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Constructing Scalar-Photon Three Point Vertex in Massless Quenched Scalar QED

Dra. Yajaira Concha Sánchez,

Michoacana University, México

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Contents

Introduction

Gauge theories of fundamental interactions have been the cornerstone of describing the physical world at the most basic level.

Gauge Invariance	⟷	Fundamental Interations

- A central problem of quantum field theories continues to be the quest to find its posible non perturbative solutions.
- Owing to the lack of Dirac matrix structure, Scalar QED provides an attractive and simple laboratory to pursue this aim.
- In Scalar QED the transverse vertex consists of only one unknown function to be fixed.

Gauge Covariance

Consequences

QED	→	Ward-Green-Takahashi Identity (WGTI)
QCD	→	Ward-Slavnov-Taylor Identity

Relation between Green functions

J. C. Ward
Phys. Rev. **78** (1950) 182
Y. Takahashi
Nouvo Cimento **6** (1957) 371
H. S. Green
Proc. Phys. Soc. (London) A**66** (1953) 873

J. C. Taylor
Nucl. Phys. B**33** (1971) 436
A.A. Slavnov
Theor. Math. Phys. **10** (1972) 99

Gauge Covariance

Green functions in differents gauges:

Landau-Khalatnikov-Fradkin Transformations (LKFTs)

As a consequence of the LKFTs we have the Multiplicative Renormalizability of the charged fermión (or scalar) propagator.

L.D. Landau, I. M. Khalatnikov
Zh. Eksp. Teor. Fiz. **29** (1956) 89
L.D. Landau, I.M. Khalatnikov
Sov. Phys. JETP **2** (1956) 69
E.S. Fradkin
Sov. Phys. JETP **2** (1956) 361

K. Johnson, B. Zumino
Phys. Rev. Lett. **3** (1959) 351
B. Zumino
J. Math. Phys. **1** (1960) 1

Schwinger-Dyson Equations (SDEs)

The SDEs are the fundamental equations of motion of any Quantum Field Theory (QFT). They form an infinite set of coupled integral equations that relate the *n*-point Green function to the *(n+1)*-point Green function.
Unfortunately, being an infinite set of coupled equations, they are intractable without some simplifying assumptions.

Typically, in the non-perturbative regime, SDEs are truncated at the level of the two-point Green functions (propagators). We must then use an *ansatz* for the full three-point vertex.

F.J. Dyson
Phys. Rev. **75** (1949) 1736
J.S. Schwinger
Proc. Nat. Acad. Sc. **37** (1951) 452

Three–Point Vertex

Criteria for any acceptable non-perturbative vertex:

- It must satisfy the WGTI.
- It must be free of kinematic singularities.
- It must have the same transformation properties as the bare vertex, under C, P and T transformation.
- It should reduce to its perturbation theory Feynman expansion in the limit of weak coupling.
- It must ensure the MR and the LKFT of the charged fermion (scalar) propagator for any covariant gauge.

Works in QED

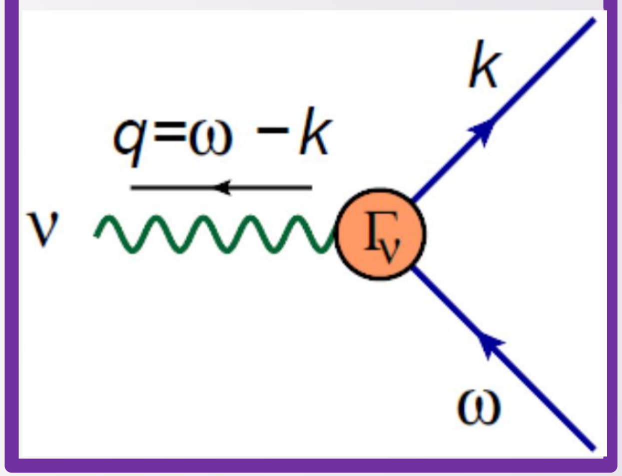
There exist a lot of literature addressing the problem of how to construct a fermion-boson vertex satisfying the WGTI and the LKF transformations, for instance

J.S. Ball, T.W. Chiu
Phys. Rev. D**22** (1980) 2542
D.C. Curtis, M.R. Pennington
Phys. Rev. D**44** (1991) 536-539
Z.H. Dong, H.J. Munczek, C.D. Roberts
Phys. Lett. B**333** (1994) 536-544
A. Bashir, Y. Concha-Sanchez, R. Delbourgo
Phys. Rev. D**76** (2007) 065009
A. Bashir, M.R. Pennington
Phys. Rev. D**50** (1994) 7679-7689
A. Kizilersu, M.R. Pennington
Phys. Rev. D**79** (2009) 125020

Longitudinal and Transverse Decomposition of the Vertex

Ward-Green-Takahashi identity

$$q_\nu \Gamma^\nu(\omega, k) = S^{-1}(\omega) - S^{-1}(k)$$



Where Γ_ν *(ω,k) is the three point vertex*, *q=ω-k*, and *S(ω)* is the fermion (or scalar) propagator. This identity allows us to decompose the vertex as a sum of

Longitudinal and Transverse Parts

$$\Gamma^\nu(\omega, k) = \Gamma^\nu_L(\omega, k) + \Gamma^\nu_T(\omega, k)$$

The *longitudinal* part satisfies the WGTI, by itself, and the *transverse* part which remains completely undetermined, is naturally constrained

$$q_\nu \Gamma^\nu_L(\omega, k) = S^{-1}(\omega) - S^{-1}(k)$$

and

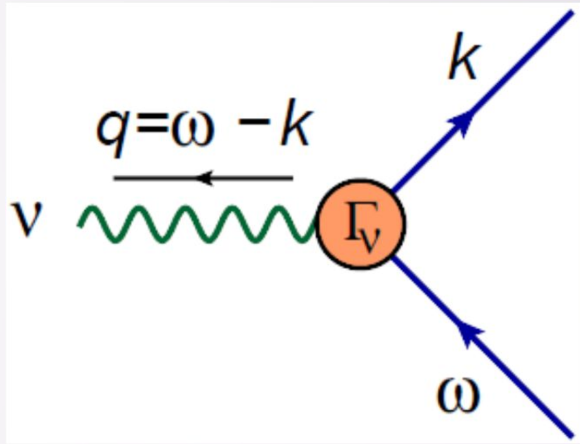
$$q_\nu \Gamma^\nu_T(\omega, k) = 0$$

J.S. Ball, T.W. Chiu
Phys. Rev. D22 (1980) 2542

Longitudinal and Transverse Parts of the Vertex in SQED

Moreover,

$$\Gamma_T^{\mu}(k,k) = 0$$



In order to satisfy the WGTI in a manner free of kinematic singularities, we follow Ball and Chiu. For SQED we have

$$\Gamma^{\nu}(\omega,k) = \frac{S^{-1}(\omega) - S^{-1}(k)}{\omega^2 - k^2}(\omega + k)^{\nu} + \tau\left(\omega^2, q^2, k^2\right) T^{\nu}(\omega,k)$$

where $T^{\nu}(\omega,k) = (\omega\cdot q)k^{\nu} - (k\cdot q)\omega^{\nu}$ is the transverse basis vector.

Multiplicative Renormalizability

In SQED, for the massless scalars, $S(k)$ can be expressed

$$\boxed{S(k) = \frac{F(k^2, \Lambda^2)}{k^2}}$$

where $F(k^2,\Lambda^2)$ is the wavefunction renormalization and Λ is the ultraviolet cut-off used to regularize the divergent integrals involved. MR of the scalar propagator requires the renormalized F_R be related to the unrenormalized F through a multiplicative factor Z by

$$\boxed{F_R(k^2, \mu^2) = Z^{-1}(\mu^2, \Lambda^2) F(k^2, \Lambda^2)}$$

Where μ plays the role of an arbitrary renormalization scale. The MR restricts $F(k^2,\Lambda^2)$ to be of the form

$$\boxed{F(k^2, \Lambda^2) \equiv F(k^2) = \left(\frac{k^2}{\Lambda^2}\right)^{\beta}}$$

Multiplicative Renormalizability

Where the anomalous dimension β is unknown at the non perturbative level. Perturbation theory tells us that

$$\frac{1}{F(k^2)} = 1 + \frac{\alpha(\xi - 3)}{4\pi} \ln\left(\frac{\Lambda^2}{k^2}\right)$$

which suggests

$$\beta = \frac{\alpha}{4\pi}(\xi - 3)$$

in

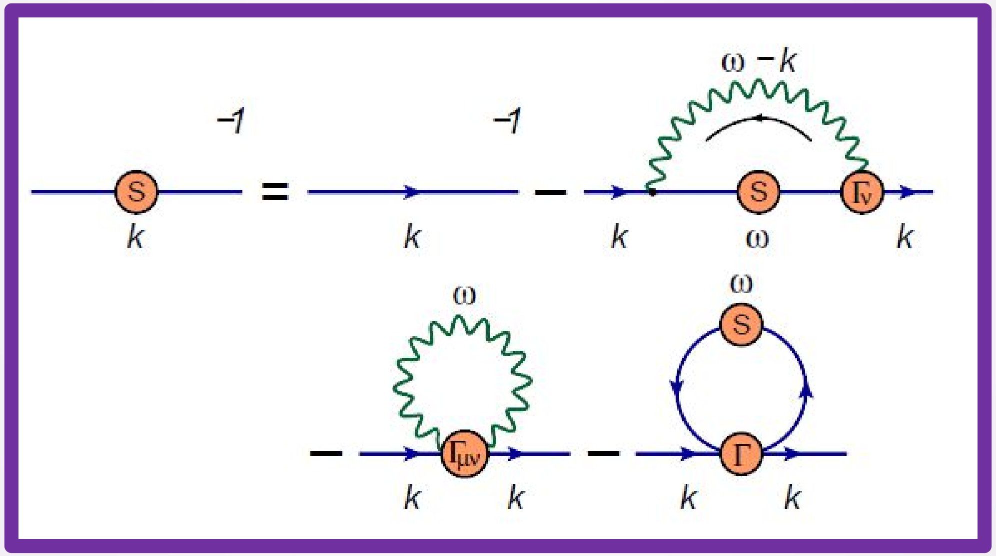
$$F(k^2) = \left(\frac{k^2}{\Lambda^2}\right)^{\beta}$$

This power behavior of $F(k^2)$, with β, is the solution of

$$\frac{1}{F(k^2)} = 1 + \frac{\alpha}{4\pi}(\xi - 3)\int_{k^2}^{\Lambda^2} \frac{d\omega^2}{\omega^2}\frac{F(\omega^2)}{F(k^2)}$$

Gap Equation

The SDE for the scalar propagator *S(k)* in SQED, in the quenched approximation, is



where

$$\Gamma^{\nu}(\omega,k)=\frac{S^{-1}(\omega)-S^{-1}(k)}{\omega^2-k^2}(\omega+k)^{\nu}+\tau\left(\omega^2,q^2,k^2\right)T^{\nu}(\omega,k)$$

Gap Equation

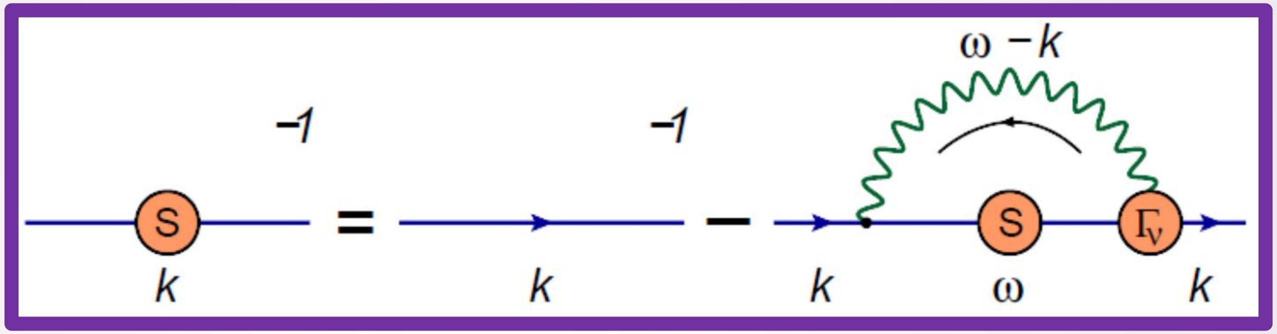
Mathematically, this is written as :

$$
\begin{aligned}
-iS^{-1}(k) \;=\;& -iS_0^{-1}(k) + e^2 \int_M \frac{d^4w}{(2\pi)^4}(w+k)^\mu S(w)\Gamma^\nu(w,k)\Delta^0_{\mu\nu}(q) \\
& -e^2 \int_M \frac{d^4w}{(2\pi)^4}\Gamma^{\mu\nu}(k,-w,k,w)\Delta^0_{\mu\nu}(w) - \int_M \frac{d^4w}{(2\pi)^4} S(w)\Gamma(w,k)
\end{aligned}
$$

where e is the electromagnetic coupling, $q = \omega - k$, and the subscript M indicates integration over the entire *Minkowski* space. $\Delta^0_{\mu\nu}(\omega)$ and $S_0(k)$ are the bare photon and scalar propagators. $S(k)$ is the full scalar propagator. We neglect the photon and the scalar bubble diagrams since they do not contribute to leading logs terms in the one loop calculation.

Gap Equation

Then



$$S^{-1}(k^2) = S_0^{-1}(k^2) + \frac{i\alpha}{4\pi^3}\int_M d^4\omega(\omega+k)^\mu S(\omega^2)\Gamma^\nu(\omega,k)\Delta^0_{\mu\nu}(q)$$

where

$$\Delta^0_{\mu\nu}(q) = \frac{1}{q^2}\left[g_{\mu\nu} + (\xi-1)\frac{q_\mu q_\nu}{q^2}\right]$$

with q=ω-k.

Gap Equation

Then Wick rotate to Euclidean space the gap equation, we have

$$\boxed{\begin{aligned}\frac{1}{F(k^2,\Lambda^2)} &= 1-\frac{\alpha}{4\pi^3}\frac{1}{k^2}\int_E d^4\omega\frac{1}{q^2}\left\{\left[1-\frac{S(\omega)}{S(k)}\right]\right.\\ &\times\left[1+(\xi-1)\frac{\omega^2-k^2}{q^2}+2\frac{k^2}{\omega^2-k^2}\right.\\ &\left.\left.+2\frac{\omega\cdot k}{\omega^2-k^2}\right]-2S(\omega)\tau(\omega^2,k^2,q^2)\Delta^2\right\},\end{aligned}}$$

where

$$\boxed{\Delta^2=(\omega\cdot k)^2-\omega^2k^2,\quad \alpha=e^2/4\pi}$$

and the subscript *E* indicates integration over the whole Euclidean space.

Gap Equation

At this stage, it appears imposible to proceed because of the dependence of τ on the angle between the incoming and outgoing momenta ω and k of the scalar particle. We shall asume that the transverse vertex has no dependence on this angle. Angular integration leads us to

$$\boxed{\begin{aligned}\frac{1}{F(k^2,\Lambda^2)} &= 1-\frac{\alpha}{4\pi}\int_0^{k^2} d\omega^2\frac{\omega^2}{k^2}\left[1-\frac{S(\omega)}{S(k)}\right] \\ &\qquad\qquad \times\left[\frac{(2-\xi)}{k^2}+\frac{1}{\omega^2-k^2}\left(2+\frac{\omega^2}{k^2}\right)\right] \\ &-\frac{\alpha}{4\pi}\int_{k^2}^{\Lambda^2} d\omega^2\left[1-\frac{S(\omega)}{S(k)}\right]\left\{\frac{3}{\omega^2-k^2}+\frac{\xi}{k^2}\right\} \\ &+\frac{\alpha}{8\pi}\int_0^{k^2} d\omega^2\omega^2 S(\omega)\tau(\omega^2,k^2)\left(\frac{\omega^4}{k^4}-3\frac{\omega^2}{k^2}\right) \\ &+\frac{\alpha}{8\pi}\int_{k^2}^{\Lambda^2} d\omega^2\omega^2 S(\omega)\tau(\omega^2,k^2)\left(\frac{k^2}{\omega^2}-3\right).\end{aligned}}$$

MR Constraint on Tau

Recall the prescription of MR on $F(k^2)$

$$\frac{1}{F(k^2)} = 1 + \frac{\alpha}{4\pi}(\xi - 3)\int_{k^2}^{\Lambda^2} \frac{d\omega^2}{\omega^2}\frac{F(\omega^2)}{F(k^2)}$$

This equation imposes the following restriction on the transverse vertex

$$\begin{aligned}
&-2\int_0^{k^2} d\omega^2 \left\{\frac{3}{k^2} + \frac{(3-\xi)}{k^2}\frac{\omega^2}{k^2} + \frac{3}{\omega^2 - k^2} \right.\\
&\qquad \left. + \frac{(\xi - 3)}{k^2}\frac{F(\omega^2)}{F(k^2)} - \frac{3}{\omega^2 - k^2}\frac{F(\omega^2)}{F(k^2)}\right\}\\
&-2\int_{k^2}^{\Lambda^2} d\omega^2 \left\{\frac{3}{\omega^2 - k^2} - \frac{3}{\omega^2 - k^2}\frac{F(\omega^2)}{F(k^2)} + \frac{\xi}{k^2}\right\}\\
&\qquad + \int_0^{k^2} d\omega^2 F(\omega^2)\tau(\omega^2, k^2)\left(\frac{\omega^4}{k^4} - 3\frac{\omega^2}{k^2}\right)\\
&\qquad + \int_{k^2}^{\Lambda^2} d\omega^2 F(\omega^2)\tau(\omega^2, k^2)\left(\frac{k^2}{\omega^2} - 3\right) = 0
\end{aligned}$$

Recall that in the above equation, we have neglected the contributions of the photon and the scalar bubble diagrams since they do not contribute to the one loop LLA.

MR Constraint on Tau

Introducing the variable x, where

$$x = \omega^2/k^2 \text{ para } 0 < \omega^2 < k^2$$
$$\text{y } x^{-1} = \omega^2/k^2 \text{ para } k^2 < \omega^2 < \Lambda^2$$

The resulting restricción can be rewritten as

$$\int_0^1 dx\, W(x) = 0$$

with

$$W(x) = -6x\frac{\left(1-x^{\beta}\right)}{x-1} + 6x^{-1}\frac{\left(1-x^{-\beta}\right)}{x-1} + 2\xi\left(1-x^{\beta}\right) + (x-3)\left(x^{\beta}+x^{-2}\right)h(x)\,,$$

Note that we have again kept only those terms which contribute to the LLA.

MR Constraint on Tau

Moreover, we have introduced the definition

$$\boxed{h(x) \equiv xk^2 F(k^2)\tau(xk^2, k^2)}$$

which is a dimensionless function satisfying the property

$$\boxed{h(1/x) = x^{\beta-1} h(x)\,.}$$

with $\beta = (\xi - 3)/\pi$. Then we write that

$$\boxed{\begin{aligned} W(x) - W(x^{-1}) \;=\;& 4\,(x-1)\left(x^{\beta} + x^{-2}\right) h(x) \\ & +6x\left(1 - x^{\beta}\right) - 6x^{-1}\left(1 - x^{-\beta}\right) \\ & +2\xi\left[\left(1 - x^{\beta}\right) - \left(1 - x^{-\beta}\right)\right]\,. \end{aligned}}$$

MR Constraint on Tau

Taking $x=p^2/k^2$, using the symmetry $\tau(p^2,k^2)=\tau(k^2,p^2)$ and Wick rotating back Minkowski space, the τ acquires the following form

$$
\begin{aligned}
\tau(k^2,p^2) \;=\; & -\frac{3}{2}\frac{1}{(k^2-p^2)}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right] \\
& -\frac{\xi}{2}\frac{1}{(k^2-p^2)}\frac{F(k^2)+F(p^2)}{s(k^2,p^2)}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right] \\
& +\frac{1}{4}\frac{1}{(k^2-p^2)}\frac{1}{s(k^2,p^2)}\left[W\left(\frac{k^2}{p^2}\right)-W\left(\frac{p^2}{k^2}\right)\right]
\end{aligned}
$$

where

$$
s(k^2,p^2)=F(k^2)\frac{k^2}{p^2}+F(p^2)\frac{p^2}{k^2}
$$

MR Constraint on Tau

The exact form of the function *W* remains unknown. In order to ensure MR, we choose the trivial solution *W(x)=0* (for any dimensionless ratio x *of momenta*):

$$\int_0^1 dx\, W(x) = 0$$

$$\tau(k^2,p^2) = -\frac{3}{2}\frac{1}{(k^2-p^2)}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right] - \frac{\xi}{2}\frac{1}{(k^2-p^2)}\frac{F(k^2)+F(p^2)}{s(k^2,p^2)}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right]$$

The scalar structure has been first reported by Curtis and Pennington.

$$\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right]$$

D.C. Curtis, M.R. Pennington
Phys. Rev. D**44** (1991) 536-539

BCD Vertex

The scalar-photon vertex has already been calculated in one loop perturbation theory by me *et al.*, using dimensional regularization, in arbitrary gauge ξ and dimension *d*. *For massless case*, in *d=4*, we report

$$
\begin{aligned}
\tau_{BCD}(k^2,p^2,q^2) \;=\; & \frac{\alpha}{8\pi\Delta^2}\Bigg\{(k^2+p^2-4k\cdot p)\left(k\cdot p J_0^{4,0}+\ln\frac{q^4}{p^2k^2}\right) \\
& +\frac{(k^2+p^2)\,q^2-8p^2k^2}{p^2-k^2}\ln\frac{k^2}{p^2} \\
& +(\xi-1)\Bigg[k^2p^2J_0^{4,0}+2\frac{k^2p^2+(k\cdot p)\,(k^2+p^2)}{k^2-p^2}\ln\left(\frac{p^2}{k^2}\right) \\
& +2\frac{(k\cdot p)}{(k^2-p^2)}\left[k^2\ln\left(\frac{q^2}{p^2}\right)-p^2\ln\left(\frac{q^2}{k^2}\right)\right]\Bigg]\Bigg\}
\end{aligned}
$$

where

$$
J_0^{4,0}=\frac{2}{i\pi^2}\int d^dw\,\frac{1}{w^2\,(p-w)^2(k-w)^2}
$$

$$
\Delta^2\equiv(k\cdot p)^2-k^2p^2
$$

A. Bashir, Y. Concha-Sanchez, R. Delbourgo
Phys. Rev. D**76** (2007) 065009

Asymptotic Limit $k^2 \gg p^2$

In order to compare the vertex *ansatz,* based upon multiplicative renormalizability, against its one loop perturbative form, it is convenient to take the asymptotic limit $k^2 >> p^2$ *of external momenta* in τ_{BCD}, we have

$$\boxed{\tau_{\mathrm{BCD}}^{\mathrm{asym}}(k^2, p^2) \overset{k^2 \gg p^2}{=} -3\frac{\alpha}{4\pi}\frac{1}{k^2}\ln\left(\frac{k^2}{p^2}\right)}$$

Expectedly, it is independent of q^2 and hence we drop this dependence from its argument. Note that this expression is also independent of the covariant gauge parameter ξ. It is unlike spinor QED where the leading asymptotic vertex is proportional to ξ.

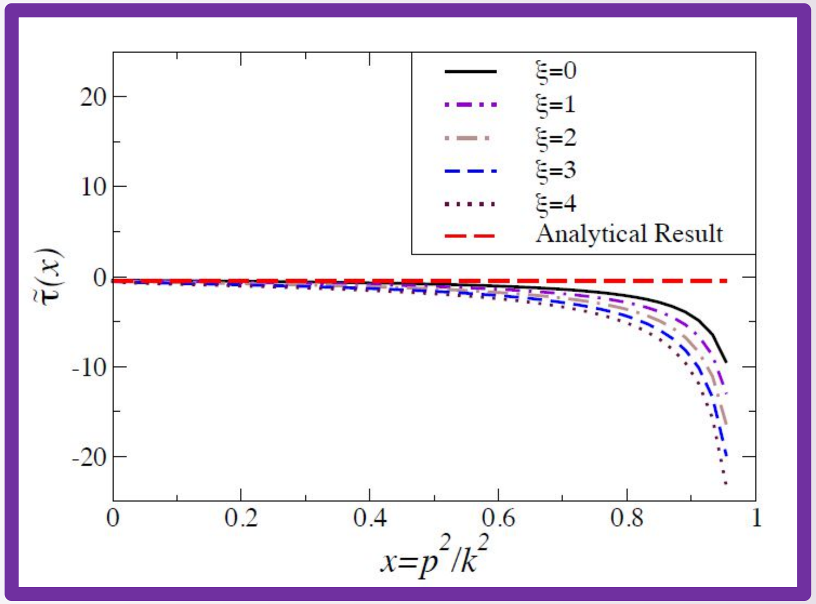
Asymptotic Limit $k^2 \gg p^2$

For a numerical check, we define

$$\tilde{\tau}(x) = -\frac{k^2\, \tau(k^2, xk^2)}{\alpha \ln x}$$

where $x = p^2/k^2$ and we have suppressed the q^2 dependence for notational simplification. Thus

$$\tilde{\tau}^{\rm asym}_{\rm BCD}(x) = -\frac{3}{4\pi}\,.$$



We plot $\tilde{\tau}^{asym}_{BCD}$(x) and $\tilde{\tau}_{BCD}(x)$ as a function of x, the latter for different values of the gauge parameter ξ and for a fixed value of q^2, chosen arbitrarily. In the asymptotic limit, all curves converge to a single value, as expected.

Asymptotic Limit $k^2 \gg p^2$

On the other hand, using the perturbative expression for $F(k^2)$

$$F(k^2,\Lambda^2) \;=\; 1+\frac{\alpha(\xi-3)}{4\pi}\ln\left(\frac{k^2}{\Lambda^2}\right)$$

In our result for tau:

$$\tau(k^2,p^2) \;=\; -\frac{3}{2}\frac{1}{(k^2-p^2)}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right] -\frac{\xi}{2}\frac{1}{(k^2-p^2)}\frac{F(k^2)+F(p^2)}{s(k^2,p^2)}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right]$$

And taking the asymptotic limit $k^2 \gg p^2$, we have

$$\tau^{\mathrm{asym}}(k^2,p^2) \overset{k^2 \gg p^2}{=} \frac{3}{2}\frac{\alpha}{4\pi}\frac{(\xi-3)}{k^2}\ln\left(\frac{k^2}{p^2}\right)$$

Our results, are in agreement in the Feynman gauge (ξ=*1*).

Perturbative Constraints on $W(x)$

In order for them to be the same in an arbitrary gauge ξ, we must seek a non-trivial *W*-function in tau:

$$\boxed{\begin{aligned}\tau(k^2,p^2) &= \frac{3}{2}\frac{1}{(k^2-p^2)}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right] \\ &\quad +\frac{\xi}{2}\frac{1}{(k^2-p^2)}\frac{F(k^2)+F(p^2)}{s(k^2,p^2)}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right] \\ &\quad -\frac{1}{4}\frac{1}{(k^2-p^2)}\frac{1}{s(k^2,p^2)}\left[W\left(\frac{k^2}{p^2}\right)-W\left(\frac{p^2}{k^2}\right)\right]\end{aligned}}$$

Still satisfying restriction

$$\boxed{\int_0^1 dx\, W(x) = 0}$$

Perturbative Constraints on $W(x)$

Then, perturbation theory demands the simplest choice for *W*

$$\boxed{W\left(\frac{k^2}{p^2}\right) = \lambda \frac{k^2}{p^2} \ln\left(\frac{k^2}{p^2}\right) + \frac{\lambda}{2}\frac{k^2}{p^2}}$$

with

$$\boxed{\lambda = -\frac{3\alpha(\xi - 1)}{2\pi}}$$

In the Feynman gauge ($\xi = 1$) *W*=0. Introducing the variable $x=k^2/p^2$, we have

$$\boxed{W(x) = \lambda x \ln x + \frac{\lambda}{2} x}$$

so that

$$\boxed{\int_0^1 W(x) dx = \lambda \int_0^1 \left(x \ln x + \frac{x}{2}\right) = \lambda \frac{x^2}{2} \ln x|_0^1 = 0}$$

Non-Perturbative Tau

This choice for *W(x)* in the vertex leads to

$$\tau(k^2,p^2) \;=\; -\frac{3}{2}\frac{1}{(k^2-p^2)}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right]$$
$$-\frac{\xi}{2}\frac{1}{(k^2-p^2)}\frac{F(k^2)+F(p^2)}{s(k^2,p^2)}\left[\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right]$$
$$-\frac{(\xi-1)}{(k^2-p^2)\,s(k^2,p^2)}\frac{3\alpha}{8\pi}\left[\frac{k^2}{p^2}+\frac{p^2}{k^2}\right]\ln\left(\frac{k^2}{p^2}\right)\,.$$

- It agrees with the perturbative limit at one-loop.
- It ensures the MR of the two point scalar propagator, in other words it guarantees the LKFT property of the scalar propagator.

Conclusions

- In massless quenched SQED, **we have constructed a Non-Perturbative Three-Point Vertex** satisfying the WGTI. This vertex involves a function *W(x)* whose integral restriction guarantees the MR of the scalar propagator to all orders in perturbation theory.

$$\boxed{\int_0^1 dx\, W(x) = 0}$$

  - The trivial choice W(x)=0 leads to a vertex that is in agreement with one loop perturbation theory in Feynman gauge.
  - We propose an *ansatz* consistent with one-loop perturbation theory in arbitrary covariant gauge.

Conclusions

- It reduces to its one loop perturbation theory Feynman expansión in the limit of small coupling and asymptotic values of momenta $k^2 \gg p^2$.
- It has the same symmetry properties as the bare vertex under charge conjugation, parity and time reversal, which imply symmetry $k \leftrightarrow p$.
- It is free of any kinematic singularities when $k^2 \to p^2$, i.e.,

$$\boxed{\lim_{k^2 \Rightarrow p^2} (k^2 - p^2)\tau(k^2, p^2) = 0}$$

Thank you for your attention …

Colaborators:

M. C. Luis Albino Fernández Rangel

Dr. Adnan Bashir

Dra. Laura Xiomara Gutiérrez Guerrero