Spin one matter fields

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Universidad de Guanajuato
Mexican Workshop on Particles and Fields

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Outline of the Talk

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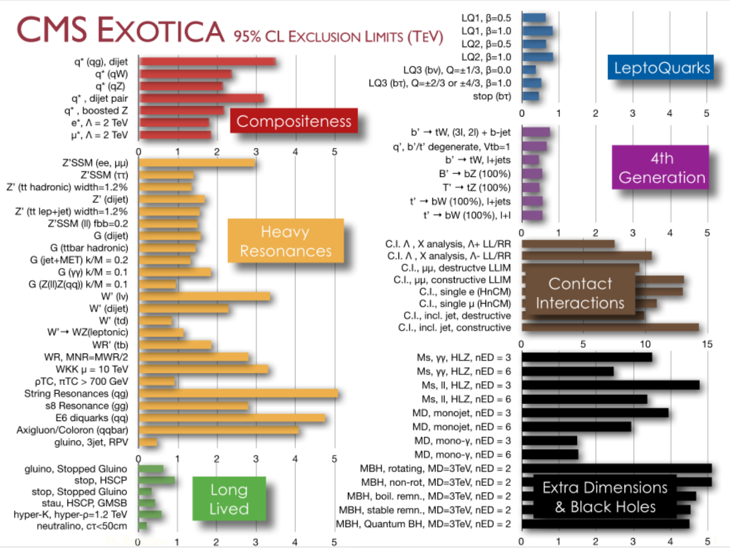
3. Spin 1 algebra and equations of motion in momentum space.
4. Dynamics and constrictions
5. Quantum Field Theory
6. Conclusions and remarks

We may need to look in other direction to extend the SM



Fields transform under th HLG

$$(0,0)$$

$$\left(\tfrac{1}{2},0\right) \qquad \left(0,\tfrac{1}{2}\right)$$

$$(1,0) \qquad \left(\tfrac{1}{2},\tfrac{1}{2}\right) \qquad (0,1)$$

$$\left(\tfrac{3}{2},0\right) \qquad \left(1,\tfrac{1}{2}\right) \qquad \left(\tfrac{1}{2},1\right) \qquad \left(0,\tfrac{3}{2}\right)$$

$$(2,0) \qquad \left(\tfrac{3}{2},\tfrac{1}{2}\right) \qquad (1,1) \qquad \left(\tfrac{1}{2},\tfrac{3}{2}\right) \qquad (0,2)$$

Fields transform under th HLG

- They can be used in effective theories of compound systems ($R_\chi PT$, hadron physics).
- They can give alternative routes to study dark matter.
- Possible extensions to the standard model.

Fields transform under th HLG

The Poincaré algebra has two algebraic invariants

$$C_2 = P_\mu P^\mu \quad C_4 = W_\mu W^\mu \quad with \quad W_\mu = \frac{1}{2}\varepsilon_{\mu\sigma\tau\rho} M^{\sigma\tau} P^\rho$$

One particle state satisfy

$$C_2|\Psi\rangle = m^2|\Psi\rangle \quad C_4|\Psi\rangle = -m^2 j(j+1)|\Psi\rangle$$

where we call m the mass and j the spin of Ψ.
The quantum fields, the basic elements of a QFT allow us to calculate expectation values, are built from operators that create or destroy this states

$$\Psi_l = \int d\Gamma \left[e^{ipx}\omega_l(\Gamma)\, a^\dagger(\Gamma) + e^{-ipx}\omega_l^c(\Gamma)\, a(\Gamma) \right]$$

the field coefficients ω, transform in the representations of the Lorentz group.

HLG and parity

The HLG is an homomorphism of $SU(2) \otimes SU(2)$. Thus the representations can be labeled by two angular momenta (j_A, j_B).
But, under parity $(j_A, j_B) \to (j_B, j_A)$.
To have a state with well defined parity we must extend our space to

$$(j_A, j_B) \oplus (j_B, j_A) .$$

Then to describe high spin matter fields we choose $j_A = j$ and $j_B = 0$.

Covariant basis

It was proven by S. Gomez and M. Napsuciale[1] that the parity based covariant basis for a general$(j,0)\oplus(0,j)$contains:

- Two Lorentz scalars.
- Six operators transforming in $(1,0)\oplus(0,1)$ forming a second rank antysimmetric tensor.
- A pair of symmetric traceless matrices $S^{\mu_1\mu_2\cdots\mu_j}$
- A series of matrix tensor operators, wich transform in the representation $(2,0)\oplus(0,2)\,,(3,0)\oplus(0,3)\,,\ldots(2j,0)\oplus(0,2j)$.

[1] 10.1103/PhysRevD.88.096012

The $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$equation of motion in momentum space: Dirac Equation

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As an example let us take $j=\frac{1}{2}$. Now take the projection over parity eigenstates

$$\Pi u(\mathbf{0}) = \pm u(\mathbf{o})$$

Now we can apply a boost to this equation to obtain

$$\left(B(\mathbf{p})\,\Pi B^{-1}(\mathbf{p}) \pm 1\right) u(\mathbf{p}) = 0$$

it turns out to be that

$$B(\mathbf{p})\,\Pi B^{-1}(\mathbf{p}) = \frac{\gamma^{\mu} p_{\mu}}{m}$$

then we recover the Dirac equation

$$\left(\gamma^{\mu} p_{\mu} \pm m\right) u(\mathbf{p}) = 0$$

in principle we can apply the same procedure for different spins.

The $(1,0)\oplus(0,1)$equation of motion in momentum space

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We can get the equation of motion by boosting the rest-frame parity-projection, but now the fields transform in the representation $(1,0)\oplus(0,1)$of the LG. This will give us

$$\left(\frac{S^{\mu\nu}p_{\mu}p_{\nu}}{m^2}\pm\mathbb{I}\right)\psi\left(\mathbf{p}\right)=0\rightarrow\Lambda^{\pm}\psi\left(\mathbf{p}\right)=\psi\left(\mathbf{p}\right),$$

where $S^{\mu\nu}$ is a traceless tensor of rank 2 and

$$\Lambda^{\pm}\equiv\pm\frac{1}{2}\left(\frac{S^{\mu\nu}p_{\mu}p_{\nu}}{m^2}\pm\mathbb{I}\right),$$

is the projector.

We have to be careful when the system is out of shell.

.
To get the projector out of the mass shell we replace m^2 by p^2

$$\frac{1}{2}\left(\frac{S^{\mu\nu}p_\mu p_\nu}{p^2} \mp \mathbb{I}\right)\psi(\mathbf{p}) = \mp\psi(\mathbf{p}),$$

and to have a local theory we project over the Poincaré orbit $p^2 = m^2$so we obtain:

$$\frac{1}{2}\left(S^{\mu\nu}p_\mu p_\nu \mp \eta^{\mu\nu}p_\mu p_\nu\right)\psi(\mathbf{p}) = \mp m^2\psi(\mathbf{p}),$$

and if we define a new operator $\Sigma^{\mu\nu} \equiv \frac{1}{2}\left(S^{\mu\nu} \mp \eta^{\mu\nu}\right)$ we obtain:

$$\left(\Sigma^{\mu\nu}p_\mu p_\nu \pm m^2\right)u(\mathbf{p}) = 0.$$

The S tensor have some interesting properties

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$S^{\mu\nu}$ fulfills some Jordan algebra, which is analogous to the Clifford algebra of the γ^{μ}in Dirac theory:

$$\left\{S^{\mu\nu}, S^{\alpha\beta}\right\} = \frac{4}{3}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\nu\alpha}\eta^{\mu\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)$$

$$-\frac{1}{6}\left(C^{\mu\alpha\nu\beta} + C^{\mu\beta\nu\alpha}\right),$$

the tensor $C^{\mu\alpha\nu\beta}$ satisfies $C^{\mu\alpha\nu\beta} = -C^{\alpha\mu\nu\beta} = C^{\alpha\mu\beta\nu}$, $C^{\mu\alpha\nu\beta} = C^{\nu\beta\mu\alpha}$ and the Bianchi identity.

The commutator is, on the other hand:

$$\left[S^{\mu\nu}, S^{\alpha\beta}\right] = -i\left(\eta^{\mu\alpha}M^{\nu\beta} + \eta^{\nu\alpha}M^{\mu\beta} + \eta^{\nu\beta}M^{\mu\alpha} + \eta^{\mu\beta}M^{\nu\alpha}\right).$$

It is clear from here that $S^{2}\left(\mathbf{p}\right) \equiv S^{\mu\nu}S^{\alpha\beta}p_{\mu}p_{\nu}p_{\alpha}p_{\beta} = p^{4}$, analogous to $\gamma^{\mu}\gamma^{\nu}p_{\mu}p_{\nu} = p^{2}$ for Dirac.

We choose the parity basis for S

.

To study the dynamics of our equations we need to write the $S^{\mu\nu}$ in some specific basis, for simplicity and clarity we choose the parity basis:

$$S^{00} = \Pi = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad S^{0i} = \begin{pmatrix} 0 & -J^i \\ J^i & -I \end{pmatrix},$$

$$S^{ij} = \begin{pmatrix} \eta^{ij} + \{J^i, J^j\} & 0 \\ 0 & -\eta^{ij} - \{J^i, J^j\} \end{pmatrix},$$

where $J^i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ are the conventional spin one matrices.

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As we have seen previously the momentum space equation of motion is

$$\left(\Sigma^{\mu\nu} p_\mu p_\nu - m^2\right) u\left(\mathbf{p}\right) = 0,$$

in configuration space this would read

$$\left(\Sigma^{\mu\nu} \partial_\mu \partial_\nu + m^2\right) \Psi\left(x\right) = 0,$$

from here It turns out that we can get this equation of motion from

$$\mathcal{L} = \partial_\mu \bar{\Psi} \Sigma^{\mu\nu} \partial_\nu \Psi - m^2 \bar{\Psi}\Psi.$$

Using the explicit representation

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To use the representation of the S it is convenient to write Ψ as

$$\Psi = \begin{pmatrix} \phi \\ \xi \end{pmatrix}, \quad \varsigma = \begin{pmatrix} \pi & , \tau \end{pmatrix}$$

where the canonical momentum are

$$\pi_a = \frac{\delta \mathcal{L}}{\delta \left(\partial_0 \phi_a \right)} = \partial \phi_a^\dagger - \frac{1}{2} \left(\partial_i \xi^\dagger J^i \right)_a ,$$

and

$$\tau_a = \frac{\delta \mathcal{L}}{\delta \left(\partial_0 \xi_a \right)} = -\frac{1}{2} \left(\partial_i \xi^\dagger J^i \right)_a ,$$

form here it is clear that we have the restrictions:

$$\rho_a = \tau_a + \frac{1}{2} \left(\partial_i \xi^\dagger J^i \right)_a \quad \rho_a^\dagger = \tau_a^\dagger + \frac{1}{2} \left(J^i \partial_i \xi \right)_a .$$

The new Hamiltonian with constraints

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Now, following Dirac, the time evolution of the system is given by H^* defined as

$$H^* = \int d^3x \mathcal{H} + \lambda_a \rho_a + \lambda_a^\dagger \rho_a^\dagger .$$

The Hamilton equations that are modified with this change of Hamiltonian are:

$$\partial_0 \xi_a = \frac{\delta H^*}{\delta \tau_a} = \lambda_a$$

$$\partial_0 \tau_a = -\frac{\delta H^*}{\delta \xi_a} = \frac{1}{2}\partial_i \left(\pi J^i\right)_a - \frac{3}{4}\left(\partial_i \partial_j \xi^\dagger J^i J^j\right) + m^2 \xi_a^\dagger ,$$

Secondary constraints

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In our particular case we define the Possion brackets as

$$\{A(x), B(y)\} = \int d^3\mathbf{x}' \left[\frac{\delta A(\mathbf{x})}{\delta \Psi_a} \frac{\delta B(\mathbf{y})}{\delta \varsigma_a} - \frac{\delta A(\mathbf{y})}{\delta \varsigma_a} \frac{\delta B(\mathbf{x})}{\delta \Psi_a} \right],$$

It is very easy to prove that

$$\{\phi_a(\mathbf{x}), \pi_b(\mathbf{y})\} = \delta_{ab}\delta^3(\mathbf{x} - \mathbf{y}) \quad \{\xi_a(\mathbf{x}), \tau_b(\mathbf{y})\} = \delta_{ab}\delta^3(\mathbf{x} - \mathbf{y})$$

The constraints that we obtained before must be satisfied at any time this implies that

$$\partial_o \rho_a^{(\dagger)} = \left\{ \rho_a^{(\dagger)}, H^* \right\} = 0$$

This will produce secondary constraints in our theory

$$\chi_a^{(\dagger)} = \partial_i (\pi J_i)_a^{(\dagger)} - \frac{1}{2} \left(\partial_i \partial_j \xi^\dagger J^i J^j \right)_a^{(\dagger)} + m^2 \xi_a^{(\dagger)} = 0.$$

there are not any other secondary constraints.

Poisson Brackets

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Accordingly to Dirac we must calculate the matrix of Poisson brackets

$$\Delta_{ab}(\mathbf{x},\mathbf{y}) = \{f_a(\mathbf{x}), f_b(\mathbf{y})\} = m^2\delta^3(\mathbf{x}-\mathbf{y})\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which has an inverse (this implies that the constraints are second class):

$$\Delta^{-1}_{ab}(\mathbf{y},\mathbf{z}) == \frac{1}{m^2}\delta^3(\mathbf{y}-\mathbf{z})\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

To go from classical to quantum mechanics we perform the transformation $\{A,B\}_D \to i\hbar[A,B]$ where

$$\{A,B\}_D = \{A,B\} - \int d^3\mathbf{z}d^3\mathbf{y}\,\{A, f_a(\mathbf{z})\}\,\Delta^{-1}_{ab}(\mathbf{z},\mathbf{y})\,\{f_b(\mathbf{y}), A\}.$$

Commutation relations for the theory

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The Possion brackets for the fields and their respective momentum are

$$\{\phi_a(\mathbf{x}),\pi_b(\mathbf{y})\}_D=\left[1-\frac{(\mathbf{J}\cdot\nabla)^2}{2m^2}\right]_{ab}\delta^3(\mathbf{x}-\mathbf{y}),$$

$$\{\xi_a(\mathbf{x}),\tau_b(\mathbf{y})\}_D=\frac{(\mathbf{J}\cdot\nabla)^2_{ab}}{2m^2}\delta^3(\mathbf{x}-\mathbf{y})$$

$$\{\xi_a(\mathbf{x}),\pi_b(\mathbf{y})\}_D=\{\phi_a(\mathbf{x}),\tau_b(\mathbf{y})\}_D=0$$

and in a spinorial language If we calculate

$$\{\Psi_a(\mathbf{x}),\varsigma_b(\mathbf{y})\}_D=\left[\Sigma^{00}-\frac{(\mathbf{J}\cdot\nabla)^2}{2m^2}S^{00}\right]_{ab}\delta^3(\mathbf{x}-\mathbf{y}),$$

then to go to the quantum theory we only must include a i. and equate this to the commutator.

Fourier Expansion

The first step of canonical quantization is to expand the fields as a Fourier series:

$$\Psi(x) = \sum_{\mathbf{p},r} \alpha(\mathbf{p}) \left[c_r(\mathbf{p}) u_r(\mathbf{p}) e^{-ipx} + d_r^{+}(\mathbf{p}) u_r^{c}(\mathbf{p}) e^{ipx} \right]$$

know we calculate all the physical quantities by imposing the usual commutation relations to the coefficients

$$\left[c_r(\mathbf{p}), c_s^{\dagger}(\mathbf{p}) \right] = \delta_{rs}\delta_{\mathbf{pp}} \quad \left[d_r(\mathbf{p}), d_s^{\dagger}(\mathbf{p}) \right] = \delta_{rs}\delta_{\mathbf{pp}}$$

now we can calculate the conjugated momenta which turns out to be

$$\bar{\varsigma}_d = \frac{\partial \mathcal{L}}{\partial \bar{\Psi}_{d,0}} = \Sigma^{0\mu}_{da} \left(\partial_\mu \Psi \right)_a$$

$$\varsigma_d = \frac{\partial \mathcal{L}}{\partial \Psi_{d,0}} = \left(\partial_\mu \bar{\Psi} \right)_a \Sigma^{0\mu}_{ad}$$

Commutation relations

Using the on shell projector we get the following result for the equal time commutation relations

$$\left[\varsigma_d\left(\mathbf{x}_1\right),\Psi_b\left(\mathbf{x}_2\right)\right]_{x_{12}^0=0}=-i\sum_{\mathbf{p}}\frac{p_\mu}{2Vp_0}\Lambda\left(\mathbf{p}\right)_{ba}\Sigma_{ad}^{\mu 0}\left(e^{ip_i\left(x_1^i-x_2^i\right)}-e^{ip_i\left(x_1^i-x_2^i\right)}\right)$$

now changing $\mathbf{p}\to-\mathbf{p}$ in the second term and using the algebra of the S tensor we get

$$\left[\varsigma_d\left(\mathbf{x}_1\right),\Psi_b\left(\mathbf{x}_2\right)\right]_{x_{12}^0=0}=-i\sum_{\mathbf{p}}\frac{e^{ip_i\left(x_1^i-x_2^i\right)}}{V}\left(\Sigma^{00}+\frac{\left(S^{ij}+g^{ij}S^{00}\right)}{4m^2}p_ip_j\right)$$

making use again of the algebra we get finally

$$\left[\varsigma_d\left(\mathbf{x}_1\right),\Psi_b\left(\mathbf{x}_2\right)\right]_{x_{12}^0=0}=-i\sum_{\mathbf{p}}\left(\Sigma^{00}+\frac{\left(\mathbf{J}\cdot\mathbf{p}\right)^2S^{00}}{2m^2}\right)\frac{e^{ip_i\left(x_1^i-x_2^i\right)}}{V}$$

$$\left[\varsigma_d\left(\mathbf{x}_1\right),\Psi_b\left(\mathbf{x}_2\right)\right]_{x_{12}^0=0}=-i\left(\Sigma^{00}-\frac{\left(\mathbf{J}\cdot\nabla\right)^2S^{00}}{2m^2}\right)\delta^3\left(\mathbf{x}_1-\mathbf{x}_2\right)$$

Energy-momentum and charge of the field

The energy momentum tensor and current are obtained as usual

$$T^{\mu}_{\nu} = \partial_{\nu}\bar{\Psi}\Sigma^{\mu\alpha}\partial_{\alpha}\Psi + \partial_{\alpha}\bar{\Psi}\Sigma^{\alpha\mu}\partial_{\nu}\Psi - \eta^{\mu}_{\nu}\left(\partial_{\alpha}\bar{\Psi}\Sigma^{\alpha\nu}\partial_{\alpha}\Psi - m^{2}\bar{\Psi}\Psi\right)$$

$$J^{\alpha} = iq\left((\partial_{\mu}\bar{\Psi})S^{\mu\alpha}\Psi - \bar{\Psi}S^{\alpha\nu}(\partial_{\nu}\Psi)\right)$$

By substituting the Fourier expansion in this expression we have proved that

$$P_{\mu} = \sum_{\mathbf{p},r}[c^{+}_{r}(\mathbf{p})c_{r}(\mathbf{p}) + d^{+}_{r}(\mathbf{p})d_{r}(\mathbf{p})]p_{\mu}$$

$$Q = q\sum_{\mathbf{p},r}\left(d^{+}_{r}(\mathbf{p})d_{r}(\mathbf{p}) - c^{+}_{r}(\mathbf{p})c_{r}(\mathbf{p})\right)$$

which is the expected result form a well behaved theory. It is important to remark that some factors are only reduced using *the algebra of the S tensor*.

2-point Green Function

The two point Green Function $i\Gamma_F(x-y)_{ab}$ is the time ordered vacuum expectation value of the fields at different spacetime points.

$$\Gamma_F(x-y)_{ab} \equiv \langle 0| T\left\{\phi_a(x)\bar{\phi}_b(y)\right\}|0\rangle$$

for our fields we have we have

$$i\Gamma_F(x-y)_{ab} = \begin{cases} \sum_{\mathbf{p}} \frac{1}{2V\omega_{\mathbf{p}}}\Lambda(\mathbf{p})_{ab}\, e^{-ip_i\left(x_1^i - x_2^i\right)} & x_0 > y_0 \\ \sum_{\mathbf{p}} \frac{1}{2V\omega_{\mathbf{p}}}\Lambda(\mathbf{p})_{ab}\, e^{ip_i\left(x_1^i - x_2^i\right)} & y_0 > x_0 \end{cases}$$

2-point Green Function

After going to the complex plane we get that the propagator obtained from quantum field theory is :

$$i\Gamma_F(x-y) = \frac{i}{(2\pi)^4}\int \frac{\left(S(k) + m^2 - \left(p^2 - m^2\right)\right) e^{-ik(x-y)} d^4k}{2m^2\left(k^2 - m^2 + i\varepsilon\right)} + \frac{\left(S^{00} - 1\right)\delta^4(x-y)}{2m^2}$$

the last term is a contact term. Accordingly to Weinberg the correct Feynman rules are obtained by eliminating this term. Then

$$i\Gamma_F(x-y) = \frac{i}{(2\pi)^4}\int \frac{\left(S(k) + m^2 - \left(p^2 - m^2\right)\right) e^{-ik(x-y)} d^4k}{2m^2\left(k^2 - m^2 + i\varepsilon\right)}$$

Conclusions and remarks

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1. The dirac formalism can be interpreted as a projection on to parity eigenstates in $(j, 0) \oplus (0, j)$
2. We have generalized this to $j = 1$ based on the parity based covariant basis construction.
3. The formalism yields a constraint dynamics, all constraints being second class.
4. We performed the canonical quantization following Dirac´s guidelines.
5. The algebra of the S tensor is fundamental for the calculations in QFT.
6. The commutator of the fields gives a non conventional result that comes from the constraints of the theory.
7. The propagator involves not only the on shell polarization sum, but also involves terms proportional to $p^2 - m^2$.
8. Possible extensions and applications ongoing...