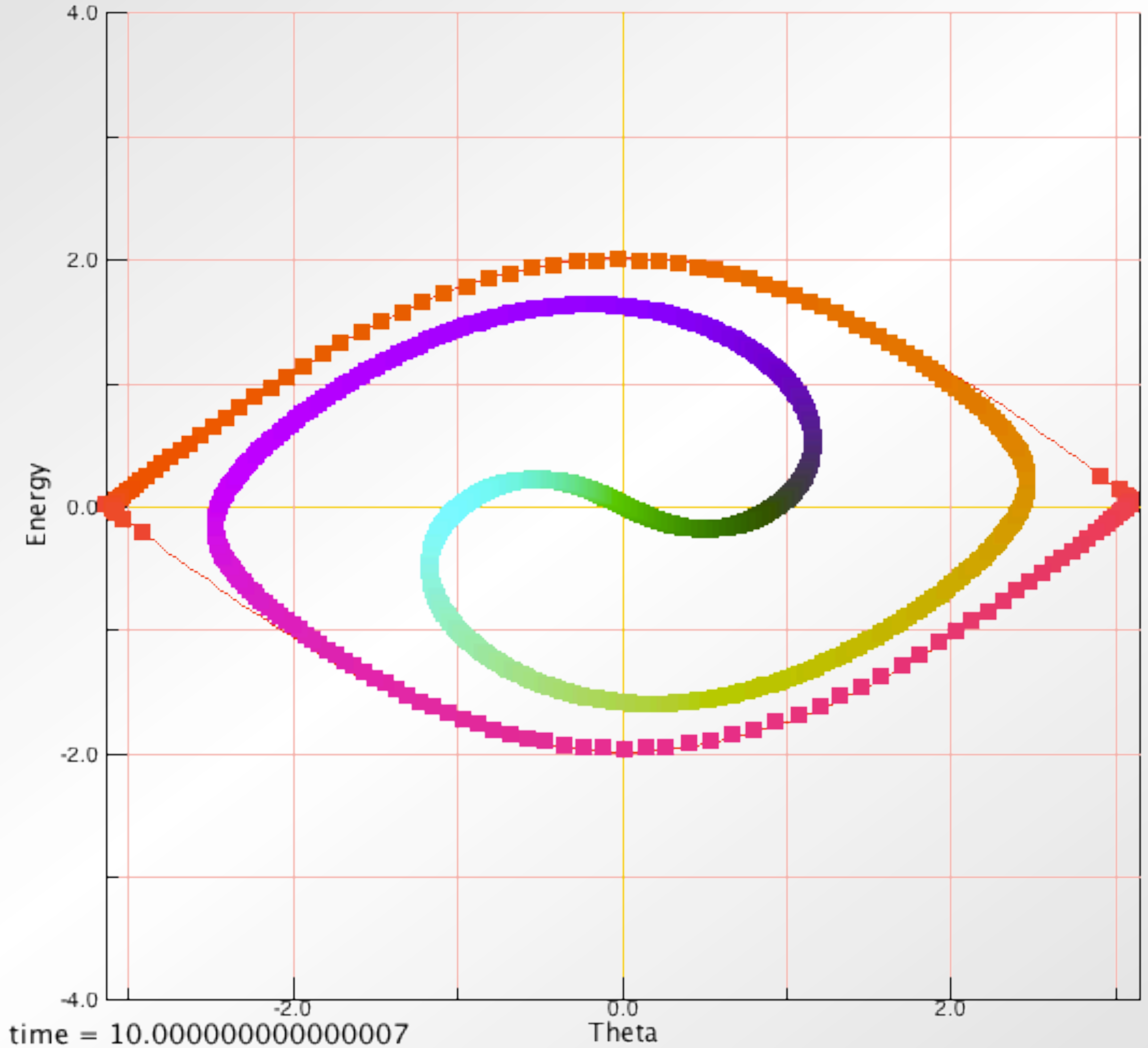
Simulation with 200 equally distributed pendula



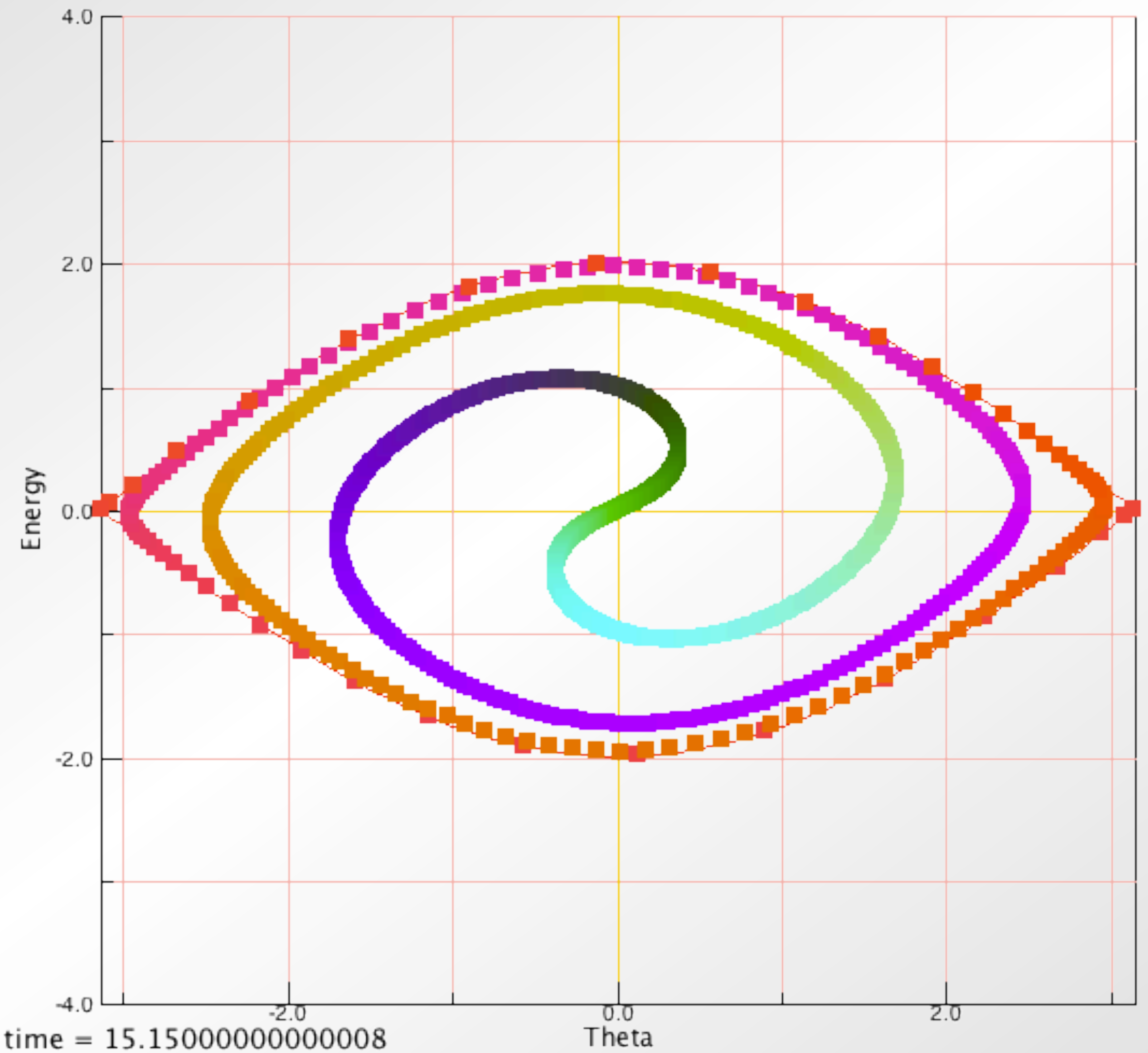

4.0
2.0
0.0
-2.0
-4.0
Energy
-2.0
0.0
2.0
Theta
time = 3.2499999999999964



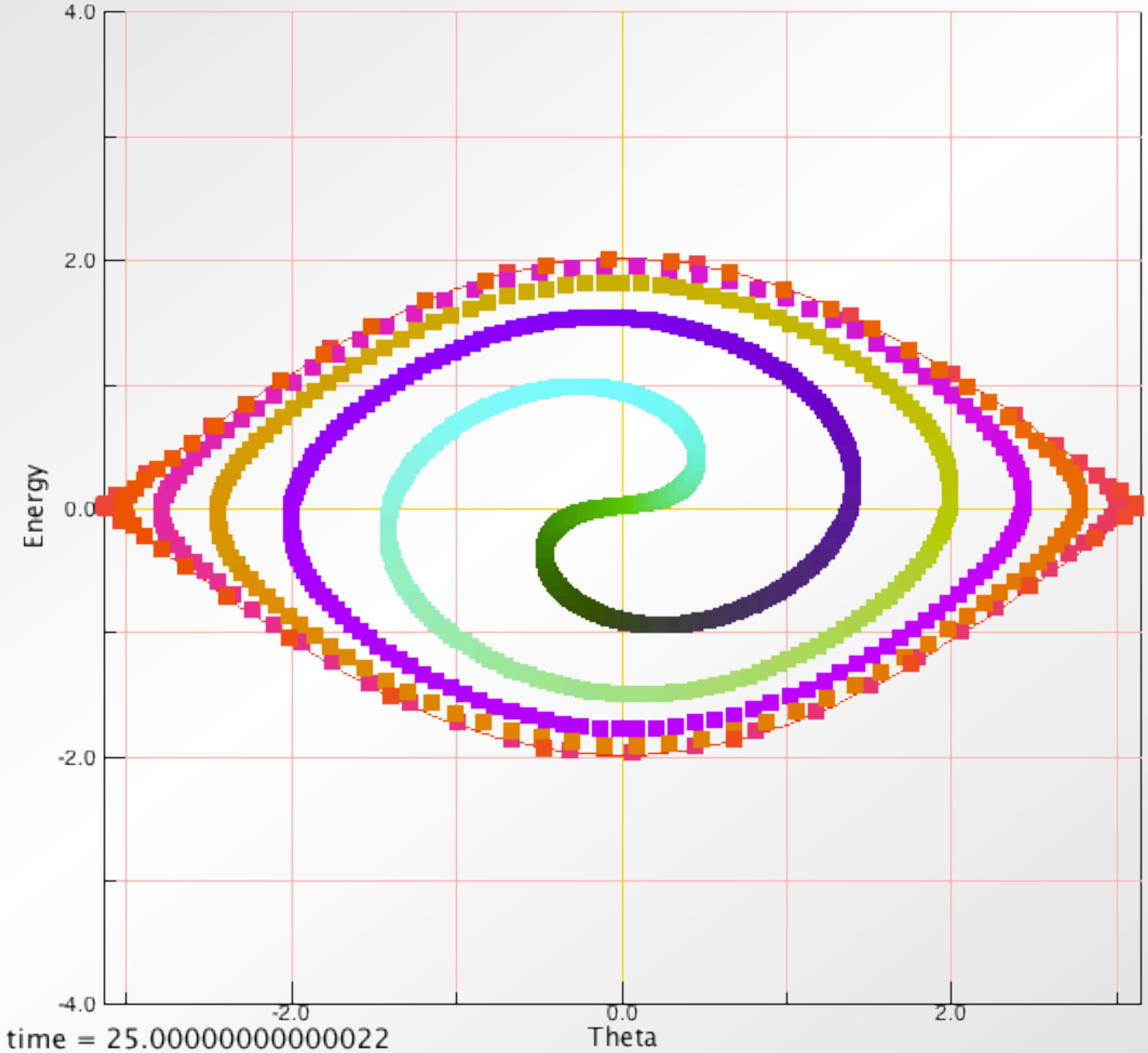
4.0
2.0
0.0
-2.0
-4.0
Energy
-2.0
0.0
2.0
Theta
time = 5.14999999999999

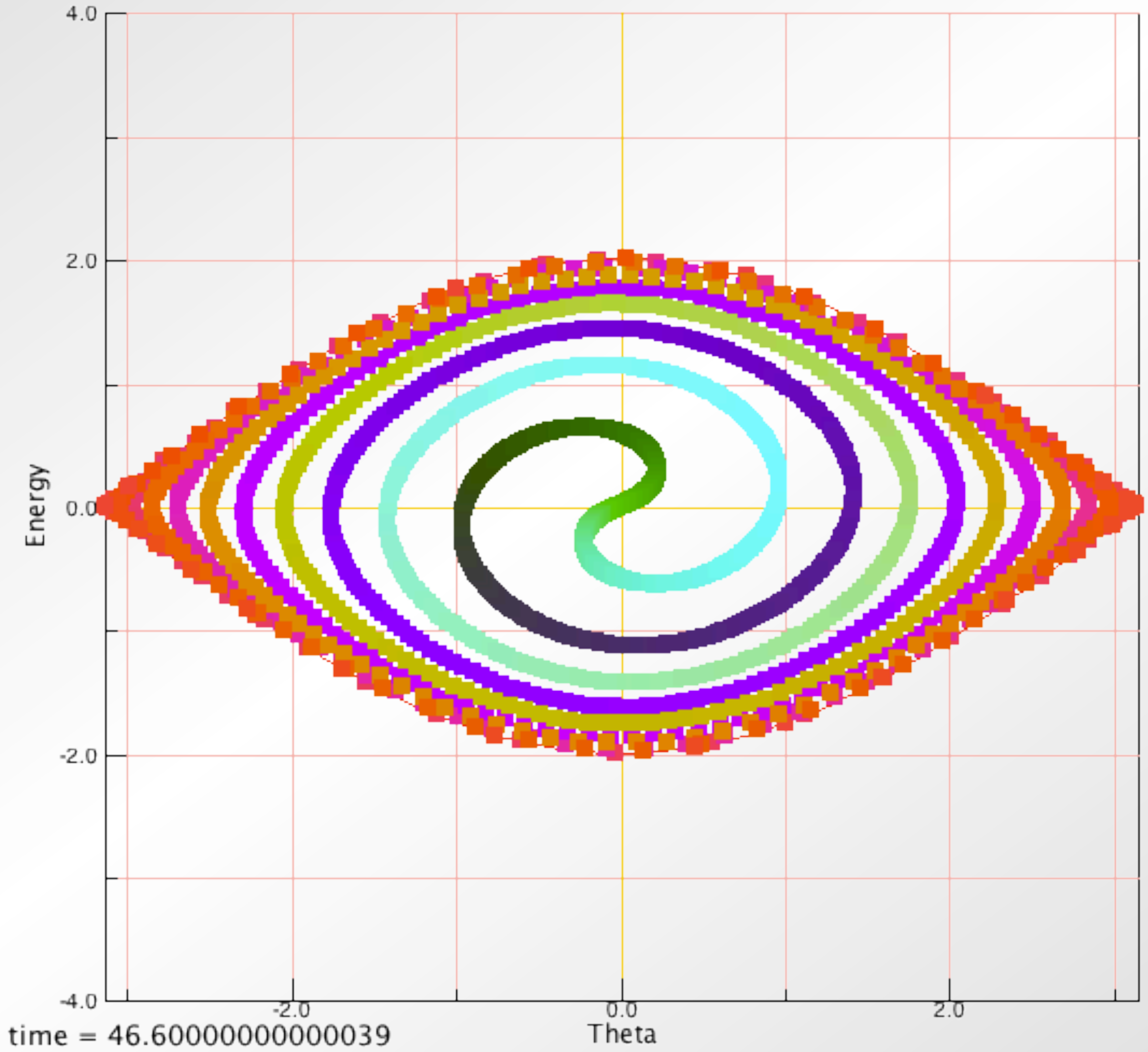

4.0
2.0
0.0
-2.0
-4.0
Energy
-2.0
0.0
2.0
Theta
time = 10.000000000000007

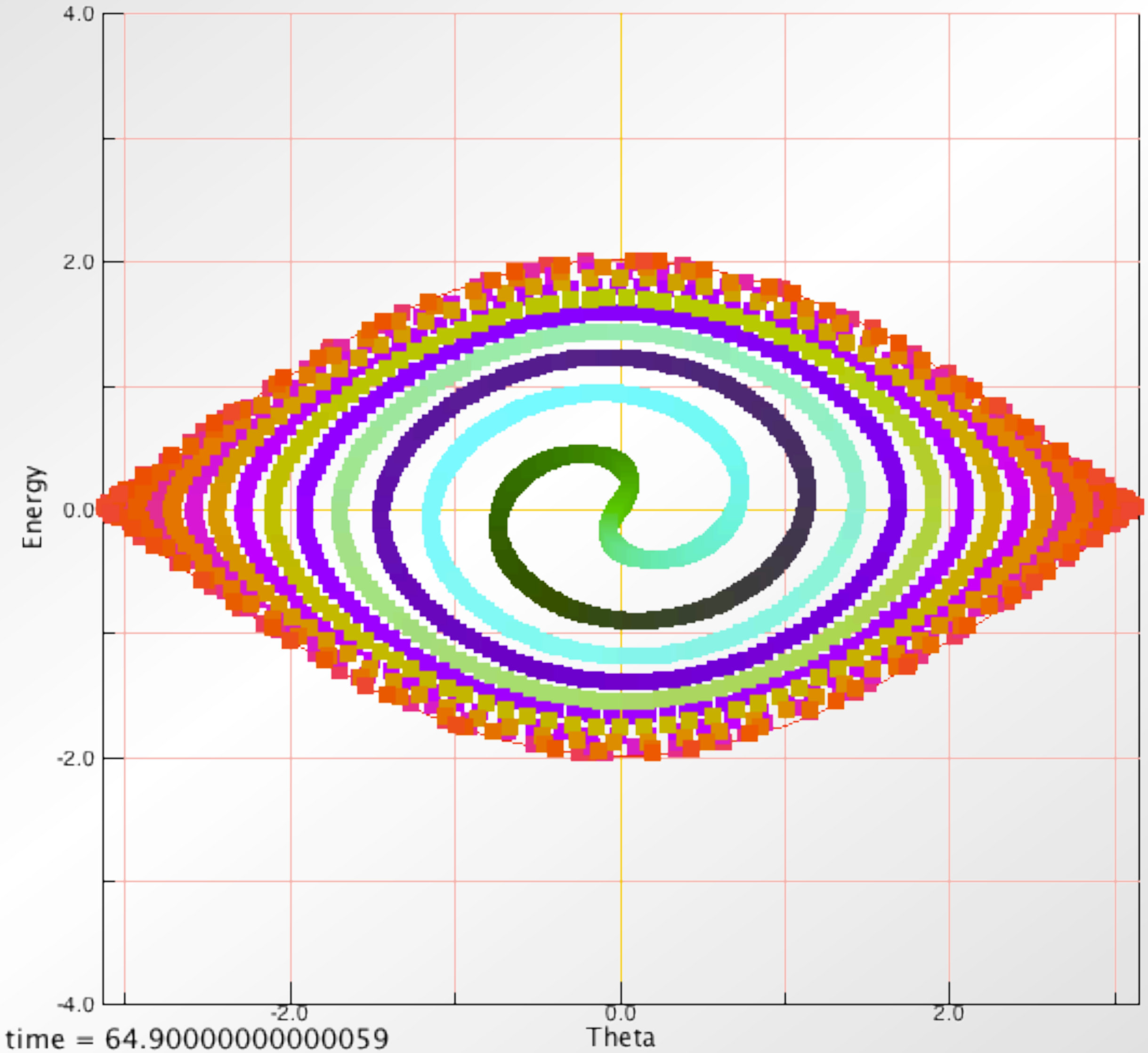
Simulation with 1000 equally distributed pendula



Simulation with 2000 equally distributed pendula

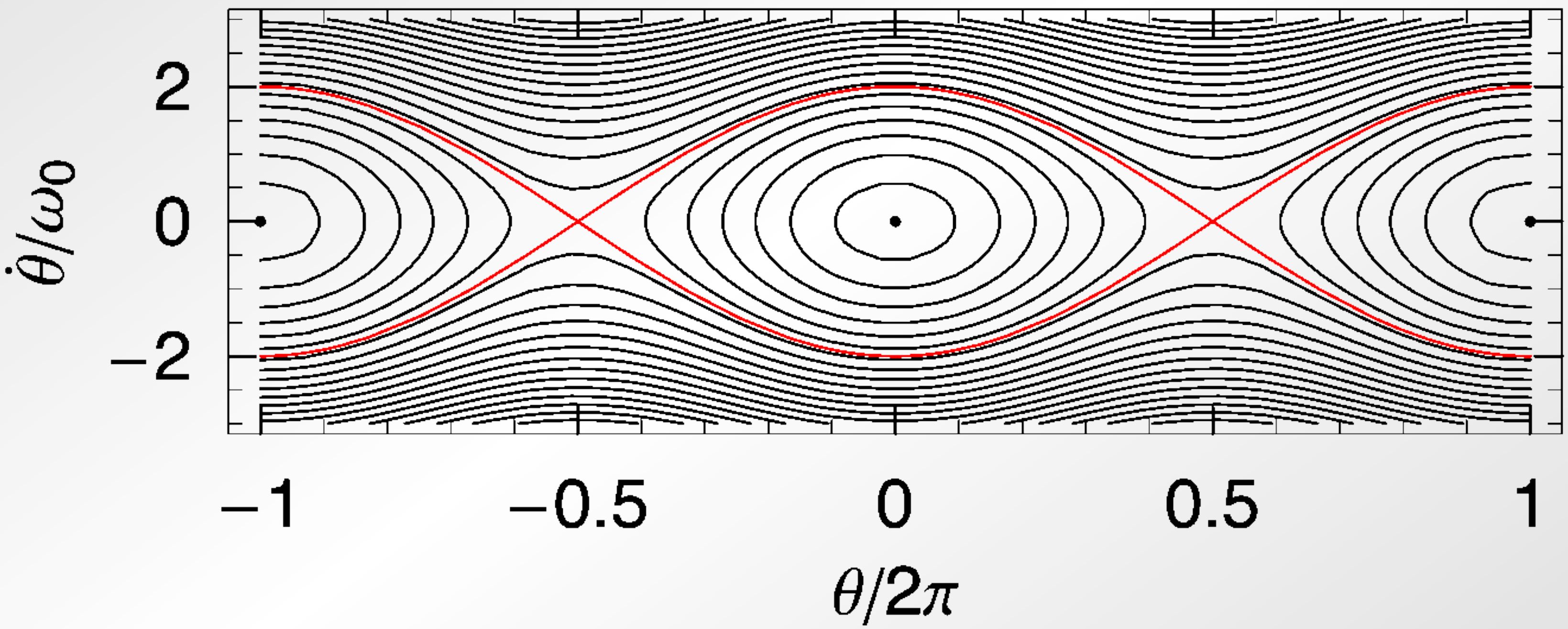



4.0
2.0
0.0
-2.0
-4.0
Energy
-2.0
0.0
2.0
Theta
time = 46.60000000000039


4.0
2.0
0.0
-2.0
-4.0
Energy
-2.0
0.0
2.0
Theta
time = 64.90000000000059

Recall the solution to the ODE

$$\frac{1}{2\omega_o^2}\dot{x}^2 - \cos x = \text{constant} = \text{ energy of the system} = E$$



With $x = \theta$

Beams subject to non-linear forces
are commonplace in accelerators

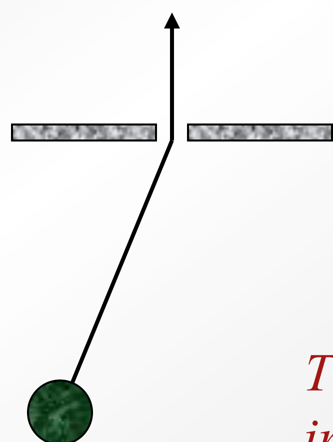
- Examples include
  - Space charge forces in beams with non-uniform charge distributions
  - Forces from magnets higher order than quadrupoles
  - Electromagnetic interactions of beams with *external* structures
    - Free Electron Lasers
    - Wakefields

Properties of harmonic oscillators

- Total energy is conserved

$$U = \frac{p^2}{2m} + \frac{m\omega_o^2 x^2}{2}$$

- If there are *slow* changes in m or ω, then $I = U/\omega_o$ *remains invariant*



$$\frac{\Delta\omega_o}{\omega_o} = \frac{\Delta U}{U}$$

This effect is important as a diagnostic in measuring resonant properties of structures

Lorentz force on a charged particle

- Force, **F**, on a charged particle of charge q in an electric field **E** and a magnetic field, **B**

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

- $E =$ electric field with units of force per unit charge, newtons/coulomb = volts/m.
- B = magnetic flux density or magnetic induction, with units of newtons/ampere-m = Tesla = Weber/m^2.

A simple problem - bending radius

- Compute the bending radius, R, of a non-relativistic particle particle in a uniform magnetic field, B.
  - Charge = q
  - Energy = $mv^2/2$

$$F_{Lorentz} = q\frac{v}{c}B = F_{centripital} = \frac{mv^2}{\rho}$$

$$\Rightarrow \rho = \frac{mvc}{qB} = \frac{pc}{qB}$$

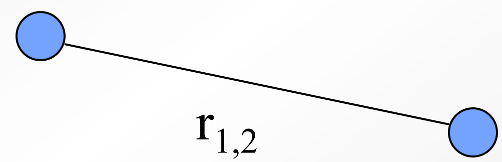
$$\boxed{\rho(\mathrm{m}) = 3.34\left(\frac{p}{1\ \mathrm{GeV/c}}\right)\left(\frac{1}{q}\right)\left(\frac{1\ \mathrm{T}}{B}\right)}$$

10 minute exercise from Whittum

- **Exercise:** A charged particle has a kinetic energy of 50 keV. You wish to apply as large a force as possible. You may choose either an electric field of 500 kV/m or a magnetic induction of 0.1 T. Which should you choose
  - (a) for an electron,
  - (b) for a proton?

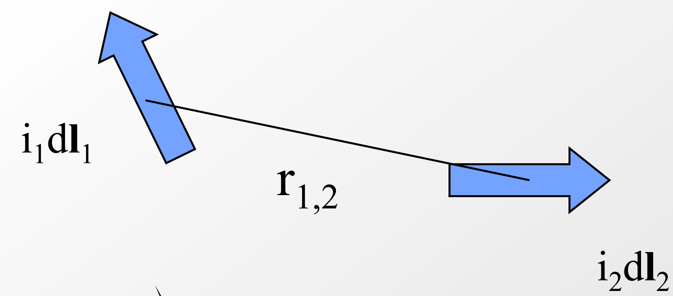
The fields come from charges & currents

❖ Coulomb's Law



$$\mathbf{F}_{1\to 2} = q_2\left(\frac{1}{4\pi\varepsilon_o}\frac{q_1}{r_{1,2}^2}\hat{\mathbf{r}}_{1\to 2}\right) = q_2\mathbf{E}_1$$

❖ Biot-Savart Law

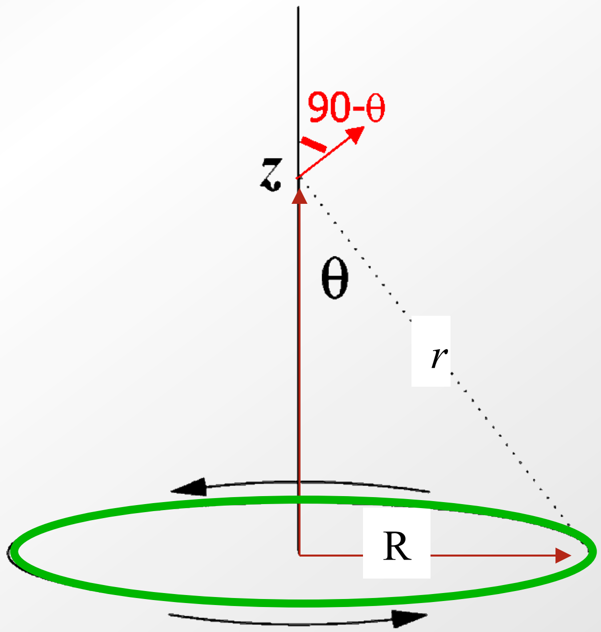


$$d\mathbf{F}_{1\to 2} = i_2 d\mathbf{l}_2 \times \left(\frac{\mu_0}{4\pi}\frac{\left(i_1 d\mathbf{l}_1 \times \hat{\mathbf{r}}_{12}\right)}{r_{1,2}^2}\right) = i_2 d\mathbf{l}_2 \times \mathbf{B}_2$$

Compute the B-field from current loop

- On axis there is only B_z by symmetry
- The Biot-Savart law says

$$B = \int_{wire} \left(d\vec{B} \right)_z = \int_{wire} \frac{I}{cr^2} \left| d\vec{l} \times \hat{r} \right| \sin\theta$$



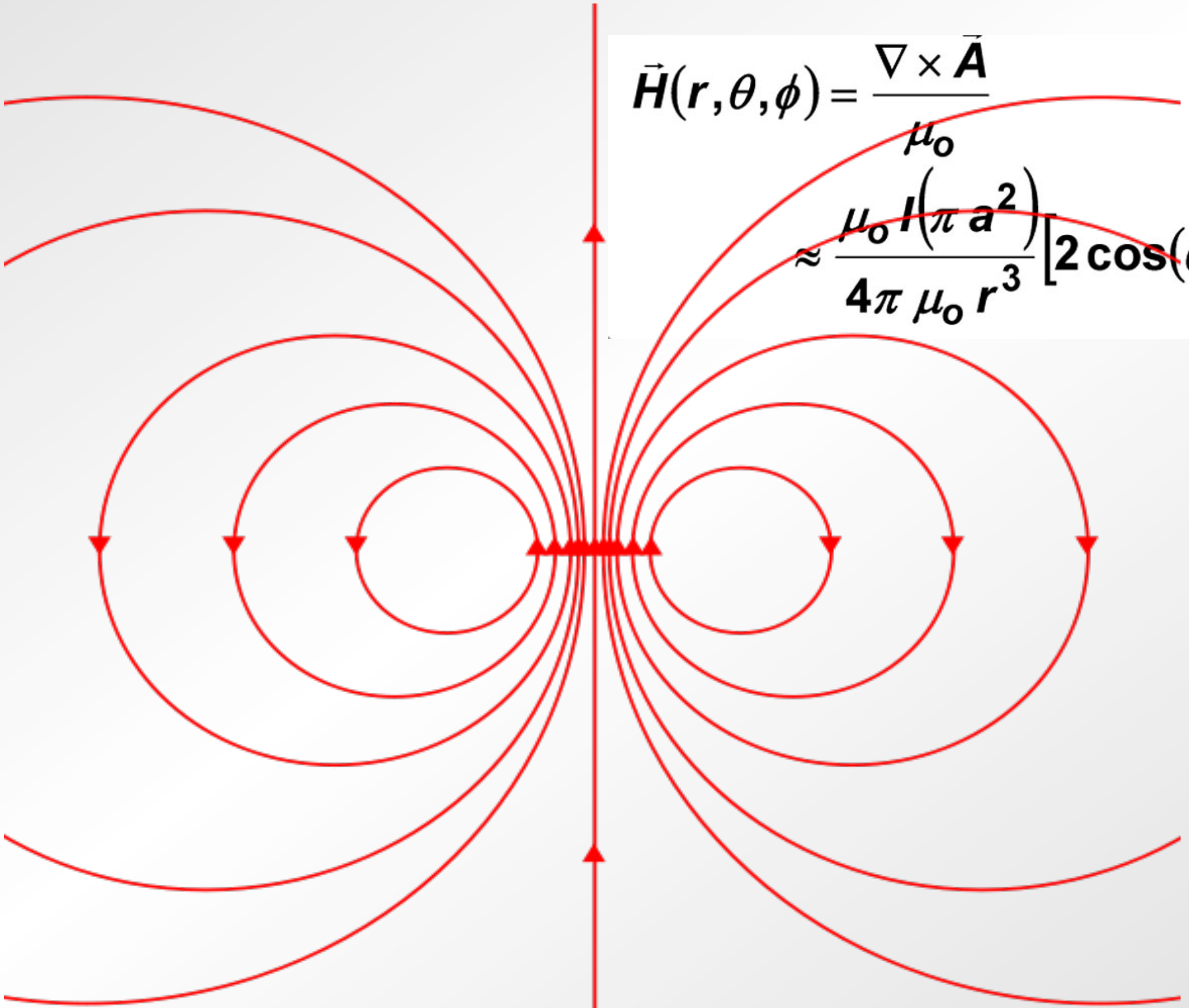
$$\left| d\mathbf{l} \times \hat{\mathbf{r}} \right| = \left| d\mathbf{l} \right| = R d\varphi$$

$$\sin\theta = R/r \quad \text{and} \quad r = \sqrt{R^2 + z^2}$$

$$\mathbf{B} = \frac{I}{cr^2} R \sin\theta \int_0^{2\pi} d\varphi\, \hat{\mathbf{z}} = \frac{2\pi I R^2}{c\left(R^2 + z^2\right)^{3/2}} \hat{\mathbf{z}}$$

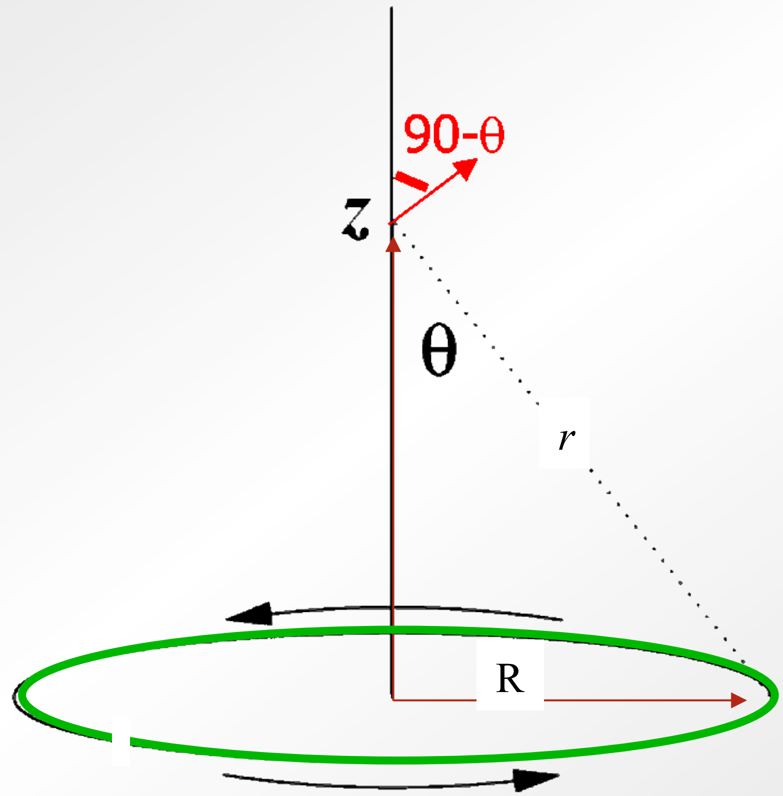
The far field B-field has a static dipole form

$$\vec{H}(r,\theta,\phi) = \frac{\nabla \times \vec{A}}{\mu_o} \approx \frac{\mu_o I \left(\pi a^2\right)}{4\pi\, \mu_o\, r^3}\left[2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right]$$



Importantly the ring of current does not radiate

Question to ponder:
What is the field from this situation?



Electric displacement & magnetic field

In vacuum,

- The electric displacement is $\mathbf{D} = \varepsilon_o \mathbf{E}$,

- The magnetic field is $\mathbf{H} = \mathbf{B}/\mu_o$

Where

$$\varepsilon_o = 8.85\text{x}10^{-12} \text{ farad/m} \quad \& \quad \mu_o = 4\,\pi\,\text{x}10^{-7} \text{ henry/m.}$$

Maxwell's equations (1)

- Electric charge density ρ is source of the electric field, **E** (Gauss's law)

$$\nabla \cdot \mathbf{E} = \rho$$

- Electric current density **J** = ρ**u** is source of the magnetic induction field **B** (Ampere's law)

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_0 \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

If we want big magnetic fields, we need large current supplies

Maxwell's equations (2)

- Field lines of **B** are closed; i.e., no magnetic monopoles.

$$\nabla \bullet \mathbf{B} = 0$$

- Electromotive force around a closed circuit is proportional to rate of change of **B** through the circuit (Faraday's law).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Maxwell's equations: integral form

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0} \quad => \quad \oint_S \vec{E} \bullet d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_0} \quad \text{Gauss' Law}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad => \quad \oint_C \vec{E} \bullet d\vec{l} = -\oint_S \frac{\partial \vec{B}}{\partial t} \bullet d\vec{a} \quad \text{Faraday's Law}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad =>$$

Displacement current

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_0 I_{enclosed} + \mu_0 \varepsilon_0 \oint_S \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a} \quad \text{Ampere's Law}$$

Boundary conditions for a perfect conductor: $\sigma = \infty$

1. If electric field lines terminate on a surface, they do so normal to the surface
   a) any tangential component would quickly be neutralized by lateral motion of charge within the surface.
   b) The E-field must be normal to a conducting surface

2. Magnetic field lines avoid surfaces
   a) otherwise they would terminate, since the magnetic field is zero within the conductor
      i. The normal component of B must be continuous across the boundary for $\sigma \neq \infty$

Exercise from Whittum

- **Exercise:** A charged particle has a kinetic energy of 50 keV. You wish to apply as large a force as possible. You may choose either an electric field of 500 kV/m or a magnetic induction of 0.1 T. Which should you choose
  - (a) for an electron,
  - (b) for a proton?

Lorentz transformations of E.M. fields

$$E'_{z'} = E_z$$

$$E'_{x'} = \gamma\left(E_x - vB_y\right)$$

$$E'_{y'} = \gamma\left(E_y + vB_x\right)$$

$$B'_{z'} = B_z$$

$$B'_{x'} = \gamma\left(B_x + \frac{v}{c^2}E_y\right)$$

$$B'_{y'} = \gamma\left(B_y - \frac{v}{c^2}E_x\right)$$

*Fields are **invariant** along the direction of motion, z*

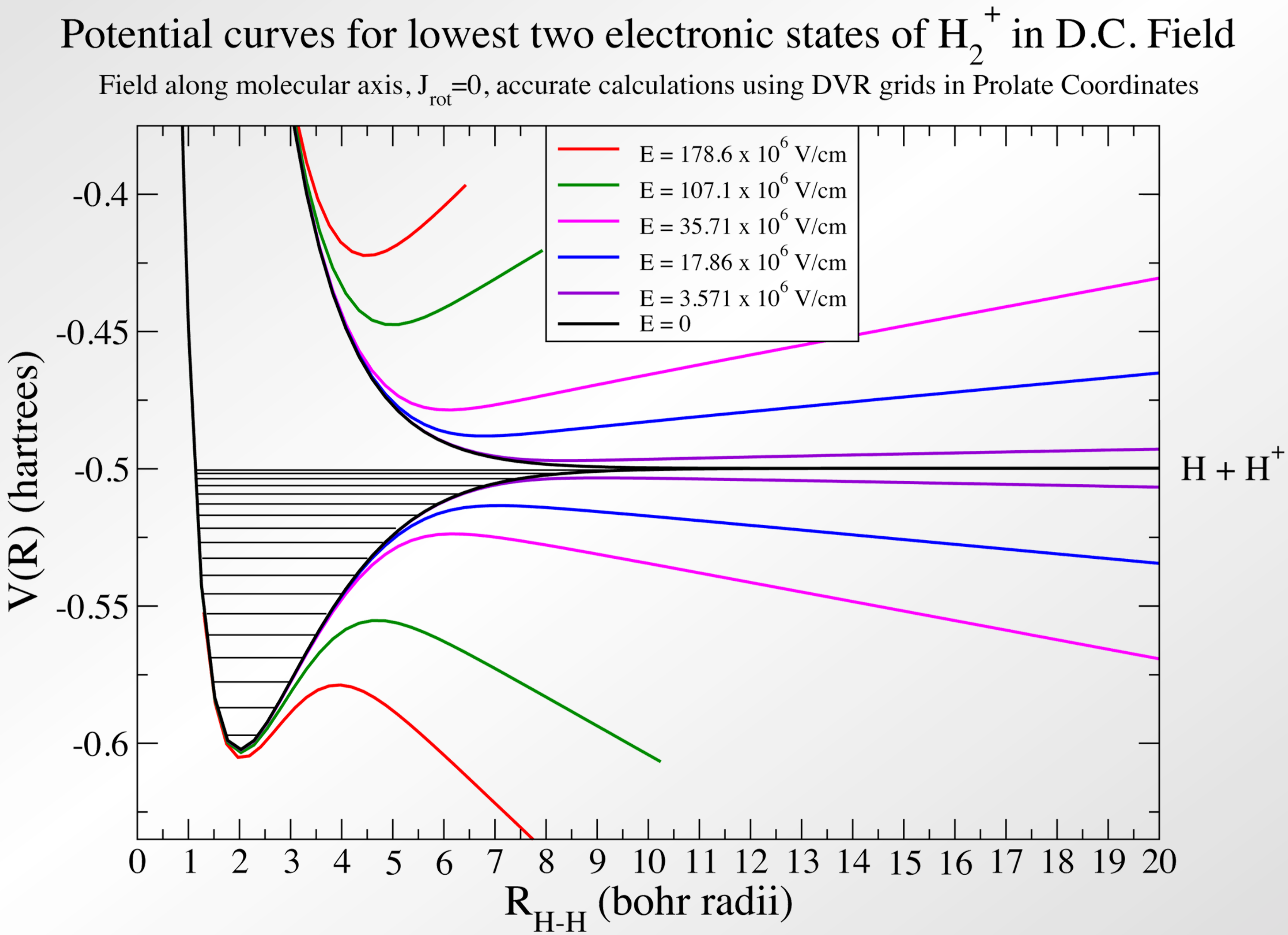
Example: Lorentz stripping & dissociation

- An ion moving in a magnetic field B experiences a Lorentz force that bends its trajectory & also tends to break it up
  - the protons & electrons are bent in opposite directions
  - The binding energy of the extra electron is only 0.755 eV.
  - The breakup is a probabilistic, quantum mechanical process

- In the ion rest frame, the stripping force is effected by the electric field E that is the Lorentz-transform of the magnetic field B in the lab,

$$E = \kappa \beta \gamma B, \text{ where } \kappa = 0.3 \text{ GV/T-m}.$$

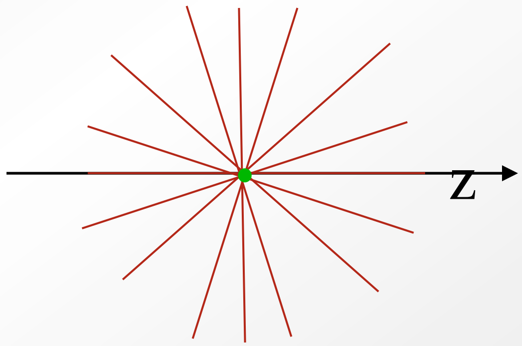
Example for H_2^+:
The huge field distorts ion potential energy



Fields of a relativistic point charge

- Let's evaluate the EM fields from a point charge q moving ultra-relativistically at velocity v in the lab
- In the rest frame of the charge, it has a static **E** field only:

$$\mathbf{E}' = \frac{1}{4\pi\varepsilon_o}\frac{q\mathbf{r}'}{r'^3}$$



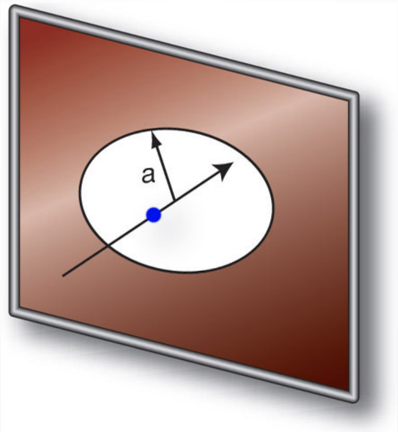
where **r** is the vector from the charge to the observer

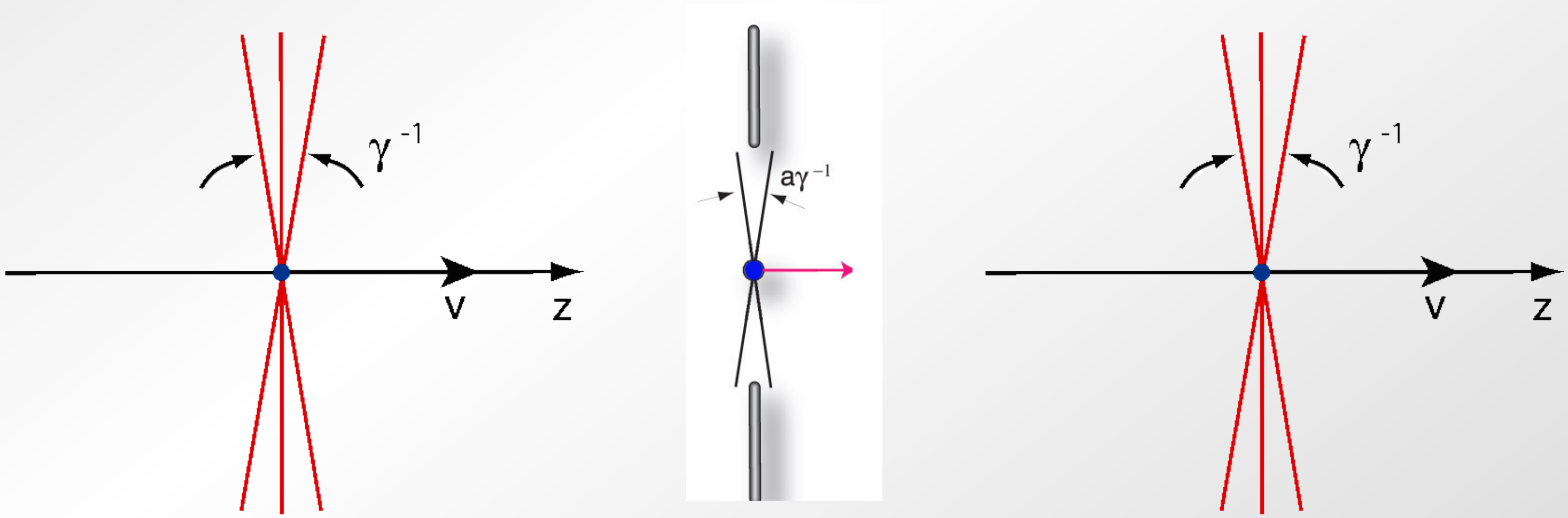
- To find **E** and **B** in the lab, use the Lorentz transformation for coordinates time and the transformation for the fields

This effect is offers us
a non-destructive beam diagnostic

- Pass the charge through a hole in a conducting foil
- The foil clips off the field for a time $\Delta t \sim a/c\gamma$
- The fields should look restored on the other side

==> radiation from the hole





Stupakov: Ch.16.4

The vector potential, A_μ

- The Electric and magnetic fields can be derived from a four-vector potential, $A_\mu = (\ \phi, \mathbf{A}\ )$

$$\mathbf{E} = \nabla\phi$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

- A_μ transforms like the vector (ct, **r**)

$$\phi' = \gamma\left(\phi - vA_z\right)$$

$$A'_x = A_x$$

$$A'_y = A_y$$

$$A'_z = \gamma\left(A_z - \frac{v}{c^2}\phi\right)$$

Energy balance & the Poynting theorem

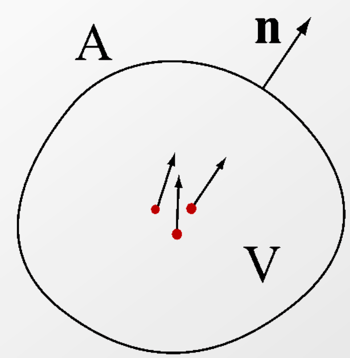
❖ The energy/unit volume of E-M field is

$$u = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}) = \frac{\epsilon_0}{2}(E^2 + c^2B^2)$$

❖ The Poynting vector, $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ = energy flux

❖ The Poynting theorem says

$$\frac{\partial}{\partial t}\int_V u dV = -\int_V \boldsymbol{j}\cdot\boldsymbol{E} dV - \int_A \boldsymbol{n}\cdot\boldsymbol{S} dA$$

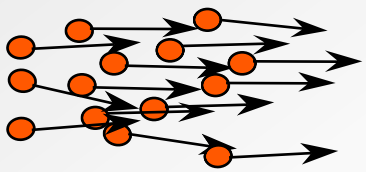


Charges moving inside volume V

rate of change of EM energy due to interaction with moving charges = - work done by **E** on moving charges - EM energy flow through the enclosing surface

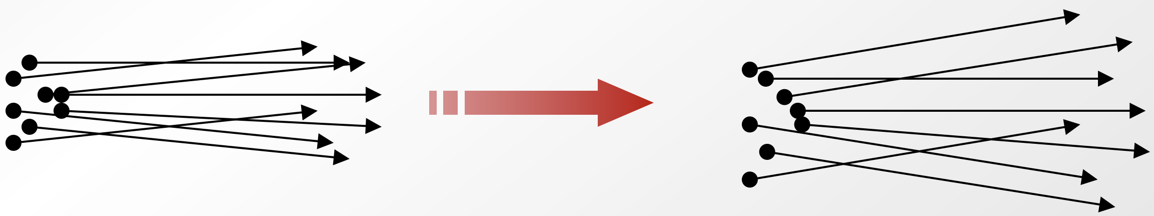
Some other characteristics of beams

- Beams particles have random (thermal) ⊥ motion



$$\vartheta_x = \left\langle \frac{p_x^x}{p_z^2} \right\rangle^{1/2} > 0$$

- Beams must be confined against thermal expansion during transport



Beams have internal (self-forces)

- Space charge forces
  - Like charges repel
  - Like currents attract

- For a long thin beam

$$E_{sp}(V/cm) = \frac{60\, I_{beam}(A)}{R_{beam}(cm)}$$

$$B_{\theta}(gauss) = \frac{I_{beam}(A)}{5\, R_{beam}(cm)}$$

Net force due to transverse self-fields

In vacuum:

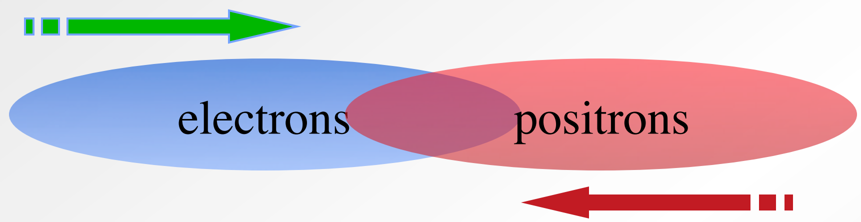
Beam's transverse self-force scale as $1/\gamma^2$

- Space charge repulsion: $E_{sp,\perp} \sim N_{beam}$
- Pinch field: $B_\theta \sim I_{beam} \sim v_z N_{beam} \sim v_z E_{sp}$

$$\therefore F_{sp,\perp} = q\,(E_{sp,\perp} + v_z \times B_\theta) \sim (1-v^2)\, N_{beam} \sim N_{beam}/\gamma^2$$

Beams in collision are ***not*** in vacuum (beam-beam effects)

Example: Megagauss fields in linear collider



At Interaction Point space charge cancels; currents add

==> strong beam-beam focus

==> Luminosity enhancement

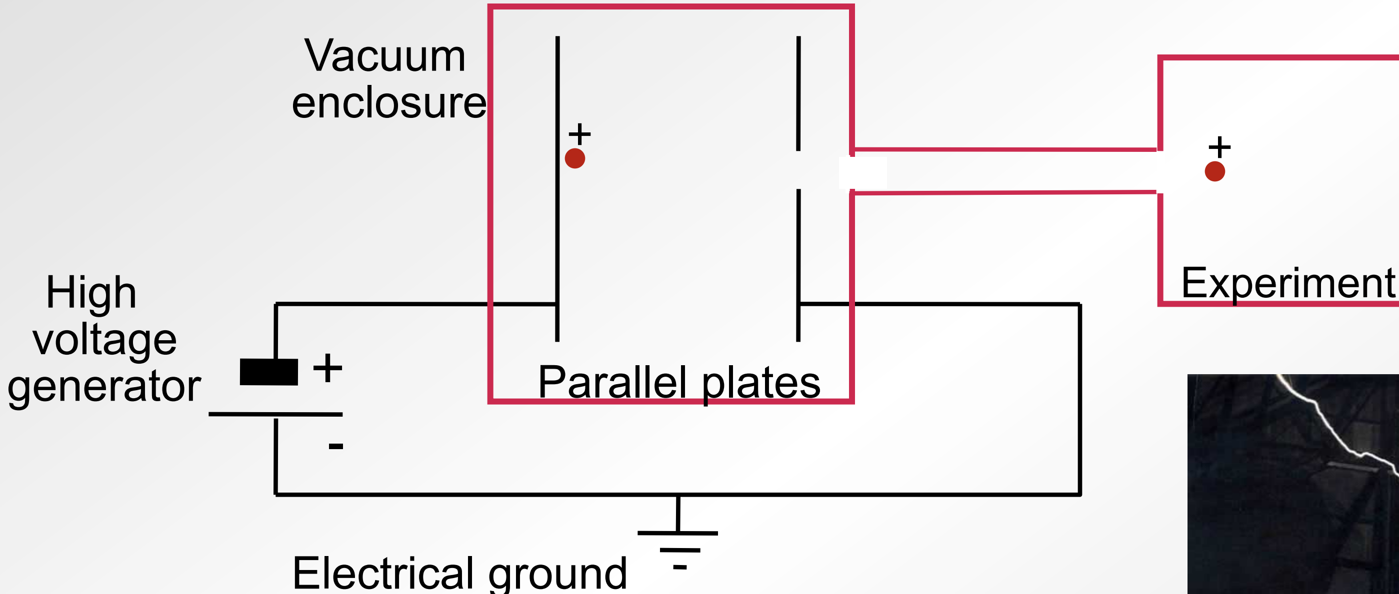
==> Strong synchrotron radiation

Consider 250 GeV beams with 1 kA focused to 100 nm

$$B_{peak} \sim 40 \text{ Mgauss}$$

Accelerators

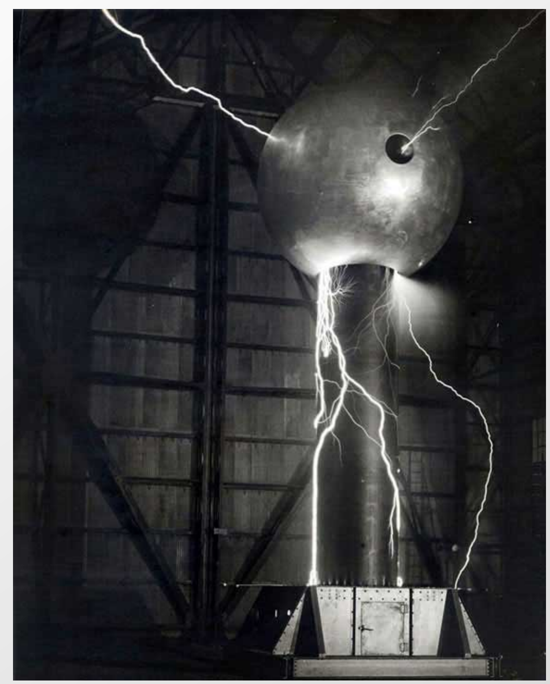
The first accelerators:
DC (electrostatic) accelerators



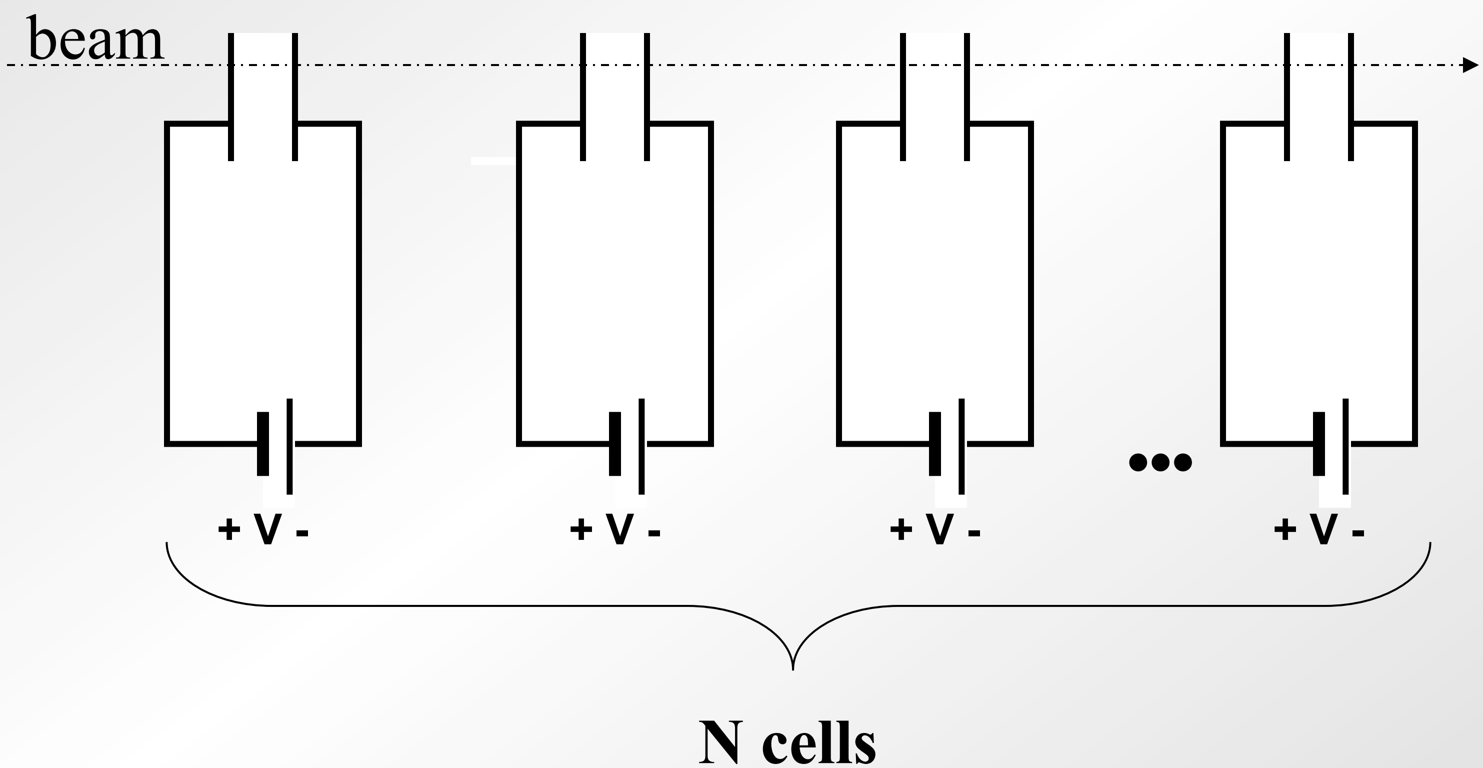
Note the exposed high voltage hazard

The energy is limited

by high voltage break down



What is final energy of the beam?



The "magnetic salad bowl"
Possible high energy DC accelerator?

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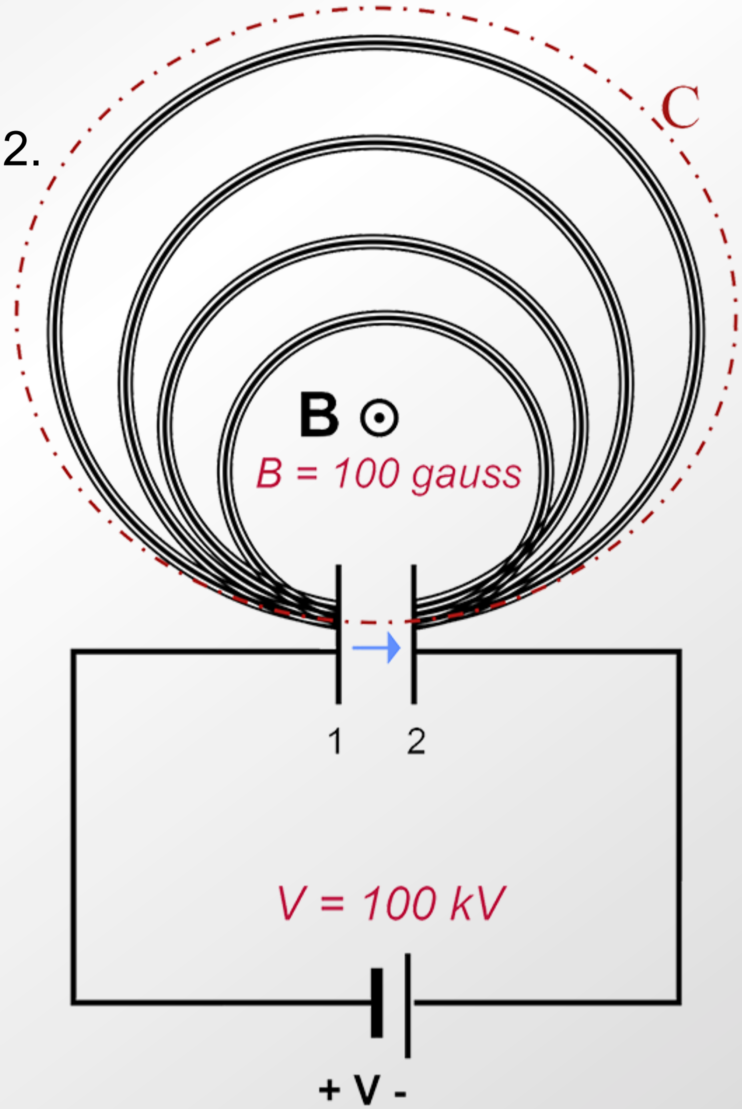
At t = 0 the ion source at 1 injects a proton of energy E_o in the gap pointed at a hole in plate 2. The entire device is imbedded in a constant magnetic (dipole) field, B, pointing out of the surface.

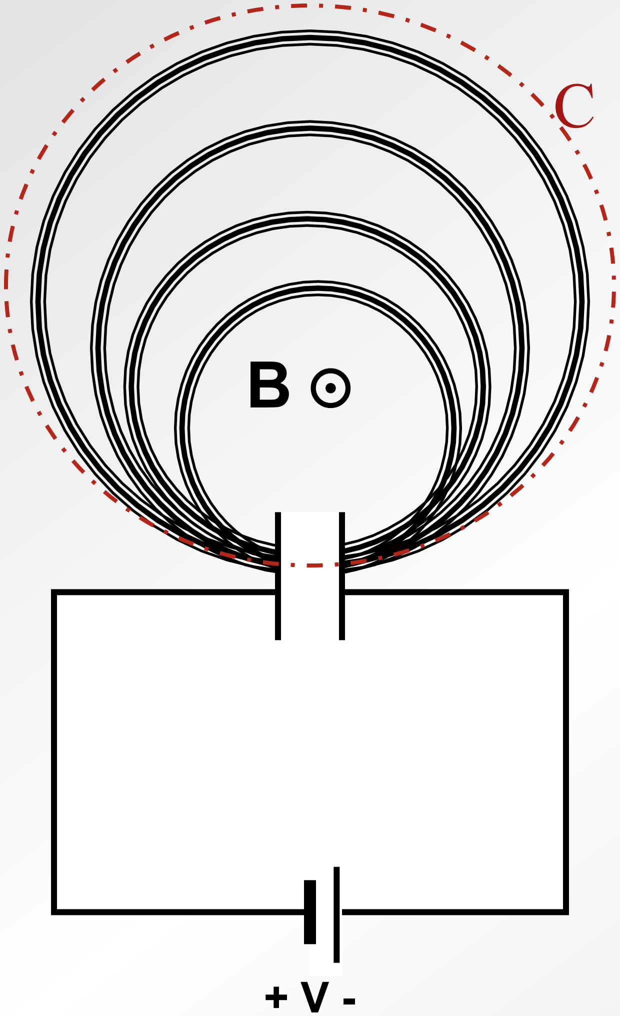
Exiting the plate 2, the proton enters the innermost virtual beam pipe.

If B = 100 Gauss and E_o = 100 keV, what is the radius of the first orbit?

After 10,000 revolutions, what is the energy of the proton as it leaves plate 2.



Maxwell forbids this!



$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

or in integral form

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot \mathbf{n}\, da$$

∴ There is no acceleration without time-varying magnetic flux

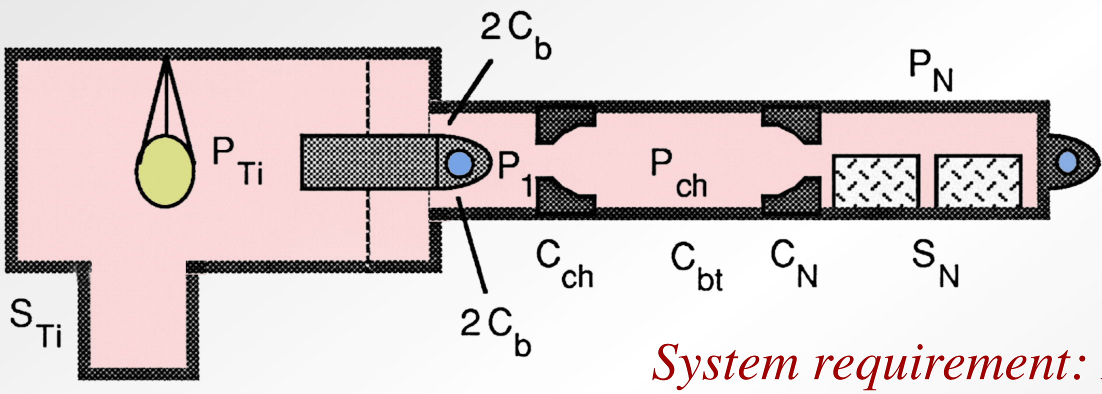
Circuit theory

Accelerator physicists often use
network (circuit) analogs of accelerator systems

1) RF systems
2) Vacuum systems
3) Control systems

Example: Vacuum design storage ring
Synchrotron radiation in hard bends of CESR-B

Estimate the pumping speed needed for Titanium pumps & NEG pumps



System requirement: P_{ch} < 1.7 nTorr

Figure 5. Schematic of the pumping scheme and beam chamber in the hard bend transition region of the high energy ring of CESR-B

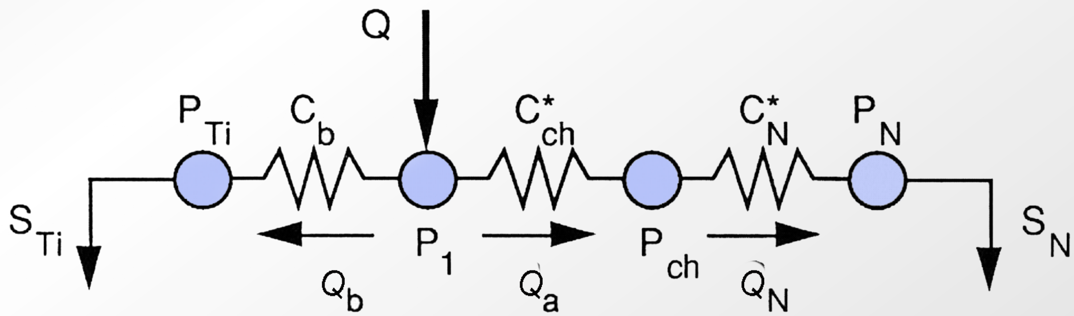
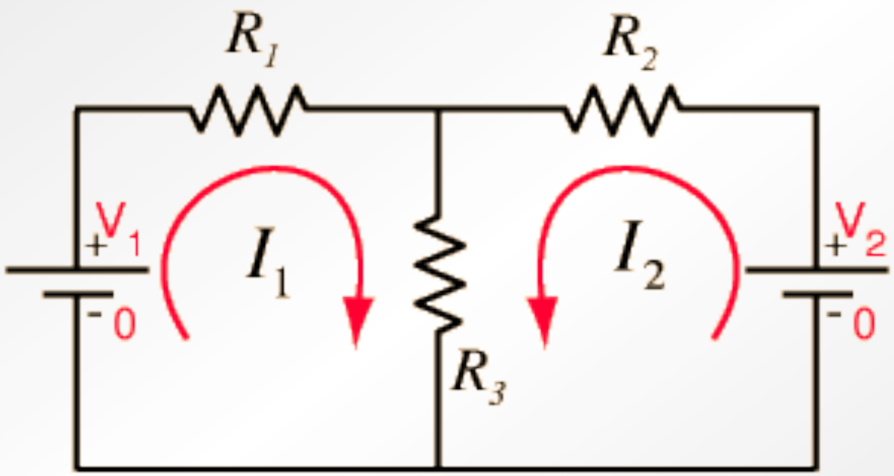


Figure 5. Circuit model of the pumping in the HER transition section

Basic concepts:
Start with dc circuits

- Kirchoff's law's
  - The sum of Voltage drops around any loop equals zero
  - The sum of the currents into any node equals zero



- Ohm's law:
  - The voltage drop across a resistance: $V = I R$

Ohm's Law Generalized

- Basic approach is the Fourier analysis of a circuit
- Start with

$$\tilde{V} = Ve^{j(\omega t+\varphi)}$$

- Instead of V = IR where the quantities are real we write

$$\tilde{V}(\omega) = \tilde{I}(\omega)\overset{\leftrightarrow}{Z}(\omega)$$

- We know this works for resistors.

$V(t) = R\ I(t) \Longrightarrow Z_R$ is real $= R$

- What about capacitors & inductors?

Impedance of Capacitors

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- For a capacitor

$$I = C\left(\frac{dV}{dt}\right) \Rightarrow \tilde{I} = C\frac{d}{dt}Ve^{j(\omega t+\varphi)} = j\omega C\tilde{V}$$

- So our generalized Ohm's law is

$$\tilde{V} = \tilde{I}\vec{Z}_C$$

where

$$\tilde{Z}_C = \frac{1}{j\omega C}$$

Impedance of Inductors

- For a capacitor

$$V = L\left(\frac{dI}{dt}\right) \Rightarrow \tilde{V} = L\frac{d}{dt}Ie^{j(\omega t+\varphi)} = j\omega L\tilde{I}$$

- So our generalized Ohm's law is

$$\tilde{V} = \tilde{I}\tilde{Z}_L$$

Where

$$\tilde{Z}_L = j\omega L$$

Combining impedances

- The algebraic form of Ohm's Law is preserved

==> impedances follow the same rules for combination in series and parallel as for resistors

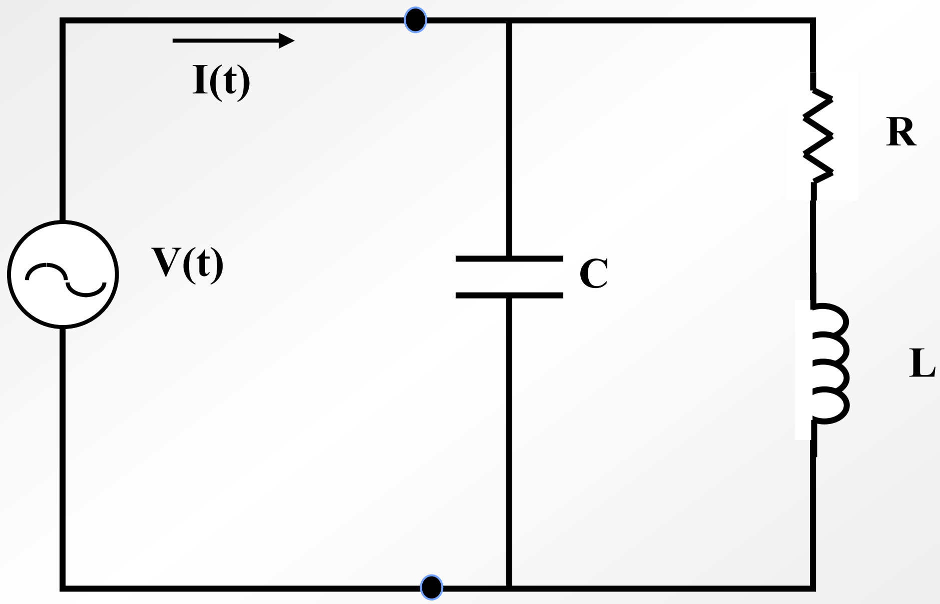
- For example

$$Z_{series} = Z_1 + Z_2$$

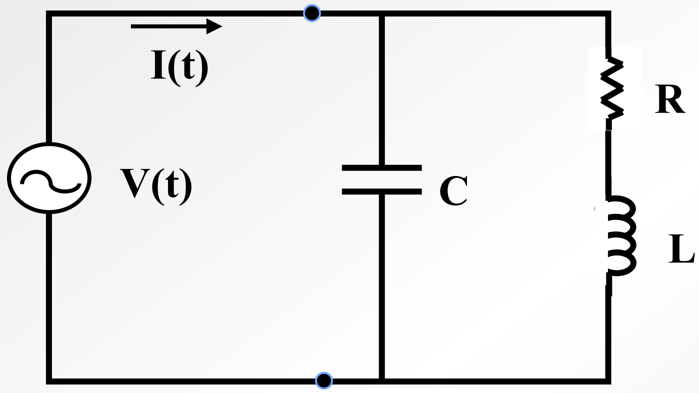
$$Z_{parallel} = \left[{}^{1}\!/_{Z_1} + {}^{1}\!/_{Z_2} \right]^{-1} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

- We can now solve circuits using Kirkhoff's laws, *but in the frequency domain*

Exercise: Compute the impedance Z
looking into the terminals (10 miutes)



Looking into the terminals, we have



$$Z(\omega) = \left[j\omega C + (j\omega L + R)^{-1} \right]^{-1}$$

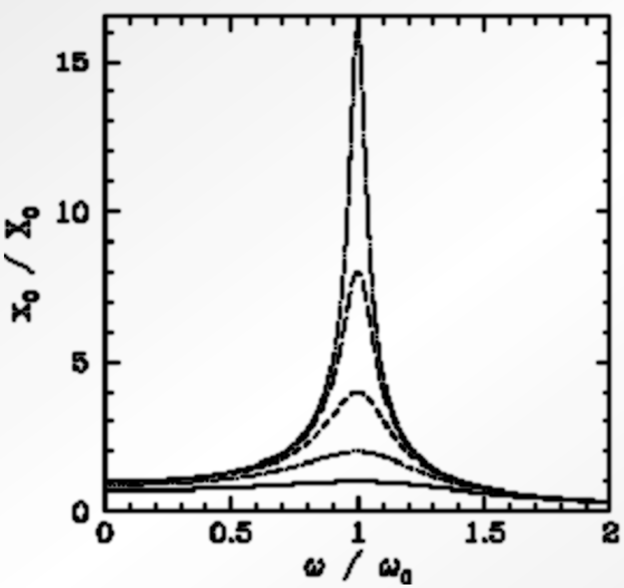
$$Z(\omega) = \frac{1}{j\omega C + (j\omega L + R)^{-1}} = \frac{(j\omega L + R)}{(j\omega L + R)\, j\omega C + 1} = \frac{(j\omega L + R)}{(1 - \omega^2 LC) + j\omega RC} = X + j\varphi$$

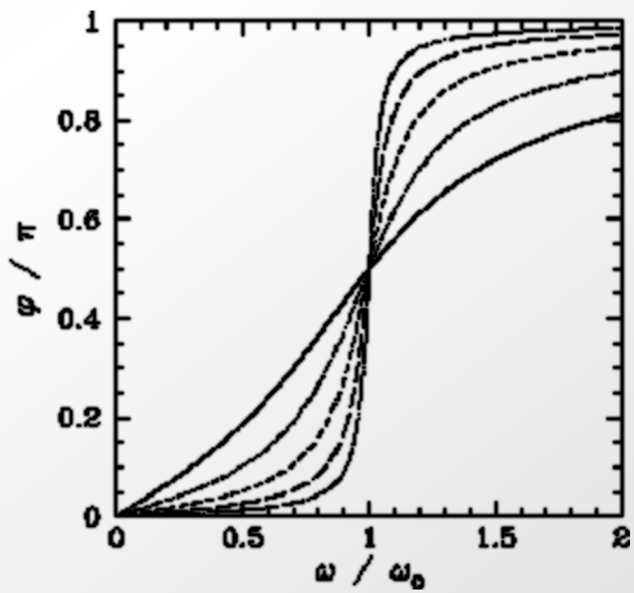
The resonant frequency is $\omega_o = 1 / \sqrt{LC}$

Resonant behavior of the lumped circuit

Converting the denominator of Z to a real number we see that

$$\left| Z(\omega) \right| \sim \left[\left(1 - \frac{\omega^2}{\omega_o^2} \right)^2 + (\omega RC)^2 \right]^{-1}$$





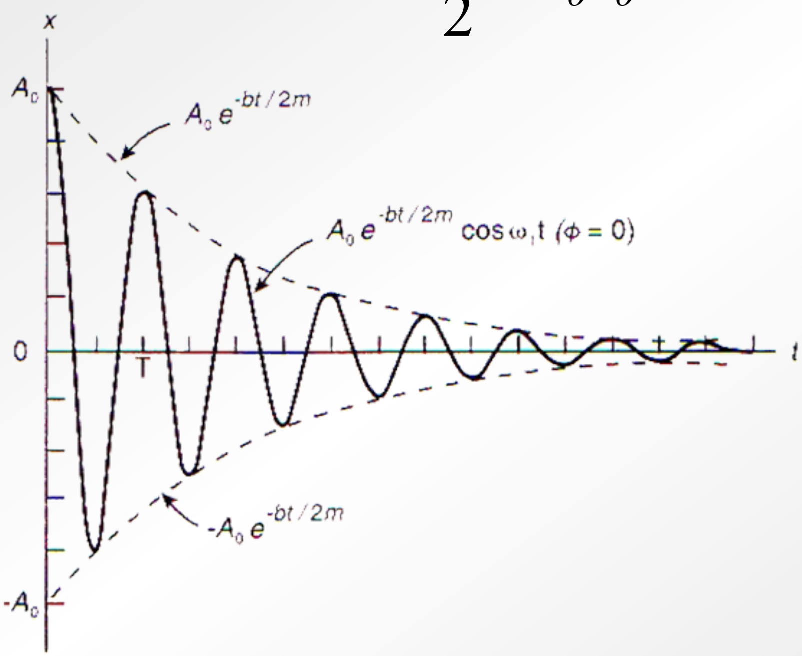
The width is $\dfrac{\Delta\omega}{\omega_0} = \dfrac{R}{\sqrt{L/C}}$

More basics from circuits - Q

$$Q = \frac{\omega_o \circ \; Energy\ stored}{Time\ average\ power\ loss} = \frac{2\pi \circ Energy\ stored}{Energy\ per\ cycle}$$

and

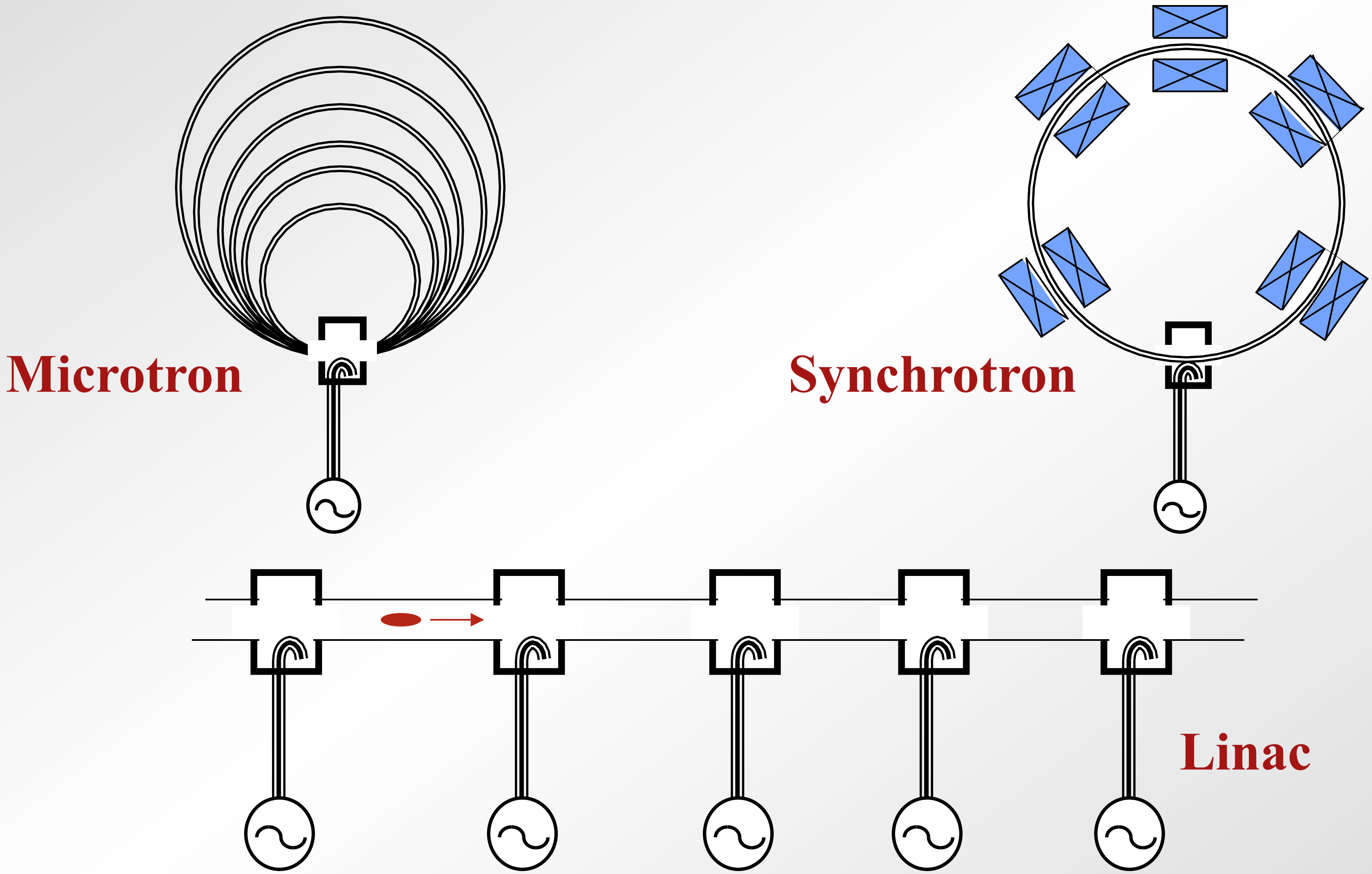
$$\mathscr{E} = \frac{1}{2} L\, I_o I_o^* \qquad \langle \mathscr{P} \rangle = \langle \mathrm{i}^2(t) \rangle R = \frac{1}{2} I_o I_o^* R_{surface}$$



$$\therefore \quad Q = \frac{\sqrt{L/C}}{R} = \left(\frac{\Delta\omega}{\omega_o} \right)^{-1}$$

RF-cavities

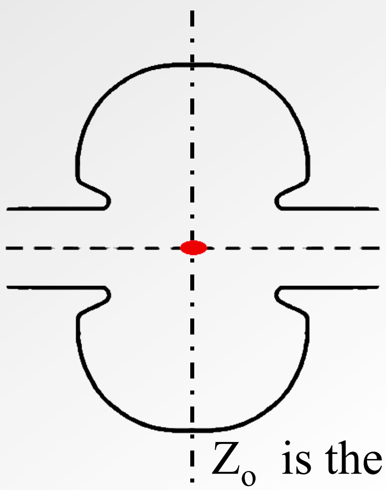
RF-cavities for acceleration:
The heart of modern accelerators



RF cativties: Basic concepts

- Fields and voltages are complex quantities.
  - For standing wave structures use phasor representation

$$\tilde{V} = V e^{i\omega t} \quad \text{where} \quad V = \left|\tilde{V}\right|$$



At t = 0 particle receives maximum voltage gain

Z_o is the reference plane

- For cavity driven externally, phase of the voltage is

$$\theta = \omega t + \theta_o$$

- For electrons $v \approx c$; therefore $z = z_o + ct$

Basic principles and concepts

- Superposition
- Energy conservation
- Orthogonality (of cavity modes)
- Causality

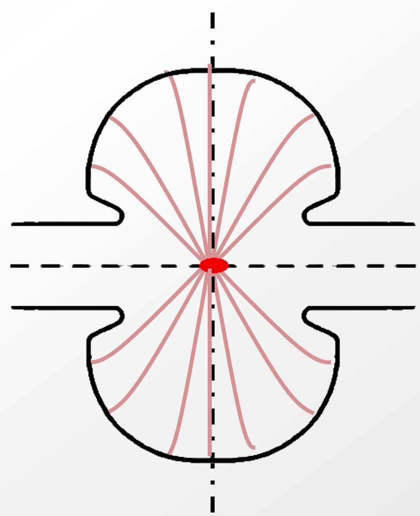
Basic principles:
Reciprocity & superposition

- If you can kick the beam, the beam can kick you

==>

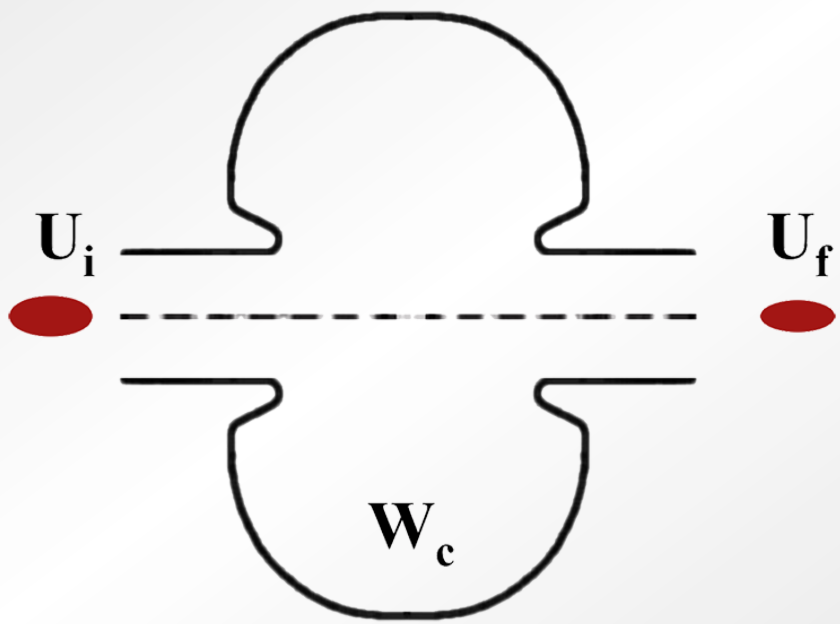
$$\text{Total cavity voltage} = V_{\text{generator}} + V_{\text{beam-induced}}$$

$$\text{Fields in cavity} = \mathbf{E}_{\text{generator}} + \mathbf{E}_{\text{beam-induced}}$$



Basic principles: Energy conservation

- Total energy in the particles and the cavity is conserved
  - Beam loading



$$\Delta W_c = U_i - U_f$$

Basics: Orthogonality of normal modes

- Maxwell's equations are linear
  - The EM field is NOT a source of EM fields
- Therefore,
  - Each mode in the cavity can be treated independently in computing fields induced by a charge crossing the cavity.
  - The total stored energy is equals the sum of the energies in the separate modes.
  - The total field is the phasor sum of all the individual mode fields at any instant.

Basic principles: Causality

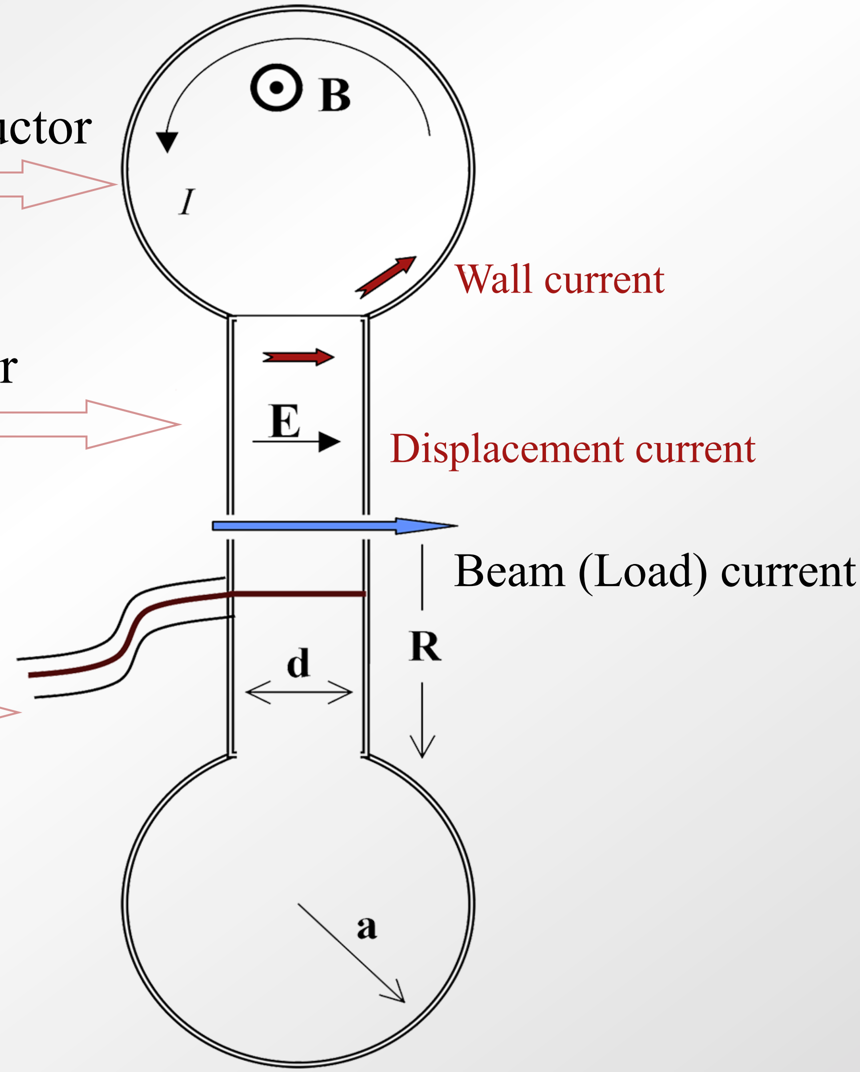
- No disturbance ahead of a charge moving at $v \approx c$
- In a mode analysis of the growth of beam-induced fields, field must vanish ahead of the moving charge *for each mode*

Basic components of an RF cavity

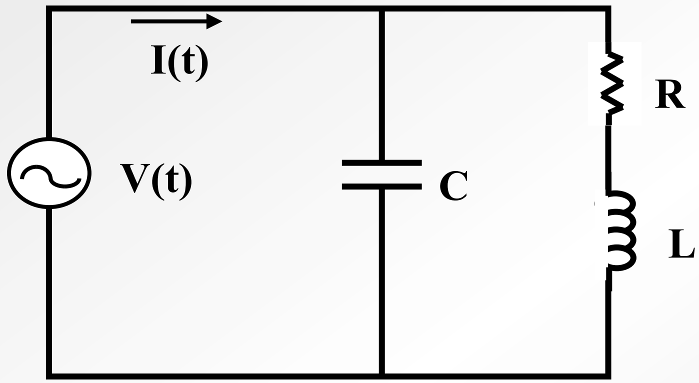
Outer region: Large, single turn Inductor

Central region: Large plate Capacitor

Power feed from rf - generator



We have already solved this circuit
Lumped circuit analogy of resonant cavity



$$Z(\omega) = \left[j\omega C + (j\omega L + R)^{-1} \right]^{-1}$$

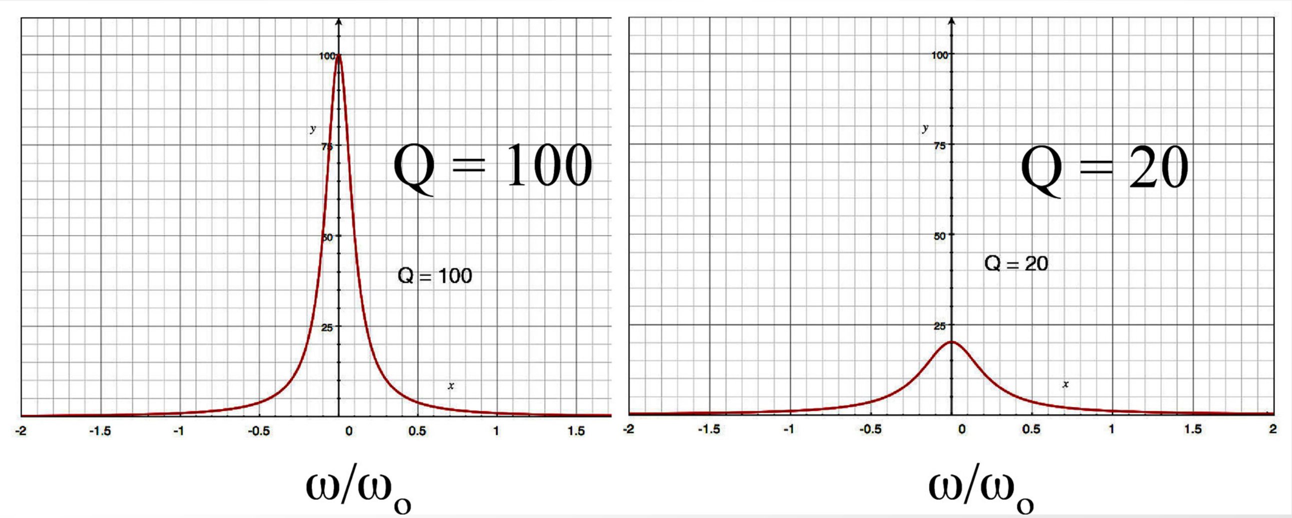
$$Z(\omega) = \frac{1}{j\omega C + (j\omega L + R)^{-1}} = \frac{(j\omega L + R)}{(j\omega L + R) j\omega C + 1} = \frac{(j\omega L + R)}{(1 - \omega^2 LC) + j\omega RC}$$

The resonant frequency is $\omega_o = 1/\sqrt{LC}$

Q of the lumped circuit analogy

Converting the denominator of Z to a real number we see that

$$\left| Z(\omega) \right| \sim \left[\left(1 - \frac{\omega^2}{\omega_o^2} \right)^2 + (\omega RC)^2 \right]^{-1}$$



The width is $\frac{\Delta\omega}{\omega_0} = \frac{R}{\sqrt{L/C}}$

Translate circuit model to a cavity model:
Directly driven, re-entrant RF cavity

Outer region: Large, single turn Inductor

$$L = \frac{\mu_o \pi a^2}{2\pi(R+a)}$$

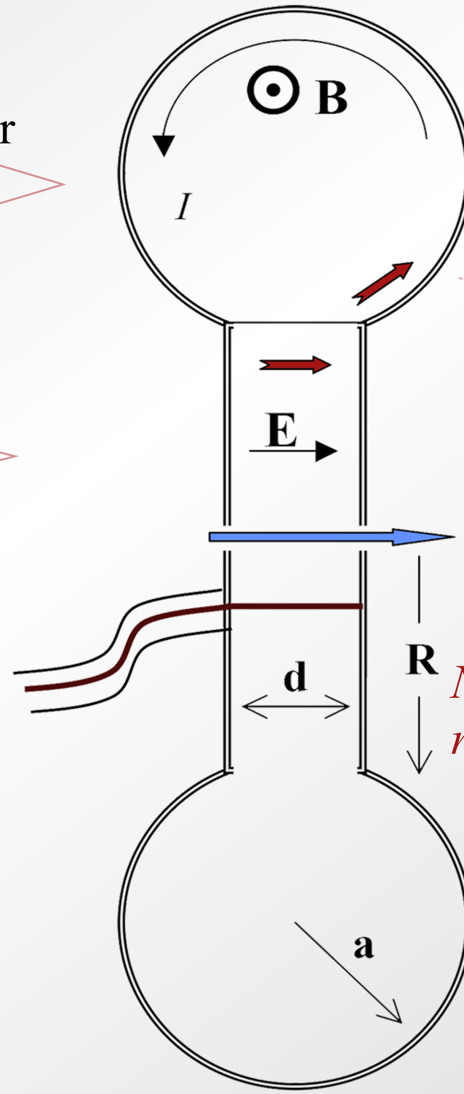
Central region: Large plate Capacitor

$$C = \varepsilon_o \frac{\pi R^2}{d}$$

$$\omega_o = 1/\sqrt{LC} = c\left[\frac{2((R+a)d}{\pi R^2 a^2}\right]^{1/2}$$

Q – set by resistance in outer region

$$Q = \sqrt{L/C}\Big/R$$

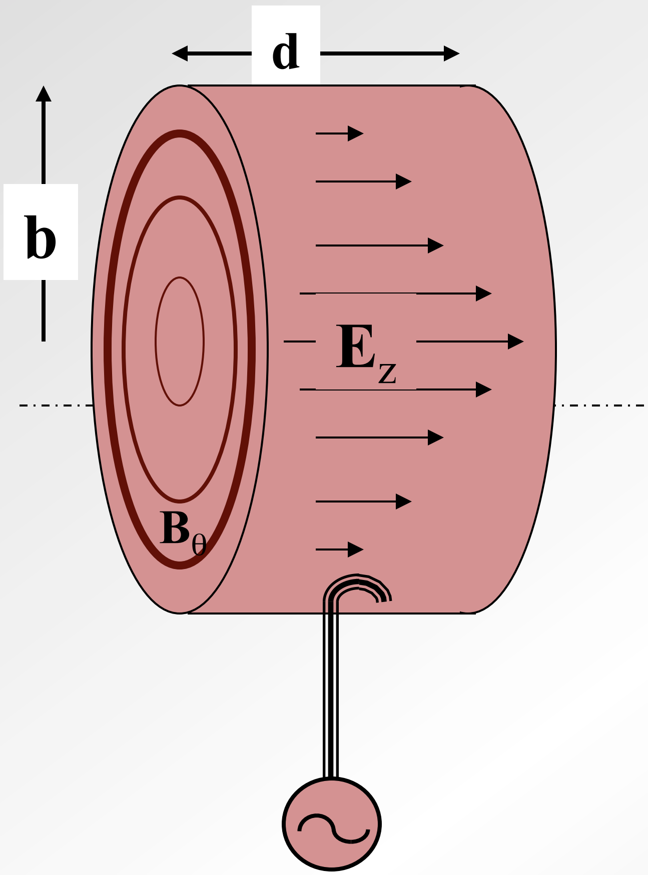


Expanding outer region raises Q

Narrowing gap raises shunt impedance

Source: Humphries, Charged Particle Accelerators

Properties of the RF pillbox cavity



$$\sigma_{walls} = \infty$$

- We want lowest mode: with only $\mathbf{E}_z$ & $\mathbf{B}_\theta$
- Maxwell's equations are:

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right) = \frac{1}{c^2}\frac{\partial}{\partial t}E_z \quad \text{and} \quad \frac{\partial}{\partial r}E_z = \frac{\partial}{\partial t}B_\theta}$$

- Take derivatives

$$\frac{\partial}{\partial t}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right)\right] = \frac{\partial}{\partial t}\left[\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r}\right] = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

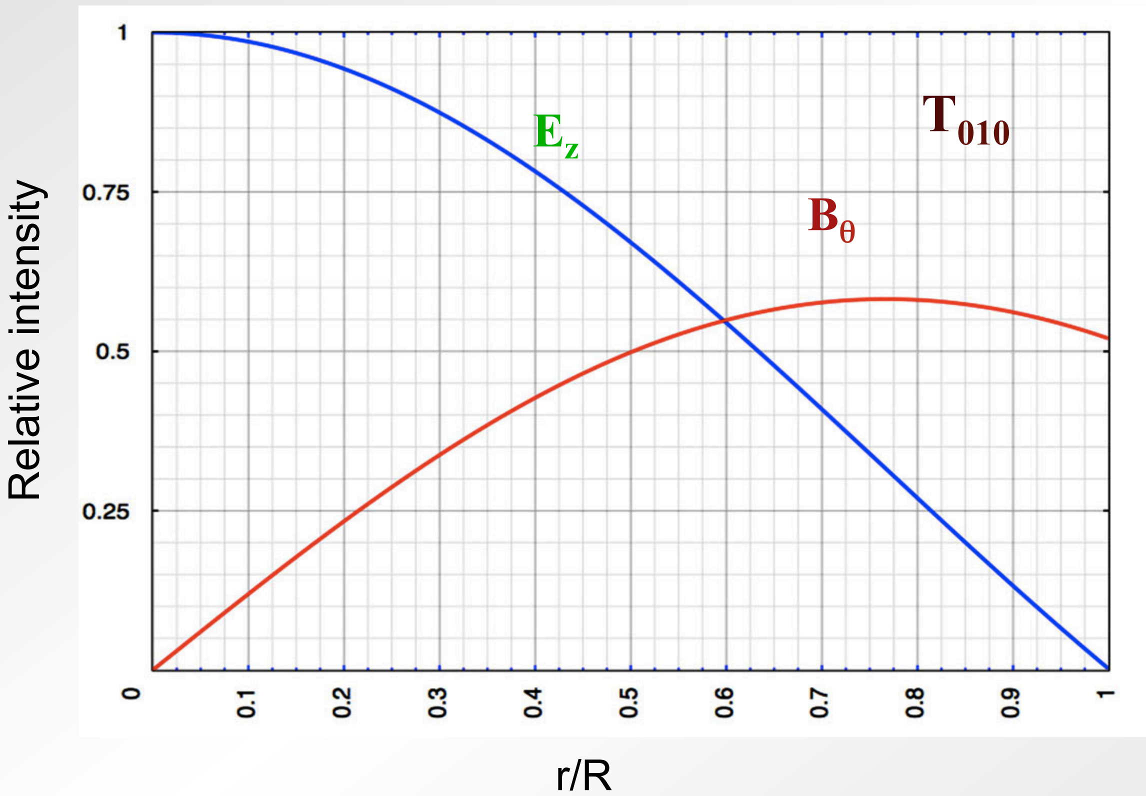
$$\frac{\partial}{\partial r}\frac{\partial E_z}{\partial r} = \frac{\partial}{\partial r}\frac{\partial B_\theta}{\partial t}$$

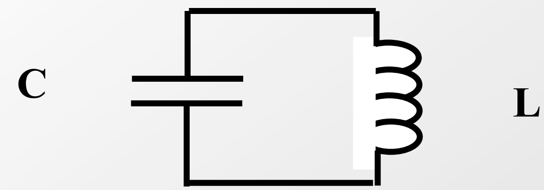
$$==> \quad \boxed{\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}}$$

For a mode with frequency ω

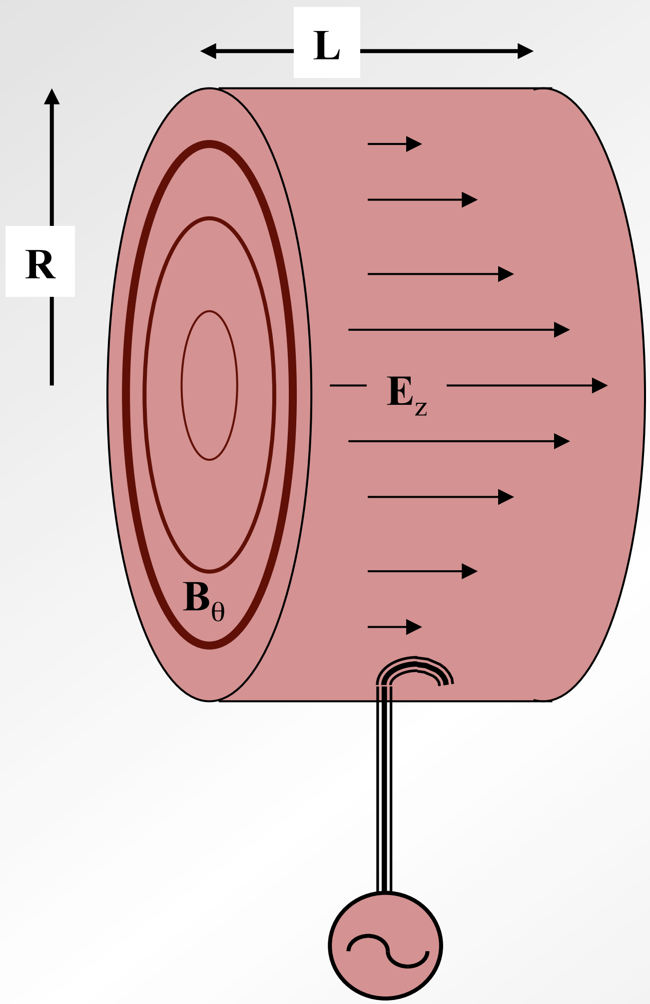
- $$E_z(r,t) = E_z(r)\, e^{i\omega t}$$
- Therefore, $$E_z'' + \frac{E_z'}{r} + \left(\frac{\omega}{c}\right)^2 E_z = 0$$
  - (Bessel's equation, 0 order)
- Hence, $$\boxed{E_z(r) = E_o\, J_o\left(\frac{\omega}{c} r\right)}$$
- Apply boundary condition for conducting walls, $E_z(R) = 0$, therefore $$\boxed{\frac{2\pi f}{c} b = 2.405}$$

E-fields & equivalent circuit: T_{on1o} mode





Simple consequences of pillbox model



- Increasing R lowers frequency

  ==> Stored Energy, $\mathscr{E} \sim \omega^{-2}$

- $\mathscr{E} \sim E_z^2$

- Beam loading lowers E_z for the next bunch

- Lowering ω lowers the fractional beam loading

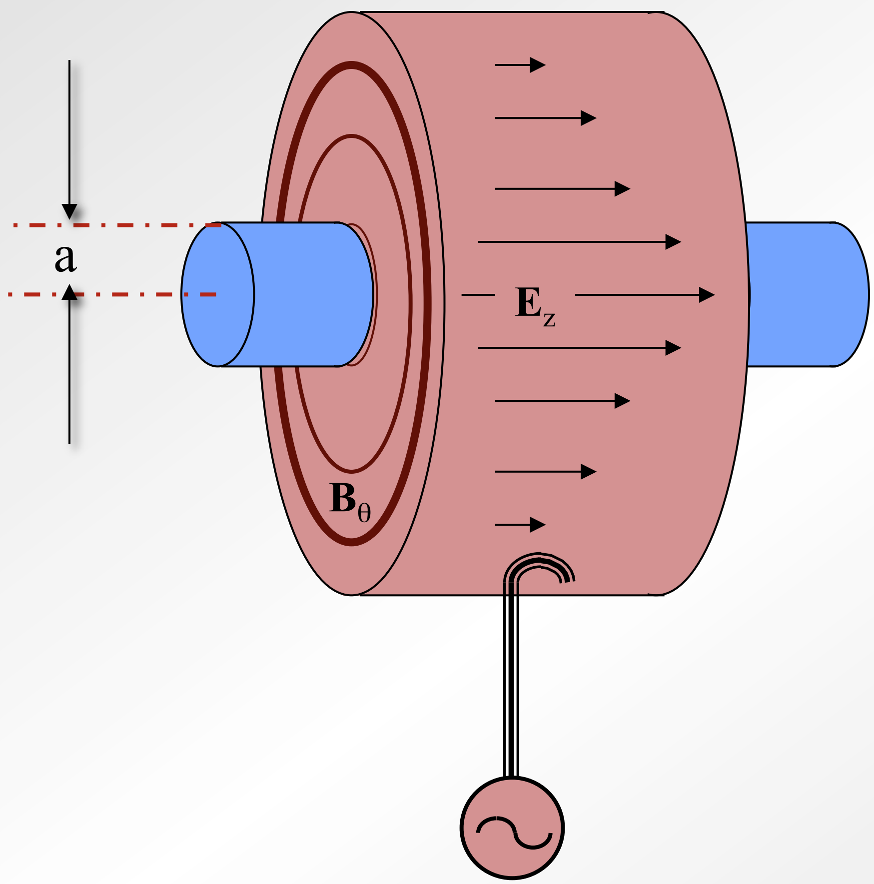
- Raising ω lowers $Q \sim \omega^{-1/2}$

- If time between beam pulses,

  $$T_s \sim Q/\omega$$

  almost all $\mathscr{E}$ is lost in the walls

The beam tube complicates the field modes
(& cell design)



- Peak E no longer on axis
  - $E_{pk} \sim 2 - 3 \times E_{acc}$
  - FOM = E_{pk}/E_{acc}
- ω_o more sensitive to cavity dimensions
  - Mechanical tuning & detuning
- Beam tubes add length & €' s w/o acceleration
- Beam induced voltages ~ a^{-3}
  - Instabilities