1. THE EFFECTIVE ACTION AND THE EFFECTIVE POTENTIAL

1. The effective action and the effective potential

In Quantum Field Theory (QFT), the effective action and potential play a central role in the study of symmetries (and their breaking) at the quantum level. (In fact "quantum effective action/potential" would be better names.)

The effective action, Γ, is closely related to the Ward identities that encapsulate the symmetries obeyed by the theory Green's functions and also plays an important role in proving the renormalizability of non-Abelian QFTs.

The effective potential, V, very closely related to the effective action, Γ, is the quantity that determines the vacuum of the theory as the lowest energy state and is crucial in the study of spontaneous symmetry breaking, the most famous example being the Higgs breaking of the SM electroweak gauge symmetry.

These objects, Γ and V, are naturally formulated in the most transparent and elegant way in the path integral language.

I will remind you a few basic QFT results required to state/understand what Γ and V are and how they can be calculated in perturbation theory.

1.1 Review of basic QFT results needed

For illustration of the main points I will use first a simple QFT example with a scalar field ϕ.

The starting point will be the generating functional for the Green functions of ϕ:

$$Z[J] = N \int \mathcal{D}\phi \; e^{iS[\phi] + i\int d^4x\, J(x)\phi(x)}$$

where

$\frac{1}{2}(\partial\phi)^2 - V(\phi)$

* $S[\phi]$ is the classical action functional, $S[\phi] = \int d^4x \, \mathcal{L}(\phi)$.
* $J(x)$ is an external source we use to probe the theory. E.g. it can be used to create/annihilate ϕ-particles and prepare initial/final states in an scattering experiment. It is linearly coupled to ϕ so that $Z[J]$ generates the correlation functions of ϕ:

Time-ordered Green function from path-integral definition

$$\int_{B.C.} \mathcal{D}\phi \, e^{iS} \phi(x_1)\dots\phi(x_n) = \langle \phi(\vec{x}', t')| \, T[\phi_H(x_1)\dots\phi_H(x_n)] \, |\phi(\vec{x}, t)\rangle$$

fields in Heisenberg picture

Boundary conditions (initial state $|\phi(\vec{x},t)\rangle$, final state $\langle\phi(\vec{x}',t')|$) we will choose in particular vacuum-to-vacuum transitions. Expanding in powers of J

$$Z[J] = \langle 0, \text{out} | 0, \text{in}\rangle_J$$

Physical vacuum

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n \, J(x_1)\dots J(x_n) \underbrace{\langle 0| T[\phi(x_1)\dots\phi(x_n)]|0\rangle}_{G^{(n)}(x_1,\dots,x_n)}$$

Or, equivalently, the Green functions can be obtained from $Z[J]$ by functional derivatives wrt the source $J(x)$.

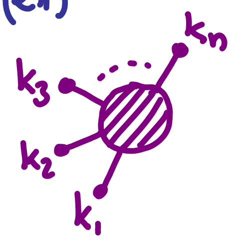
* The normalization constant is $N^{-1} = \int \mathcal{D}\phi \, e^{iS[\phi]} \Rightarrow Z[0] = 1$

Diagrammatic Expansion of $Z[J]$:

$Z[J]$ contains all the physics information of the theory, through the Green functions $G^{(n)}(x_1,\dots x_n)$, which describe the non-linear response of the system to the external source $J(x)$. In particular, $Z[J]$ contains the full S-matrix, but it's more than that: it generates off-shell Green functions:

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n \, J(x_1)\dots J(x_n) \, G^{(n)}(x_1,\dots,x_n)$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \dots \frac{d^4k_n}{(2\pi)^4} \tilde{J}(-k_1)\dots\tilde{J}(-k_n) \underbrace{\tilde{G}^{(n)}(k_1,\dots k_n)}_{\text{sum of all Feynman diags (with n external source insertions)}}$$



(Feynman rule from $i\int d^4x \, J(x)\phi(x) \Rightarrow$  $iJ(x)$)

Note that the k_i are only constrained by momentum conservation $\delta^4(k_1+k_2+\dots+k_n)$ in $\tilde{G}^{(n)}$, but are otherwise arbitrary. This off-shell formulation is quite useful to deal with renormalization or unitarity in relativistic QFT.

So we have that

$$Z[J] = \sum(\text{vacuum diagrams in presence of } J)$$

and then $= \exp\left[\sum(\text{connected v. diagrams})\right]$ as usual,

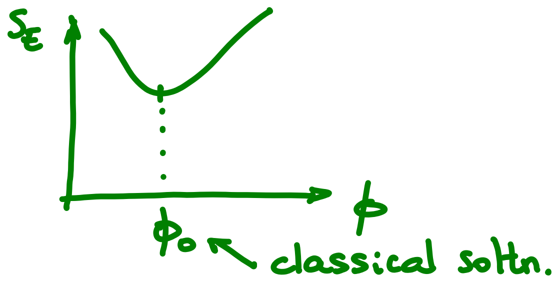
with the exponential taking into account all combinatoric factors for those diagrams composed of disconnected pieces. (Also, the normalization factor N in $Z[J]$, removes automatically all pure vacuum diagrams without J insertions).

Semiclassical Expansion

The path integral approach sheds light on the nature of the classical limit $\hbar \to 0$, explaining how for $S \gg \hbar$ quantum amplitudes are dominated by the classical solution, which corresponds to a stationary point of the classical action and leads to phase alignment.

$$\int \mathcal{D}\phi \; e^{-S_E[\phi]/\hbar}$$

Euclidean action



The semiclassical expansion is a systematic expansion in powers of $\hbar$ around that classical ($\hbar = 0$) limit.

Expanding $S_E[\phi]$ around the minimum ϕ_0:

$$S_E[\phi] = S_E[\phi_0] + \frac{1}{2}\int d^4x_1 d^4x_2 \left.\frac{\delta^2 S_E}{\delta\phi(x_1)\,\delta\phi(x_2)}\right|_{\phi_0} \delta\phi(x_1)\,\delta\phi(x_2) + \ldots$$

where $\delta\phi = \phi(x) - \phi_0(x)$, and plugging this

$$\int \mathcal{D}\phi \; e^{-S_E[\phi]/\hbar} =$$

classical term

$$= \int \mathcal{D}\phi \; e^{-S_E[\phi_0]/\hbar} \exp\left[-\frac{1}{2\hbar}\int d^4x_1 d^4x_2 \, \delta\phi(x_1) \left.\frac{\delta^2 S_E}{\delta\phi(x_1)\,\delta\phi(x_2)}\right|_{\phi_0} \delta\phi(x_2)\right]$$

$$\times \left[1 - \frac{1}{3!\,\hbar}\int \left.\frac{\delta^3 S_E}{\delta\phi\,\delta\phi\,\delta\phi}\right|_{\phi_0} \delta\phi\,\delta\phi\,\delta\phi + \ldots\right]$$

exp of quadratic term

expansion of higher order terms

so that one has integrals of polynomials × Gaussian.

Using the well known result

$$\int dx_1 \dots dx_N \, e^{-\frac{1}{2}(\vec{x}, A\vec{x})} = \prod_{n=1}^{N} \int dx_n \, e^{-\frac{1}{2}\lambda_n x_n^2} =$$

$$= \prod_{n=1}^{N} \left(\frac{2\pi}{\lambda_n}\right)^{1/2} = (2\pi)^{N/2} (\mathrm{Det}\, A)^{-1/2}$$

we get

$$\int \mathcal{D}\phi \, e^{-S_E[\phi]/\hbar} \propto \underbrace{\left[\mathrm{Det}\left(\frac{1}{\hbar} \frac{\delta^2 S_E}{\delta\phi\delta\phi}\Big|_{\phi_0}\right)\right]^{-1/2}}_{\text{determined by the "harmonic" fluctuations around the classical trajectory } \phi_0} e^{-S_E[\phi_0]/\hbar} \, [1 + O(\hbar)]$$

determined by the "anharmonic" (interaction) fluctuations

[• To see this is indeed a series in $\hbar$ rescale $\delta\phi \to \sqrt{\hbar}\,\delta\phi$.

Odd terms in the polynomial in $\delta\phi$ integrate to zero. Even terms (of order $\delta\phi^{2n}$, $n=2,\dots$) lead to

$$\int \mathcal{D}\phi \, e^{-S''\delta\phi^2} \frac{1}{\hbar} (\sqrt{\hbar})^{2n} (\delta\phi)^{2n} \sim \hbar^{n-1} \;]$$

As is well known, this semiclassical expansion in powers of $\hbar$ is the same as the Feynman weak coupling expansion. E.g. for

$$\frac{S}{\hbar} = \frac{1}{\hbar} \int d^4x \left[\frac{1}{2}\phi(-\Box - m^2)\phi - \frac{1}{4!}\lambda\phi^4 + J\phi\right]$$

$$\Big\downarrow \; \phi = \phi'/\sqrt{\lambda}$$

$$\frac{S}{\hbar} = \frac{1}{\lambda\hbar} \int d^4x \left[\frac{1}{2}\phi'(-\Box - m^2)\phi' - \frac{1}{4!}\phi'^4 + \sqrt{\lambda} J\phi\right]$$

For a particular Green function (corresponding to some power of J in $\int \mathcal{D}\phi \, e^{iS[\phi]/\hbar}$) expanding in $\hbar$ or λ is the same: the action depends only on their product.

So the Feynman diagram expansion is an expansion in small quantum fluctuations around the classical trajectory.

This result is general. When we evaluate $\int \mathcal{D}\phi\, e^{iS[\phi]/\hbar}$ diagrammatically, propagators are given by the inverse of the quadratic part of S, and therefore scale like $\hbar$. Vertices are proportional to S, and scale like $\hbar^{-1}$. So, a diagram with P propagators and V vertices scales like $\hbar^{P-V}$. Connected graphs with L loops satisfy the identity* $P-V+1=L$, and scales like $\hbar^{L-1}$. Therefore, tree graphs are order $\hbar^{-1}$ and each loop introduces an additional factor of $\hbar$.

Looking back at the diagrammatic expansion of $Z[J]$ we have

$$Z[J] = \exp\left[\Sigma(\text{connected diagrams})\right]$$

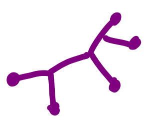
$$= \exp\Big[\underbrace{iS[\phi_0]}_{\text{tree}} + \underbrace{A\hbar}_{\text{1-loop}} + \underbrace{B\hbar^2}_{\text{2-loop}} + \dots\Big]/\hbar$$

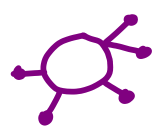
$$= e^{iS[\phi_0]/\hbar}\, e^{A}\, \left[1 + O(\hbar)\right]$$

Sum of all tree-graphs in the presence of source J

Sum of 1-loop graphs

2-loops & higher: anharmonic corrs. to semiclassical exp.





gives the logarithm of the determinant of Gaussian fluctuations around the classical solution

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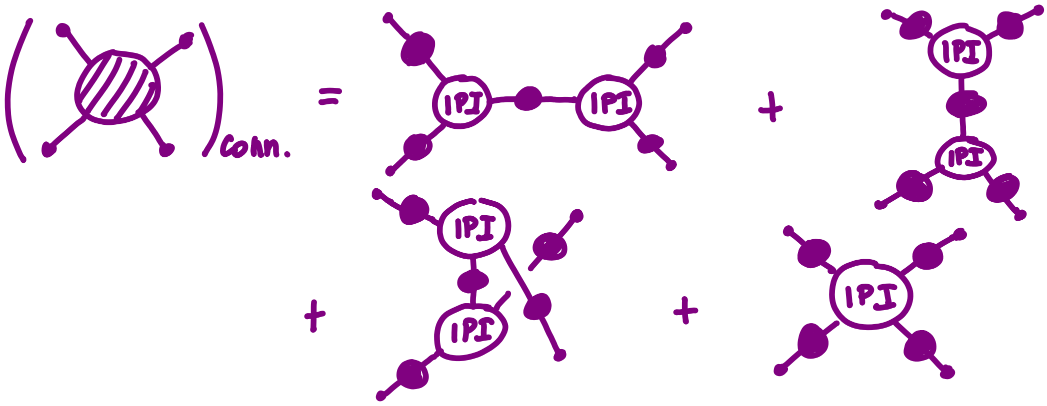
* Clear in momentum integrals: $\underbrace{\Pi \int d^4p_i}_{P}\ \underbrace{\Pi\delta(\Sigma p)}_{-\,V} = \underbrace{\delta(\Sigma_t p)}_{-1}\underbrace{\Pi\int d^4p_j}_{L}$

1.2 Relation between the Effective Action and $Z[J]$

In order to study at the quantum level the symmetries of a QFT it is most convenient to focus on the symmetry constraints (the Ward identities) obeyed by the 1PI (one-particle irreducible) Green functions, which are the building blocks of the theory, in the following sense:

The key observation is that the sum of all connected diagrams can be obtained by constructing all "tree diagrams" with the exact connected 2-point function as propagator and the complete 1PI Green functions as vertices. E.g.



All loop corrections, no matter how complicated, can be reduced to this "tree-level" form in terms of such building blocks: 1PI vertices and complete propagator.

For this reason it would be useful to construct a generating functional of 1PI Green functions, just as $Z[J]$ is the generating functional of the complete Green functions:

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n \, J(x_1) \dots J(x_n) \underbrace{G^{(n)}(x_1, \dots, x_n)}_{\text{complete Green function}}$$

Such 1PI generating functional is the effective action, Γ.

Remembering that $Z[J] = \exp[\Sigma \text{ connected diags}]$ and writing

$$Z[J] = e^{iW[J]} \Rightarrow$$

$$iW[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n \, J(x_1) \ldots J(x_n) \underbrace{G_c^{(n)}(x_1, \ldots, x_n)}_{\text{Connected Green function}}$$

In the same way, we can write

$$\Gamma[\bar{\phi}] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, \bar{\phi}(x_1) \ldots \bar{\phi}(x_n) \underbrace{\Gamma^{(n)}(x_1, \ldots, x_n)}_{\text{Connected 1PI Green function}^*}$$

where, for the time being, $\bar{\phi}$ is just a dummy variable of integration.

We want to find out what's the relation between $\Gamma[\bar{\phi}]$ and $W[J]$.

As the tree-level diagrams built with full propagator and 1PI vertices correspond to an effective field theory with action

$$S_{eff}[\bar{\phi}] = \Gamma[\bar{\phi}]$$

(which is complicated and non-local) and the tree-level approximation corresponds to the classical limit, we have

$$\int \mathcal{D}\bar{\phi} \, e^{(i\Gamma[\bar{\phi}] + i\int J\bar{\phi})/\hbar} = \exp\Big[\underset{\substack{\uparrow \\ \text{Sum of connected tree-diagrams} \\ \text{of theory with action } \Gamma[\bar{\phi}]}}{iW[J]/\hbar} \; (1 + O(\hbar))\Big]$$

In the $\hbar \to 0$ limit this path-integral is dominated by the classical trajectory that extremizes the action $\Gamma[\bar{\phi}] + \int J\bar{\phi}$, so

* For $n=2$, we define $\Gamma^{(2)}$ as the inverse of the exact propagator:

$\tilde{\Gamma}^{(2)}(p) = \dfrac{-1}{\tilde{G}_c^{(2)}(p)}$, which follows (see below) from

$$\frac{\delta^2 \Gamma[\phi]}{\delta\phi(x)\,\delta\phi(y)} = -\frac{\delta J(x)}{\delta\phi(y)}$$

$$\frac{\delta^2 W[J]}{\delta J(x)\,\delta J(y)} = \frac{\delta\phi(x)}{\delta J(y)}$$

$$W[J] = \Gamma[\bar\Phi] + \int d^4x\, J(x)\bar\Phi(x) \Big|_{\text{stationary } \bar\Phi} \qquad (1)$$

↑ found by solving the

"quantum" E.O.M. for $\bar\Phi$: $\dfrac{\delta\Gamma[\bar\Phi]}{\delta\bar\Phi(x)} = -J(x)$

So, $W[J]$ and $\Gamma[\bar\Phi]$ are related by a Legendre transformation. This is sometimes used as the starting point to define Γ, but then the definition looks cryptic and mysterious. Derived in this alternative way the result above follows nicely from the properties of Γ as generating functional of 1PI Green functions. To find $\Gamma[\bar\Phi]$ for an arbitrary $\bar\Phi$ we need to find the J that gives such $\bar\Phi$. The "EOM" for J is found by taking the functional derivative of (1) wrt $J(x)$:

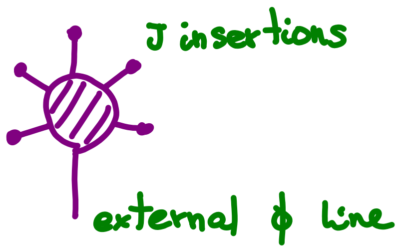
$$\frac{\delta W[J]}{\delta J(x)} = \bar\Phi(x) \qquad (2)$$

and we can invert (1) to obtain

$$\Gamma[\bar\Phi] = W[J] - \int d^4x\, J(x)\bar\Phi(x) \Big|_{\substack{\text{stationary } J \\ \text{as in (2).}}} \qquad (3)$$

as the definition of the effective action $\Gamma[\bar\Phi]$.

Eq. (2) also tells us what $\bar\Phi$ is, as $\delta W/\delta J$ is the connected one-point function $\langle 0|\phi(x)|0\rangle_J$ in the presence of the source.



That is, to compute $\Gamma[\bar\Phi]$, we choose the external source J, coupled to $\phi(x)$, so that the expectation value of $\phi(x)$ in the physical vacuum is the specified function $\bar\Phi(x)$. Then, with that source present, we calculate $\Gamma[\bar\Phi] = W[J] - \int d^4x\, J(x)\,\bar\Phi(x)$.

1.3 Symmetries and Ward Identities

We can then address how the symmetries of the classical action S translate into symmetries of the quantum effective action Γ, or, what is equivalent, into the symmetries of 1PI Green functions. Suppose the classical action is left invariant by an infinitesimal transformation of the fields of the form

$$\delta\phi(x) = \underbrace{A(\phi)}_{\text{linear function of } \phi\text{'s.}}\ \underbrace{\delta\omega(x)}_{\text{infinitesimal}}$$

which also leaves invariant the functional measure $\mathcal{D}\phi$ (it has a trivial Jacobian). Shift $\phi \to \phi + \delta\phi$ in the functional integral defining $Z[J]$

$$\int \mathcal{D}\phi\, \exp\left[iS[\phi] + i\int d^4x\, J(x)\,\phi(x)\right] \overset{\phi\to\phi+\delta\phi}{=}$$

$$= \int \mathcal{D}\phi\, \exp\left[i\, S[\phi+\delta\phi] + i\int J(\phi+\delta\phi)\right]$$

invariant

$$= \int \mathcal{D}\phi\, \exp\left[iS[\phi] + i\int J\phi\right]\left\{1 + i\int \delta\phi\left(\frac{\delta S}{\delta\phi} + J\right) + \dots\right\}$$

so that

$$0 = \int \mathcal{D}\phi\, e^{iS + i\int J\phi} \int d^4x\, \delta\phi(x)\left(\frac{\delta S}{\delta\phi} + J(x)\right)$$

Choosing again $J(x)$ so as to get $\langle\phi\rangle_J = \bar\Phi$, we have $J = -\dfrac{\delta\Gamma}{\delta\bar\Phi}$, and, substituting above, we obtain

$$\langle \delta S \rangle_J = \int d^4x \, \frac{\delta \Gamma}{\delta \bar{\phi}(x)} \langle \delta \phi(x) \rangle_J$$

As $\delta\phi$ is linear in ϕ:

$$\langle \delta\phi(x) \rangle_J = \langle A(\phi)\, \delta\omega(x) \rangle_J = A(\underbrace{\langle \phi \rangle_J}_{=\bar{\phi}})\, \delta\omega(x) = \delta\bar{\phi}$$

$$\Rightarrow \qquad \langle \delta S \rangle_J = \langle \delta \Gamma \rangle_J = 0$$

$$\Rightarrow \qquad \delta S[\bar{\phi}] = \delta \Gamma[\bar{\phi}] = 0$$

so that the effective action is also invariant. This is crucial for renormalizability so that all counterterms needed can be introduced respecting the symmetries of the classical action and all infinities can be absorbed. In this respect, it is also crucial that the regularization method used also respects the symmetries. The Green function identities implied by the symmetries of the action (Ward or Slavnov-Taylor identities) can be elegantly derived using this path-integral approach and expanding in powers of the source $J(x)$.

1.4 The Effective Potential

The non-local effective action can be Taylor-expanded, with nonlocalities giving higher order field derivatives. In our scalar field QFT example we have

$$\Gamma[\bar{\phi}] = \int d^4x \left[\Gamma_0(\bar{\phi}) + \Gamma_2(\bar{\phi})(\partial_\mu\bar{\phi})^2 + \Gamma_4(\bar{\phi})(\partial_\mu\bar{\phi})^4 + \dots \right]$$

or, in a more suggestive notation,

$$\Gamma[\bar{\phi}] = \int d^4x \left[\frac{1}{2} Z(\bar{\phi})(\partial_\mu\bar{\phi})^2 - V_{eff}(\bar{\phi}) + \text{higher der.} \right]$$

and from ouprevious discussion on the semiclassical expansion, we know that

$$Z(\bar{\Phi}) = 1 + O(\hbar) \qquad V_{eff}(\bar{\Phi}) = V_{tree}(\bar{\Phi}) + O(\hbar)$$

and $V_{eff}(\bar{\Phi})$ is the quantum version of the classical potential. In other words, if we specialize to a position-independent field ϕ_0, the (quantum) effective potential is given by:

$$I[\phi_0] = -\underbrace{\int d^4x}_{\substack{\text{4-dim spacetime} \\ \text{volume } V\cdot T}} \underbrace{V_{eff}(\phi_0)}_{\substack{\text{ordinary function} \\ \text{a density}}}$$

The vacuum of the theory, i.e. the vacuum expectation value of ϕ in the absence of external sources ($J \to 0$) is determined by the minimum of V_{eff}. Remember the relations

$$\frac{\delta W[J]}{\delta J(x)} = \bar{\Phi}(x) = \langle 0|\phi(x)|0\rangle_J \quad , \quad \frac{\delta I[\bar{\Phi}]}{\delta \bar{\Phi}(x)} = -J(x)$$

$$J = 0 \quad \Downarrow \quad \bar{\Phi}(x) = \phi_0$$

$$\underbrace{\phi_0 = \langle \phi \rangle}_{} \qquad \& \qquad \underbrace{\frac{dV_{eff}(\phi_0)}{d\phi_0} = 0}_{}$$

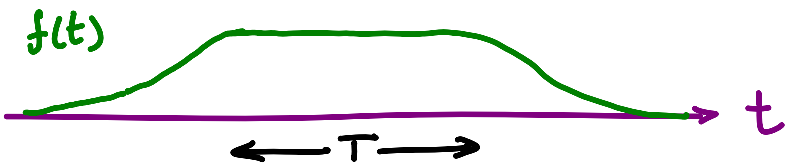
ϕ_0 is sometimes called classical field (ϕ_c)
(in the sense of coherent homogeneous field, but it's the minimum of the quantum potential).

Minimization condition determines value of $\phi_0 = v$

Quantum (or radiative) corrections can play a very important role in modifying/triggering symmetry breaking. We will discuss how to include them shortly.

1.5 Energy Interpretation of V_{eff}

To further understand the physical meaning of the effective potential, consider the following thought experiment using our external source $J(x)$. We will examine how the vacuum responds to a constant homogeneous value of $J(x)$, although we turn $J(x)$ off slowly at $t \to \pm\infty$: $J(x) = f(t)\,\rho(\vec{x})$, with



We also choose $J(x)$ in such a way that it produces an specified value of the time-independent classical field $\phi_c(\vec{x})$:

$$\langle \phi(x) \rangle_J = \phi_c(\vec{x})$$

We end up with the same vacuum state we started with, up to a phase that the system picks up while staying in its ground state during time T, with the source present. That is :

$$\langle 0, out \,|\, 0, in \rangle_J = \exp\left(-i\, E[J]\, T\right)$$

where $E[J]$ is such energy. For sufficiently long T and for slow enough turning on and off of the source, we can neglect any phase introduced in that adiabatic turn on/off. Remembering the definition of $W[J]$, this means

$$W[J] = -T \cdot E[J]$$

The vacuum in the presence of J, $|0\rangle_J$ should therefore be an eigenstate of the Hamiltonian, supplemented by the source term:

$$\left(H - \int d^3\vec{x}\, \rho(\vec{x})\, \phi(\vec{x})\right)|0\rangle_J = E[J]\,|0\rangle_J \qquad (E)$$

and $E[J]$ must be the lowest energy eigenvalue, for the specified $\rho(\vec{x})$. We can show $|0\rangle_J$ is the state that minimizes the energy, given the two constraints ${}_J\langle 0|0\rangle_J = 1$ and ${}_J\langle 0|\phi(x)|0\rangle_J = \phi_c(\vec{x})$. This is a constrained minimization problem and can be stated using Lagrange multipliers.

The quantity to be minimized is

$$\langle\psi|H|\psi\rangle - \underbrace{\lambda_1}_{\text{Lagrange multipliers}}\left(\langle\psi|\psi\rangle - 1\right) - \int d^3\vec{x}\, \lambda_2(\vec{x})\left[\langle\psi|\Phi|\psi\rangle - \phi_c(\vec{x})\right]$$

And the state $|\psi\rangle_{min}$ corresponding to that minimum must satisfy:

$$H|\psi\rangle_{min} - \lambda_1|\psi\rangle_{min} - \int d^3\vec{x}\, \lambda_2(\vec{x})\, \phi(\vec{x})\,|\psi\rangle_{min} = 0$$

But this is precisely eq. (E) above with

$$|\psi\rangle_{min} = |0\rangle_J\,; \qquad \lambda_1 = E[J]\,; \qquad \lambda_2(\vec{x}) = J(x) = \rho(\vec{x})$$

and we have

$${}_J\langle 0|H|0\rangle_J = \underbrace{E[J]}_{-\frac{1}{T}W[J]} + \underbrace{\int d^3\vec{x}}_{\frac{1}{T}\int d^4x}\, \rho(\vec{x})\, \phi_c(\vec{x})$$

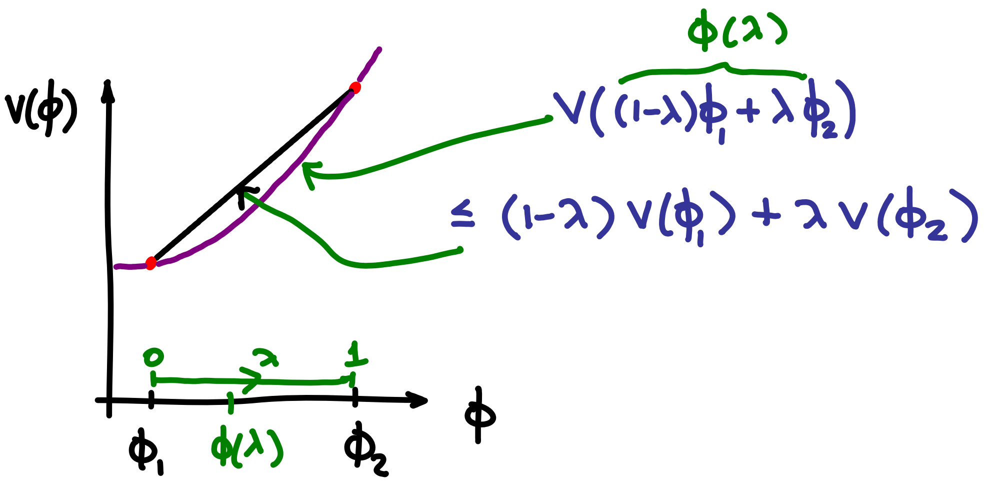
$$= \frac{-1}{T}\left[W[J] - \int d^4x\, J(x)\,\phi_c(x)\right] = -\frac{1}{T}\, \Gamma[\phi_c]$$

or $V(\phi) = {}_J\langle 0|H|0\rangle_J / V_3$.

We conclude that $V(\phi_c)$ is the minimum value of the energy density expectation value for all states that give $\langle \phi \rangle = \phi_c(x)$. In particular, for $J=0$, the vacuum state will correspond to a minimum of the effective potential.

Convexity of the Potential

From our definition $V(\phi) = -\Gamma[\phi]/V_4$ we can relate derivatives of the potential to derivatives of Γ, that is, 1PI Green functions (at zero external momentum). In particular $\partial^2 V/\partial\phi^2$ is directly related to $\tilde{\Gamma}^{(2)}$ which, as we saw in the footnote of page 1.8, is related to the inverse of the full ϕ propagator and, tracking the signs, we conclude that $\partial^2 V/\partial\phi^2$ and $\tilde{G}^{(2)} = \delta^2 W/\delta J \delta J$ have the same sign. It's easy to see, using the Euclidean formulation of the path integral, that $\partial^2 W/\partial J^2$ is positive and so one should have $\partial^2 V/\partial\phi^2 > 0$. In other words, the potential is convex:

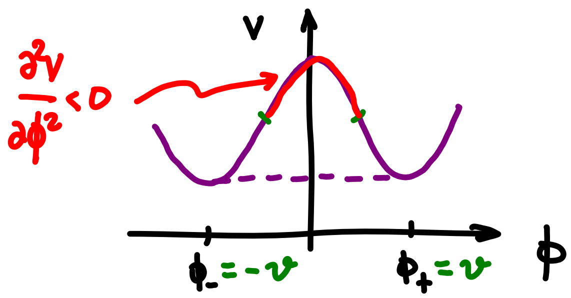


This is clearly at odds with our experience with effective potentials, which e.g. in the SM, often have non-convex shape:

Take for instance the usual Mexican-hat potential

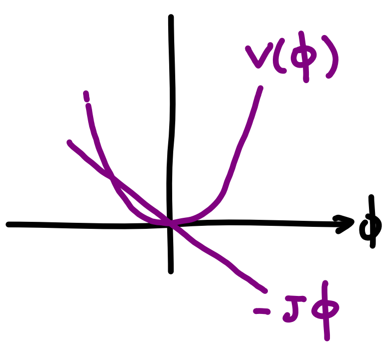
$$V(\phi) = \lambda(\phi^2 - v^2)^2$$

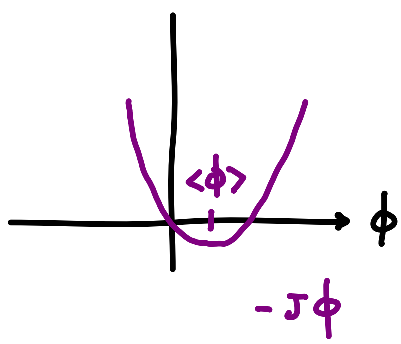
invariant under $\phi \to -\phi$, often use to explain spontaneous sym. breaking:



This contradiction comes from the fact that, for values of ϕ with $\partial^2 V/\partial\phi^2 < 0$, the stationary point of $-\Gamma(\phi) - \int J_\phi \phi$ is a maximum rather than a minimum, corresponding to a field configuration that is unstable. Not a good starting point for a perturbative expansion.

The way J fixes the expectation value of ϕ is trivial: one has to minimize $V(\phi) - J\phi$ and J tilts the potential by the right amount, e.g.:

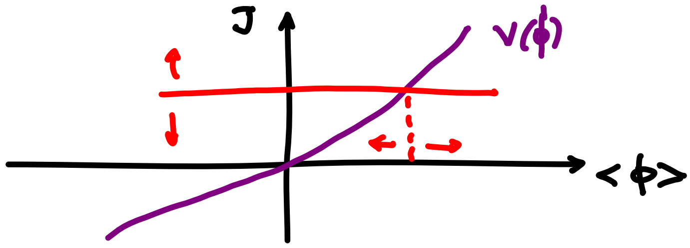




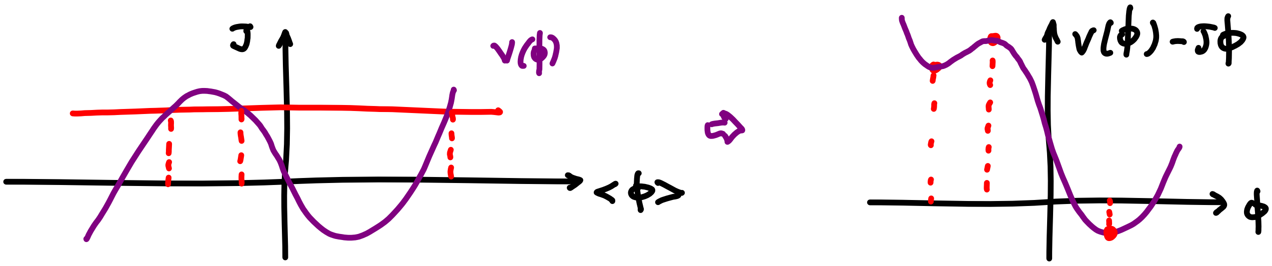
One simply adjusts J to the required $\langle\phi\rangle$, solving the minimization equation:

$$\frac{\partial}{\partial\phi}\left(V(\phi) - J\phi\right) = 0 \quad \Rightarrow \quad \frac{\partial V}{\partial \phi} = J$$

This works smoothly when $V'(\phi)$ is a monotonic function:



But it produces several stationary points whenever $V''(\phi)$ changes sign, as in the Mexican-hat case:



For $\langle\phi\rangle$ in the region with $V''<0$, $V(\phi)-J\phi$ is a maximum at $\langle\phi\rangle$, not a minimum.

The "true effective potential" for $\phi_- < \phi < \phi_+$ is flat, as indicated by the dashed line in the figure above and would correspond to the energy density in a vacuum state

$$|0\rangle \equiv \alpha|0,\phi_+\rangle + \beta|0,\phi_-\rangle = \alpha|+\rangle + \beta|-\rangle$$

with $|\alpha|^2+|\beta|^2=1$ (remember that $\langle+|-\rangle=0$ in QFT)

and

$$\langle 0|\phi|0\rangle = |\alpha|^2\phi_+ + |\beta|^2\phi_- = \phi \in (\phi_-,\phi_+)$$

obviously giving

$$\langle 0|H|0\rangle = |\alpha|^2 V(\phi_+) + |\beta|^2 V(\phi_-) = V(v)$$

The resulting non-analytic "true potential" is convex, as expected, but does not give some physically useful information that the "naive" potential offers, e.g. regarding the instability of some field configurations (as we'll discuss later on).

Moreover, remember the key difference between symmetry breaking in QFT and quantum mechanics, e.g. with a double well potential as in the picture above. In quantum mechanics the two ground states are superpositions of the vacuum vectors $|+\rangle$ and $|-\rangle$, either symmetric or antisymmetric:

$$|S\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$|A\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

While $|+\rangle$ or $|-\rangle$ break the symmetry $\phi \to -\phi$ of the potential (under that symmetry $|+\rangle \leftrightarrow |-\rangle$) both $|S\rangle$ and $|A\rangle$ are left invariant (up to an unimportant sign), so that there is no spontaneous symmetry breaking in QM.

The infinite number of degrees of freedom in QFT makes the story different, as then $\langle -|+\rangle = 0$ and then $|+\rangle, |-\rangle$ are the true ground states, that do break the symmetry. One can still use $|S\rangle$ and $|A\rangle$ as vacuum states upon which to build up the Hilbert space but this is not a good idea. The basis $|+\rangle$, $|-\rangle$ is far more convenient as local operators will have diagonal matrix elements in this basis, with $\langle +|\mathcal{O}|-\rangle = 0$, while this does not happen in the $|S\rangle, |A\rangle$ basis, e.g.

$$\langle S|\phi(x)|A\rangle = \frac{1}{2}\langle +|\phi(x)|+\rangle - \frac{1}{2}\langle -|\phi(x)|-\rangle = \frac{1}{2}\phi_+ - \frac{1}{2}\phi_- = \phi_+$$

To study local physics in the vacuum state $|+\rangle$ you never need to deal with $|-\rangle$ and vice versa so that one can construct two orthogonal Hilbert spaces, that don't communicate with each other. Moreover, in this basis one has the cluster decomposition principle:

$$\langle +|\phi(x)\phi(y)|+\rangle \xrightarrow[|x-y|\to\infty]{} \langle +|\phi(x)|+\rangle\langle +|\phi(y)|+\rangle$$

necessary for locality. But in $|S\rangle$ we would have

$$\langle S|\phi(x)\phi(y)|S\rangle = \frac{1}{2}\langle +|\phi(x)\phi(y)|+\rangle + \frac{1}{2}\langle -|\phi(x)\phi(y)|-\rangle$$

$$\xrightarrow[|x-y|\to\infty]{} \frac{1}{2}\langle +|\phi(x)|+\rangle\langle +|\phi(y)|+\rangle + \frac{1}{2}\langle -|\phi(x)|-\rangle\langle -|\phi(y)|-\rangle$$

$$= \frac{1}{2}\phi_+^2 + \frac{1}{2}\phi_-^2 = v^2$$

while, on the other hand

$$\langle S|\phi(x)|S\rangle = \frac{1}{2}\langle +|\phi(x)|+\rangle + \frac{1}{2}\langle -|\phi(x)|-\rangle = \frac{1}{2}\phi_+ + \frac{1}{2}\phi_- = 0$$

so that

$$\langle S|\phi(x)\phi(y)|S\rangle \not\xrightarrow[|x-y|\to\infty]{} \langle S|\phi(x)|S\rangle\langle S|\phi(y)|S\rangle$$

Therefore we do not care about the non-analytic true potential discussed above, although we have learned that the perturbative calculation of the potential in regions of field space with $\partial^2 V/\partial\phi^2 < 0$ corresponds to unstable field configurations, about which we will say more later on. Now we turn to such perturbative calculations of V_{eff}.

1.5 Calculation of V_{eff}

In order to calculate the quantum corrections to the classical potential of ϕ we will proceed as we did for the semi-classical expansion of $Z[J]$, considering field fluctuations over the classical background:

$$\phi(x) = \underset{\substack{\uparrow \\ \text{constant value,} \\ \text{fixed by } J}}{\phi_0} + \underset{\nwarrow \text{ fluctuation}}{\varphi(x)}$$

It is also convenient to write the effective action in terms of momentum space 1PI Green functions

one $\delta^4(\Sigma p_i)$ (connected)

$$\tilde{\Gamma}^{(n)}(p_1, \dots, p_n) \cdot (2\pi)^4 \delta^4(p_1 + p_2 + \dots + p_n) =$$

$$= \int d^4x_1 \dots d^4x_n \; e^{i(p_1 x_1 + \dots + p_n x_n)} \Gamma^{(n)}(x_1, \dots, x_n)$$

so that the Taylor expansion of $\Gamma[\phi]$ reads:

$$\Gamma = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^4p_1}{(2\pi)^4} \dots \frac{d^4p_n}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + \dots + p_n) \times$$

$$\times \tilde{\phi}(p_1) \dots \tilde{\phi}(p_n) \tilde{\Gamma}^{(n)}(p_1, \dots, p_n)$$

Now, for a constant background ϕ_0, we replace

$$\tilde{\phi}(p_i) = \phi_0 (2\pi)^4 \delta^4(p_i)$$

so that

$$\Gamma[\phi_0] = \sum_{n=0}^{\infty} \frac{1}{n!} \phi_0^n \underbrace{(2\pi)^4 \delta(0)}_{\int d^4x = V_4} \tilde{\Gamma}^{(n)}(0)$$

zero external momentum

and therefore:

$$\boxed{V(\phi_0) = -\sum_{n=0}^{\infty} \frac{1}{n!} \phi_0^n \tilde{\Gamma}^{(n)}(0)} \qquad (V.1)$$

From this, we conclude that the effective potential is the generating functional of 1PI Green functions with zero external momentum. We also get a recipe to calculate the potential, which we will follow shortly in a simple example. Eq. (V.1) corresponds to a Taylor expansion of $\Gamma[\phi]$ around the origin $\phi = 0$. But we can as well Taylor expand it around ϕ_0 instead, using $\varphi(x) = \phi(x) - \phi_0$ as fluctuating field.

Take e.g. $S[\phi] = \int d^4x\, \mathcal{L}(\phi)$ with

$$\mathcal{L}(\phi) = -\Lambda + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4 \qquad (M.1)$$

which leads to the shifted theory Lagrangian

$$\begin{aligned}\mathcal{L}(\varphi) = &-\left(\Lambda + \frac{1}{2}m^2\phi_0^2 + \frac{1}{4}\lambda\phi_0^4\right) + \frac{1}{2}(\partial\varphi)^2 \\ &- \left(m^2\phi_0 + \lambda\phi_0^3\right)\varphi - \frac{1}{2}\left(m^2 + 3\lambda\phi_0^2\right)\varphi^2 \\ &- \lambda\phi_0\varphi^3 - \frac{1}{4}\lambda\varphi^4 \qquad (M.2)\end{aligned}$$

The classical (tree-level) potential is simply

$$V_0(\phi_0) = \Lambda + \frac{1}{2}m^2\phi_0^2 + \frac{1}{4}\lambda\phi_0^4$$

This leads to new, background dependent Feynman rules, and new interactions (eg linear and cubic φ-terms). In this shifted theory, the expansion of the effective action in terms of the 1PI Green functions reads

$$I_{\phi_0}[\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_n}{(2\pi)^4} (2\pi)^4 \delta^4(p_1, \dots p_n)\, \tilde{\varphi}(p_1) \dots \tilde{\varphi}(p_n) \times$$
$$\times \tilde{I}^{(n)}_{\phi_0}(p_1, \dots, p_n) \leftarrow \text{Green functions in shifted theory.}$$

Setting $\varphi(x) = 0$ we see that only the vacuum term contributes now:

$$I_{\phi_0} = I^{(0)}_{\phi_0} = (2\pi)^4 \delta^4(0) \tilde{I}^{(0)}_{\phi_0}(0)$$

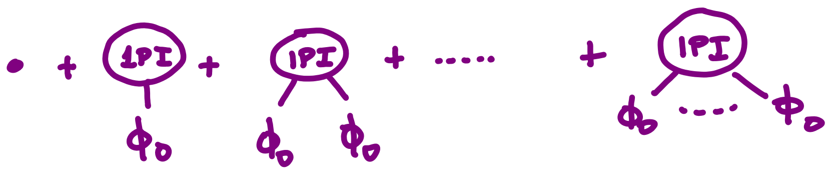
and so

$$\boxed{V(\phi_0) = -\tilde{I}^{(0)}_{\phi_0}(0)} \qquad \text{(V.2)}$$

From this we see that the potential is given by the vacuum 1PI Green function in the shifted theory.

One-loop potential (toy example)

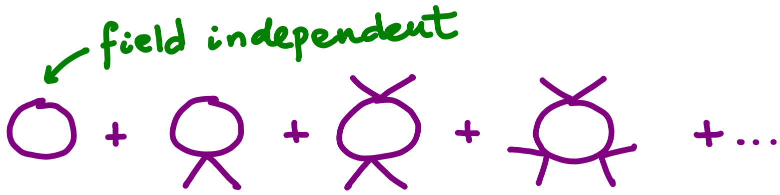
Let us now compute explicitly the 1-loop correction to the potential in the toy model we have introduced above. In the un-shifted case, (M.1), we use (V.1) and construct the series



The only interaction term is the quartic coupling:

$-6\lambda \qquad \left(\& \quad \text{———} \quad \frac{1}{m^2 - p^2} \right)$

so, the series above only contains even terms in ϕ_0, and at 1-loop is:



The term with 2n legs (n vertices) gives:

$$-\frac{1}{(2n)!}\phi_0^{2n}\,\tilde{\Gamma}^{(2n)}(0) = -\frac{1}{2n}\left(-3\lambda\phi_0^2\right)^n\int\frac{d^Dp}{(2\pi)^D i}\frac{1}{(m^2-p^2)^n}$$

Summing the series:

$$V_1(\phi_0) = V_1(0) - \frac{1}{2}\int\frac{d^Dp}{(2\pi)^D i}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{-3\lambda\phi_0^2}{m^2-p^2}\right)^n$$

$$= (\phi_0\text{-indep piece}) + \frac{1}{2}\int\frac{d^Dp}{(2\pi)^D i}\log\left(1+\frac{3\lambda\phi_0^2}{m^2-p^2}\right)$$

$$= \frac{1}{2}\int\frac{d^Dp}{(2\pi)^D i}\log\left(-p^2+m^2+3\lambda\phi_0^2\right) + (\phi_0\text{- indep. piece})$$

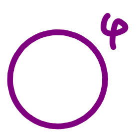
We will evaluate this explicitly later on.

Now, let us use (V.2) in the shifted case (M.2). The interaction vertices are now

(The tadpole x— φ is irrelevant for our calculation).

And the propagator is now $\longrightarrow \dfrac{1}{m^2+3\lambda\phi_0^2-p^2}$.

At 1-loop we only have a single diagram to compute:

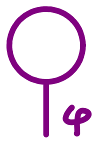


which gives

$$V(\phi_0) = \frac{1}{2}\int\frac{d^Dp}{(2\pi)^D i}\log\left(m^2+3\lambda\phi_0^2-p^2\right)$$

in agreement with our previous calculation.

To see this, notice that $-\partial V/\partial\phi_0$ is the 1PI 1-point function



$$= -6\lambda\phi_0\cdot\frac{1}{2}\int\frac{d^Dp}{(2\pi)^D i}\,\frac{1}{m^2+3\lambda\phi_0^2-p^2}$$

so that

$$V(\phi_0)=\int d\phi_0\,\frac{dV}{d\phi_0}=\frac{1}{2}\int\frac{d^Dp}{(2\pi)^D i}\log\left(m^2+3\lambda\phi_0^2-p^2\right)$$

This result could have been anticipated from our previous discussion of the semi-classical expansion of $Z[J]$, p. 1.5, in which we found that the 1-loop correction was given by

$$e^A=\left[\operatorname{Det}\left(\frac{1}{\hbar}\frac{\delta^2 S_E}{\delta\phi\delta\phi}\Big|_{\phi_0}\right)\right]^{-1/2}$$

so that

$$i\delta_1\Gamma=i\delta_1 W=-\frac{1}{2}\operatorname{Tr}\log\left(\frac{\delta^2 S_E}{\delta\phi\delta\phi}\right)\Big|_{\phi_0}$$

Going to momentum space, we would reproduce our result above. So we see again that the 1-loop correction comes from harmonic fluctuations around the classical background. That is why $V(\phi_0)$ depends only on the mass of ϕ in that background.

Regularization

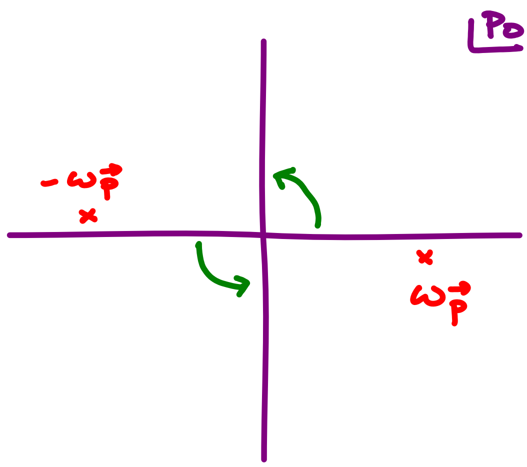
The momentum integral obtained above for $V_1(\phi_0)$:

$$V_1(\phi_0)=\frac{1}{2}\int\frac{d^Dp}{(2\pi)^D i}\log\left[-p^2+M^2(\phi_0)\right]$$

where $M^2(\phi_0)\equiv m^2+3\lambda\phi_0^2$, is obviously divergent and needs to be regularized. Let us first use a simple momentum cut-off regularization. It is also instructive to first do the integration in p_0:

$$V_1(\phi_0) = \frac{1}{2}\int\frac{d^3\vec{p}}{(2\pi)^3}\int\frac{dp_0}{(2\pi)\,i}\log\Big[-p_0^2+\underbrace{\vec{p}^{\,2}+M^2(\phi_0)}_{\omega_{\vec{p}}^2}-i\epsilon\Big]$$

Here we also introduce the notation $\omega_{\vec{p}}^2$, that reminds us that this QFT describes an infinite collection of (coupled) harmonic oscillators with fundamental frequencies $\omega_{\vec{p}}$. We have also written explicitly the $i\epsilon$ term required to properly define the propagator. We can now Wick rotate the p_0 integration contour to the imaginary axis



So that the momentum integral is now in the Euclidean momentum space:

$$V_1(\phi_0) = \frac{1}{2}\int\frac{d^3\vec{p}}{(2\pi)^3}\int_{-\Lambda}^{\Lambda}\frac{dp_{E0}}{(2\pi)}\log\left[p_{E0}^2+\vec{p}^{\,2}+M^2(\phi_0)\right]$$

Performing the p_{E0} integral, and taking the limit $\Lambda^2 \gg M^2(\phi_0)$ we get (inserting now $\hbar$)

$$\boxed{V_1(\phi_0) = \int \frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{\hbar}{2}\omega_{\vec{p}}\right) + \text{ct.}}$$

which shows even more explicitly that the one-loop potential is nothing but the sum of the ground-state energies of the (infinite number of) harmonic oscillators of which the QFT is composed.

Performing the full momentum integral we get

$$V_1(\phi_0) = \underbrace{*\Lambda^4}_{M^2\text{-indep.}} + \frac{\Lambda^2 M^2}{32\pi^2} + \frac{M^4}{64\pi^2}\left(\log\frac{M^2}{\Lambda^2} - \frac{1}{2}\right) + O\left(\frac{M^6}{\Lambda^2}\right)$$

where $M^2 = M^2(\phi_0) = m^2 + 3\lambda\phi_0^2$. Alternatively, we can use dimensional regularization, with $D = 4 - 2\epsilon$, and get

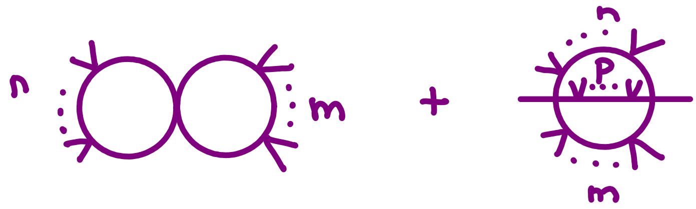
$$V_1(\phi_0) = \frac{M^4}{64\pi^2}\left(\log M^2 - \frac{3}{2} - C_{UV}\right)$$

where $C_{UV} = \frac{1}{\epsilon} - \gamma_E + \log 4\pi$, leads in the $\overline{MS}$ scheme to $\log Q^2$, where Q is the renormalization scale.

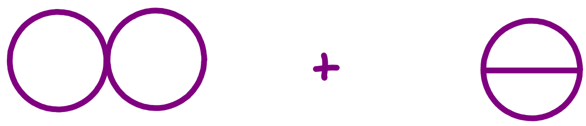
$$\boxed{V_1(\phi_0) = \frac{M^4}{64\pi^2}\left(\log\frac{M^2}{Q^2} - \frac{3}{2}\right)}$$

Higher orders

Corrections to V_{eff} beyond 1-loop take into account the effect of interaction terms (anharmonic effects). In principle we could compute such corrections either in the unshifted or in the shifted formulation, although the latter is simpler. It's easy to see that both methods would agree. Take our toy example at two loops. In the unshifted theory we would need to sum the series



while in the shifted theory we just have the two diags.



That both approaches agree can be seen by resumming the propagators appearing in the unshifted case as

to get

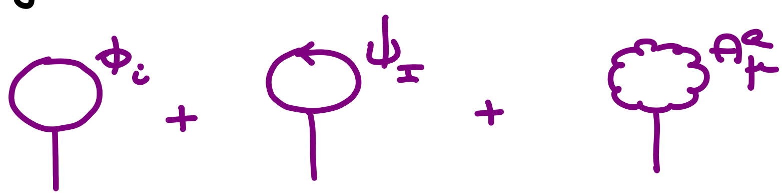
which reproduces the diagrams in the shifted case. It's clear that the same will happen to arbitrarily high order.

General case

If the scalar field ϕ (or fields) for which we are computing the potential couples to other sectors of the theory (other scalar fields, fermions, gauge bosons) those sectors will also contribute to $V(\phi)$. For the reasons explained above their contributions will only depend on their fundamental frequencies, and thus on their masses. Let us write such general mass terms as

$$-\mathcal{L} = \frac{1}{2} m^2_{ij}\phi_i\phi_j + \frac{1}{2}\left(m^{IJ}\psi_I\psi_J + h.c.\right) + \frac{1}{2} m^2_{ab} A^{\mu}_a A^b_{\mu}$$

where ϕ_i are real scalar fields, ψ_I Weyl fermions and A^{μ}_a gauge bosons and the mass matrices are functions of the background field ϕ_0. The matrices m^2_{ij}, $m^{IJ}m^{\dagger}_{JK}$, m^2_{ab} can be diagonalized and we will call $m^2_{\alpha}(\phi_0)$ the generic eigenvalues. The contribution of these fields to $V_1(\phi_0)$ can be computed using any of the methods discussed in the simple scalar toy model, for instance integrating the tadpole diagrams



In all cases the resulting expression is a momentum integral of the logarithm of the inverse propagator. In the case of the gauge boson loop, such propagator requires a choice of gauge :

$$P^{ab}_{\mu\nu}(p) = \delta^{ab}\frac{1}{p^2 - m_a^2}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2 - \xi m_a^2}\right]$$

The simplest choice is Landau gauge, $\xi = 0$. Feynman gauge ($\xi = 1$) would lead to a simpler expression for $P^{ab}_{\mu\nu}$ above but would also introduce masses m_a^2 for ghosts, and ghost-φ^2 couplings.

Putting all pieces together one gets for the 1-loop effective potential (in $\overline{MS}$ and Landau gauge):

$$\boxed{V_1(\phi_0) = \frac{1}{64\pi^2}\sum_\alpha N_\alpha\, m_\alpha^4(\phi_0)\left[\log\frac{m_\alpha^2(\phi_0)}{Q^2} - C_\alpha\right]} \qquad (V.1)$$

where α runs over all states in the theory, with N_α counting their number of degrees of freedom (taken negative for fermions, to absorb the negative sign from the fermion loop): i.e. for a real scalar ϕ_i, $N_i = 1$; for a Weyl (Dirac) fermion $N_I = -2(-4)$; for a massive (real) gauge boson A_μ^a, $N_a = 3$. The constant C_α takes the value $C_\alpha = 3/2$ for scalars or fermions, and $C_\alpha = 5/6$ for gauge bosons. The weighted sum is sometimes defined as a "supertrace":

* color/charge dof

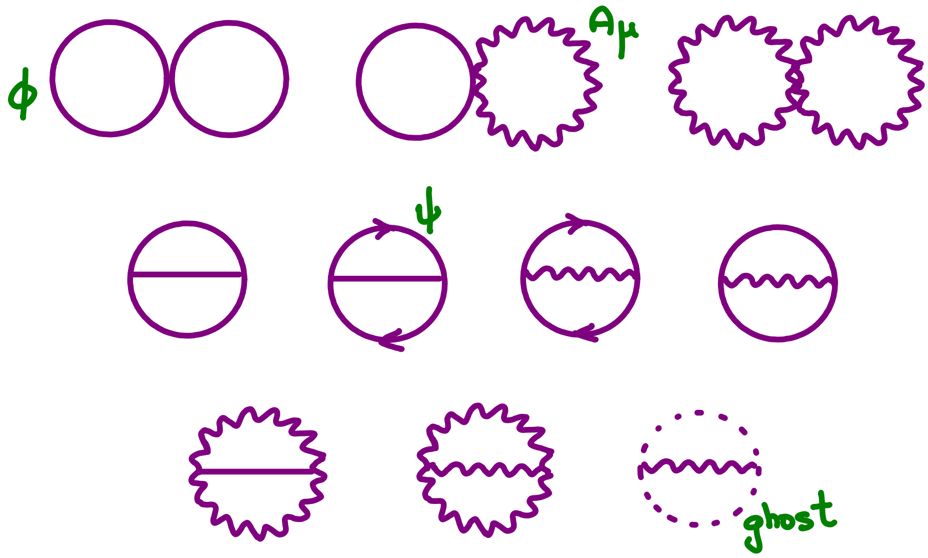
$$\mathrm{Str}\,\mathcal{O} \equiv \sum_\alpha N_\alpha \mathcal{O}_\alpha = \sum_\alpha (-1)^{2S_\alpha}(2S_\alpha+1)\, C_\alpha \mathcal{O}_\alpha$$

where $\mathcal{O}$ is some operator (eg $\mathcal{M}^2$), and S_α is the spin.

It can also be useful to write the general result when a momentum cutoff regularization is used:

$$\boxed{\delta_1 V(\phi_0) = \frac{\Lambda^2}{32\pi^2}\mathrm{Str}\, M^2 + \frac{1}{64\pi^2}\mathrm{Str}\, M^4\left(\log\frac{M^2}{\Lambda^2} - \frac{1}{2}\right)} \qquad (V.2)$$

Higher order corrections to the effective potential involve interactions. The two-loop potential for a general $\mathcal{L}$ with scalars, fermions and gauge bosons requires the calculation of the following type of vacuum diagrams (in the shifted theory):



This result is available in the literature (see bibliography at the end).

1.6 Standard Model Higgs Potential

The tree-level Higgs potential in the Standard Model (SM) has the form:

$$V_0(H) = m^2 |H|^2 + \lambda |H|^4$$

where H is the Higgs doublet:

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(h^0 + \phi + iG^0) \end{pmatrix}$$

we have written explicitly the field content of H. $G^\pm, G^0$ are the Goldstone bosons (eaten by $W^\pm$ and Z^0 to gain their masses); h^0 is the physical Higgs field; ϕ is the background value, so that V is a function of ϕ. At tree-level, minimization of the potential leads to

$$\frac{\partial V_0}{\partial \phi} = 0 \Rightarrow \langle \phi^2 \rangle = v^2 = \frac{-m^2}{\lambda}$$

The mass scale v, which determines M_Z, M_W, is fixed by m^2, which should be negative to trigger electroweak symmetry breaking.

The one-loop potential, calculated in $\overline{MS}$ and Landau gauge is of the general form (V.1), see page 1.20, with the following contributions:

Top quark: $N_t = \underset{\text{fermion}}{-}12 = -\underset{\text{color}}{N_c} \times \underset{\text{Dirac}}{4}$. $\quad M_t^2 = \frac{1}{2} \underset{\text{Yukawa coupling}}{h_t^2} \phi^2$. $\quad C_t = -\frac{3}{2}$

Other fermions (quarks and leptons) give negligible contributions, due to the smallness of their Yukawa couplings.

Z^0: $\quad N_Z = \underset{\text{3-polarizations}}{3}$. $\quad M_Z^2 = \frac{1}{4}(\underset{SU(2)_L}{g^2} + \underset{U(1)_Y}{g'^2})\phi^2$. $\quad C_Z = 5/6$

gauge couplings

$W^\pm$: $\quad N_W = 6 = 3 \times \underset{\text{charge}}{2}$. $\quad M_W^2 = \frac{1}{4} g^2 \phi^2$. $\quad C_W = 5/6$

h^0: $\quad N_h = 1$. $\quad M_{h^0}^2 = \frac{\partial^2 V_0}{\partial \phi^2} = m^2 + 3\lambda\phi^2$. $\quad C_h = 3/2$

$G^0, G^\pm$: $\quad N_G = 3$. $\quad M_G^2 = m^2 + \lambda\phi^2$. $\quad C_G = 3/2$

Notice that, at the minimum $\langle \phi \rangle = v$ we get

$$M_{h^0}^2 = 2\lambda v^2 \qquad M_G^2 = 0$$

For generic values of the background field, however, the Goldstones are massive and contribute to $V(\phi)$.

The 2-loop potential is known and not too complicated :

We split the two-loop potential in different pieces according to their diagrammatic origin. We use the short-hand notation $t \equiv m_t^2$, $w \equiv m_w^2$, $z \equiv m_z^2$, $h \equiv m_h^2$, $\chi \equiv m_\chi^2$ and we neglect the bottom Yukawa coupling. The important top Yukawa contribution is

$$V_Y = \frac{3}{2} y_t^2 \kappa^2 \left[2J_{tt} - 4J_{tg} - 2J_{th} + (4t - h)I_{tth} + 2(t-\chi)I_{tg0} - \chi I_{ttz}\right] . \tag{75}$$

There is a purely scalar piece

$$V_S = \frac{3}{4}\kappa^2 \lambda \left[5J_{gg} + 2J_{hg} + J_{hh} - 4\lambda\phi^2 (I_{hgg} + I_{hhh})\right] , \tag{76}$$

a purely gauge part

$$\begin{aligned} V_V &= \frac{e^2}{4z}\kappa^2 (z - w)\left[J_{zw} + w(I_{zw0} - I_{w00})\right] \\ &- \frac{e^2}{4}\kappa^2 w \left[2(11A_z - 25A_w) + \frac{1}{w}(24J_{ww} + 25J_{zw}) + 24I_{zww} + 10I_{zw0} - 9I_{w00} + 49w\right] \\ &+ \frac{g^2}{4}\kappa^2 w \left[\frac{58}{3}(A_z + 2A_w) + \frac{1}{w}(7J_{ww} + 15J_{zw}) + 58I_{zww} - 9I_{zw0} + I_{z00} + I_{w00} + 76w\right] \\ &+ \frac{G^2}{8}\kappa^2 \left[J_{ww} - (16w + z)I_{zww} + 2(8w + z)I_{zw0} - zI_{z00} + 4w^2\right] , \end{aligned} \tag{77}$$

a fermion-gauge boson part[3] (which includes the important QCD piece)

$$\begin{aligned} V_{FV} &= 8g_s^2\kappa^2 m_t^4 \left(3L_t^2 - 8L_t + 9\right) + \frac{16}{3}e^2\kappa^2 \left(tA_z + J_{tz} - tI_{tt0} + tI_{ttz}\right) \\ &+ \frac{g^2}{6}\kappa^2 \left\{9t^2 - 16tw - 36w^2 - 26tA_t + 6(4w - 3t)A_w + 8(t + 4w)A_z - 4J_{tt} + 8J_{tz}\right. \\ &+ \left. 8\left[(t - 2w)I_{ttz} - 6wI_{w00} - 10wI_{z00}\right] + \frac{9}{w}\left[(t - 2w)J_{tw} + (t - w)(t + 2w)I_{tw0} - t^2 I_{t00}\right]\right\} \\ &+ \frac{G^2}{6}\kappa^2 \left\{-tA_t - (17t + 40w - 20z)A_z + \frac{17}{2}J_{tt} - 17J_{tz} - \frac{1}{2}(7t - 40w + 17z)I_{ttz}\right. \\ &+ \left. \left(100w - \frac{103}{2}z\right) I_{z00} + 9t^2 + 20tw - 48w^2 - 4tz + 60wz - 30z^2\right\} , \end{aligned} \tag{78}$$

and a scalar-gauge boson part

$$\begin{aligned} V_{SV} &= g^2\kappa^2 \left\{\left[\frac{1}{2}(h + 3\chi + z) - \frac{1}{3}w\right] A_w + \frac{3}{2}w(A_h + A_g) + \frac{1}{4}(J_{gg} + J_{hg}) + \frac{(h - w)^2}{4w} I_{wh0}\right. \\ &+ \frac{1}{4w}\left[\frac{1}{2}(h - 2w)J_{ww} + (3w + \chi - h)J_{wh} + (h + 5w + z - \chi)J_{wg}\right] - \left(\frac{w}{4} - \chi\right) I_{wgg} \\ &- \left. \frac{1}{8w}\left(h^2 - 4hw + 12w^2\right) I_{wwh} - \left[\frac{1}{4w}(h + w - \chi)^2 - h\right] I_{whg} - w\left(w + \frac{h}{2}\right)\right\} \\ &+ \frac{1}{2}\left\{\begin{array}{c} w \leftrightarrow z \\ g \leftrightarrow G \end{array}\right\} + \frac{3g^2}{16w}\kappa^2 \left(8\lambda^2\phi^4 I_{hg0} - h^2 I_{h00}\right) \end{aligned}$$

$$
\begin{aligned}
&- \frac{e^2}{4wz}(z-w)\kappa^2 \left\{(w+z-\chi)J_{zw} - wJ_{zg} + \left[(w+z-\chi)^2 + 8wz\right] I_{zwg}\right. \\
&- \left.(w-\chi)^2 I_{wg0} - (z-\chi)^2 I_{zg0} + \chi^2 I_{g00}\right\} \\
&- \frac{e^2}{2}\kappa^2 \left\{\left(4\chi + w - \frac{5}{3}z - \frac{z^2}{2w}\right) A_z + wA_W - J_{wg} + \left(4 - \frac{z}{4w}\right) J_{zg} + \frac{3}{2}\chi^2 + 2z(z+\chi)\right. \\
&- \left.\frac{w}{2}(w+2\chi) - \frac{1}{4}\chi^2\left(6+\pi^2\right) + (4\chi - z)I_{zgg} - \frac{13}{4}\chi I_{gg0} + \frac{3}{2}(3w-\chi)I_{wg0}\right\} . \qquad (79)
\end{aligned}
$$

The functions A, J and I are

$$
\begin{aligned}
A_x \equiv A[x] &\equiv x(L_x - 1) , &(80)\\
J_{xy} \equiv J[x,y] &\equiv A[x]A[y] , &(81)\\
I_{xyz} \equiv I[x,y,z] &\equiv \frac{1}{2}\left[(x-y-z)L_yL_z + (-x+y-z)L_xL_z + (-x-y+z)L_xL_y\right] \\
&+ 2(xL_x + yL_y + zL_z) - \frac{5}{2}(x+y+z) - \frac{1}{2}\xi[x,y,z] , &(82)
\end{aligned}
$$

where $L_x = \ln(x/Q^2)$ and

$$
\begin{aligned}
\xi[x,y,z] &= R\left[2\ln\left(\frac{x-y+z-R}{2z}\right)\ln\left(\frac{-x+y+z-R}{2z}\right) - \ln\left(\frac{x}{z}\right)\ln\left(\frac{y}{z}\right)\right. \\
&- \left. 2\,\mathrm{Li}_2\left(\frac{x-y+z-R}{2z}\right) - 2\mathrm{Li}_2\left(\frac{-x+y+z-R}{2z}\right) + \frac{\pi^2}{3}\right] , \qquad (83)
\end{aligned}
$$

where $R^2 = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ and $\mathrm{Li}_2(x)$ is the dilogarithm function. The above expression is valid for $R^2 > 0$, while for $R^2 < 0$ the analytical continuation should be used instead. Some particular cases of the previous functions which are useful to evaluate the potential are

$$
\begin{aligned}
I[x,y,0] &= (x-y)\left[\mathrm{Li}_2\left(\frac{y}{x}\right) - \frac{\pi^2}{6} - (L_x - L_y)L_{x-y} + \frac{1}{2}L_x^2\right] \\
&\quad -\frac{5}{2}(x+y) + 2xL_x + 2yL_y - xL_xL_y , \\
I[x,x,0] &= x\left(-L_x^2 + 4L_x - 5\right) , \\
I[x,0,0] &= x\left(-\frac{1}{2}L_x^2 + 2L_x - \frac{5}{2} - \frac{\pi^2}{6}\right) . \qquad (84)
\end{aligned}
$$

1.7 Scale (in)dependence

The regularization procedure has introduced in V an explicit dependence on the renormalization scale, which is in principle arbitrary while any physical quantity derived from V should be independent of that choice of scale.

To investigate the scale dependence of V, remember that the n-point 1PI Green function $\Gamma^{(n)}(x_1,\ldots,x_n)$ for ϕ satisfies the Callan-Symanzik equation

$$\left(Q\frac{\partial}{\partial Q}+\beta_{\lambda_i}\frac{\partial}{\partial\lambda_i}+n\gamma\right)\tilde{\Gamma}^{(n)}(p_1,\ldots p_n)=0$$

where λ_i stands for all couplings or masses in the theory and

$$\beta_{\lambda_i}\equiv\frac{d\lambda_i}{d\log Q}\qquad\qquad \gamma\equiv\frac{d\phi_0}{d\log Q}$$

Remembering our definition of the effective potential in page 1.13

$$V(\phi_0)=-\sum_{n=0}^{\infty}\frac{1}{n!}\phi_0^n\,\tilde{\Gamma}^{(n)}(0)$$

we get

$$Q\frac{dV(\phi_0)}{dQ}=-\sum_{n=0}^{\infty}\frac{1}{n!}\Big[n\phi_0^{n-1}\overbrace{\frac{d\phi_0}{d\log Q}}^{\gamma\phi_0}\tilde{\Gamma}^{(n)}(0)+$$

$$+\phi_0^n\left(\beta_{\lambda_i}\frac{\partial}{\partial\lambda_i}+\frac{\partial}{\partial\log Q}\right)\tilde{\Gamma}^{(n)}(0)\Big]=0$$

In other words, the potential is indeed independent of the renormalization scale and the explicit scale dependence we found, e.g. at 1-loop, must cancel with the implicit scale dependence of the parameters entering $V(\phi_0)$. Let us see how this works in the SM at 1-loop order.

Let us write

$$V = V_0(\phi) + V_1(\phi) + O(\hbar^2)$$

The tree-level potential, V_0, depends on the ren. scale Q only implicitly:

$$\frac{d}{d\log Q} V_0(\phi) = \frac{1}{2}(\beta_{m^2} + 2\gamma m^2)\phi^2 + \frac{1}{4}(\beta_\lambda + 4\lambda\gamma)\phi^4$$

The beta and γ functions start at order $\hbar$:

$$\beta_{\lambda_i} = \beta^{(1)}_{\lambda_i} + O(\hbar^2) \qquad \gamma = \gamma^{(1)} + O(\hbar^2)$$

The one-loop potential has an explicit Q-dependence and an implicit dependence through couplings and masses

$$\frac{d}{d\log Q} V_1(\phi) = \frac{\hbar}{64\pi^2} \sum_\alpha N_\alpha \frac{d}{d\log Q}\left[M_\alpha^4\left(\log\frac{M_\alpha^2}{Q^2} - C_\alpha\right)\right]$$

$$= \frac{\hbar}{64\pi^2}\sum_\alpha N_\alpha \left[-2M_\alpha^4 + 2M_\alpha^2\left(\log\frac{M_\alpha^2}{Q^2} - C_\alpha + \frac{1}{2}\right)\underbrace{\frac{dM_\alpha^2}{d\log Q}}_{O(\hbar)}\right]$$

At $O(\hbar)$, only the explicit dependence matters as $dM_\alpha^2/d\log Q$ introduces β_{λ_i} or γ factors which are $O(\hbar)$.

If we write the generic squared masses as

$$M_\alpha^2(\phi) = \kappa'_\alpha + \kappa_\alpha \phi^2$$

we end up with

$$\frac{dV_1(\phi)}{d\log Q} = -\frac{\hbar}{32\pi^2}\sum_\alpha N_\alpha\left(\kappa_\alpha'^2 + 2\kappa_\alpha\kappa_\alpha'\phi^2 + \kappa_\alpha^2\phi^4\right) + O(\hbar^2)$$

To $O(\hbar)$ we should have $\frac{d}{d\log Q}\left[V_0(\phi)+V_1(\phi)\right]=0$, which is an identity that should hold for any ϕ. One can, therefore, split that identity in several, looking at the different powers of ϕ.

For ϕ^4 we find in this way:

$$\frac{1}{4}\beta_\lambda^{(1)} + \lambda\gamma^{(1)} = \frac{1}{32\pi^2}\sum_\alpha N_\alpha \kappa_\alpha^2$$

$$= \frac{1}{32\pi^2}\Big(\underbrace{-3h_t^4}_{\text{top}} + \underbrace{\tfrac{3}{8}g^4}_{W} + \underbrace{\tfrac{3}{16}G^4}_{Z} + \underbrace{9\lambda^2}_{h} + \underbrace{3\lambda^2}_{G^0,\,G^\pm}\Big)$$

(Here $G^2 \equiv g^2+g'^2$), which agrees with the known $\beta_\lambda^{(1)}$ and $\gamma^{(1)}$.

For ϕ^2 one finds:

$$\frac{1}{2}\beta_{m^2}^{(1)} + m^2\gamma^{(1)} = \frac{1}{16\pi^2}\sum_\alpha \kappa_\alpha'\kappa_\alpha$$

$$= \frac{1}{16\pi^2}\Big(\underbrace{3\lambda m^2}_{h} + \underbrace{3\lambda m^2}_{G^0,\,G^\pm}\Big)$$

which agrees with the known $\beta_{m^2}^{(1)}$.

Finally, $dV_1/d\log Q$ also gives a field independent piece $\sim m^4/(16\pi^2)$. Although we do not care about such terms in principle, this scale dependence can also be cancelled by a constant tree-level term we forgot to include. Calling this term $\delta V_0 = \Omega$, we learn from the Q-independence of the potential that the renormalization group equation (RGE) for Ω should be:

$$\beta_\Omega \equiv \frac{d\Omega}{d\log Q} = \frac{\hbar}{32\pi^2}\Big(\underbrace{m^4}_{h}+\underbrace{3m^4}_{G^0,G^\pm}\Big) + O(\hbar^2)$$

Notice that, if we knew $\gamma^{(1)}$ and the 1-loop potential, we would be able to obtain directly the 1-loop beta functions for λ, m^2 and Ω. In general, this provides a quick trick to calculate such RGEs in a given theory. Of course, the scale independence of the potential holds to all orders in perturbation theory. If we write

$$\left(\frac{\partial}{\partial\log Q} + \beta_{\lambda_i}\frac{\partial}{\partial\lambda_i} + \gamma\phi\frac{\partial}{\partial\phi}\right)V(\phi) \equiv 0$$

and expand the potential and $\beta_{\lambda_i}, \gamma$ in powers of $\hbar$:

$$\beta_{\lambda_i} = \sum_{k=1}^{\infty} \beta_{\lambda_i}^{(k)}\hbar^k \quad ; \quad \gamma = \sum_{k=1}^{\infty}\gamma^{(k)}\hbar^k$$

$$V(\phi) = \sum_{k=0}^{\infty} V_k(\phi)\hbar^k$$

we would get, at order $\hbar$:

$$\frac{\partial V_1}{\partial\log Q} + \beta_{\lambda_i}^{(1)}\frac{\partial V_0}{\partial\lambda_i} + \gamma^{(1)}\phi\frac{\partial V_0}{\partial\phi} = 0$$

Defining the operator

$$\mathcal{D}^{(n)} \equiv \beta_{\lambda_i}^{(n)}\frac{\partial}{\partial\lambda_i} + \gamma^{(n)}\phi\,\frac{\partial}{\partial\phi}$$

we would write the previous identity as

$$\frac{\partial V_1}{\partial\log Q} + \mathcal{D}^{(1)}V_0 = 0$$

Then, at order $\hbar$, one gets:

$$\frac{\partial V_2}{\partial \log Q} + \mathcal{D}^{(1)} V_1 + \mathcal{D}^{(2)} V_0 = 0$$

and, to order $\hbar^n$:

$$\frac{\partial V_n}{\partial \log Q} + \mathcal{D}^{(1)} V_{n-1} + \ldots + \mathcal{D}^{(n)} V_0 = 0$$

We will see in a later lecture that such identities, which relate contributions to the potential of different orders in $\hbar$, are of great practical value to resum potentially large corrections to $V(\phi)$.

Finally, notice also that in practice we will always work with a potential calculated up to some finite value of $\hbar$. Such "truncated" potentials are scale-independent only up to higher order corrections in $\hbar$. In such cases the choice of renormalization scale should be done with care, so that the error in neglecting higher orders is minimized. Usually one can determine a good choice for Q by studying the scale dependence of the quantity of interest x and choosing Q in the region where such scale-dep is minimized, $dx/d\log Q = 0$. This will usually happen for Q of order of the typical mass scales that dominate the radiative corrections to V.

For instance, in the SM case, $Q \sim M_t$ is typically a good choice. Needless to say, the residual scale dependence of a truncated potential approximation will be smaller, the higher order in $\hbar$ is used for V.

1.8 Gauge Dependence of V_{eff}

As we saw in a previous section, for theories with gauge fields, the expression of the effective potential is in general gauge dependent while we know that physical quantities should be gauge independent. By looking at how the effective action changes under a change in the gauge-fixing parameter ξ,

$$\mathcal{L}_{g.f.} = -\frac{1}{2\xi} F^2$$

where F is the gauge-fixing function, e.g.

$$F = \partial_\mu A^\mu \qquad \text{(Lorentz gauge)}$$

$$F = \partial_\mu A^\mu + ig\xi\left(\langle\phi\rangle^* \phi - \phi^* \langle\phi\rangle\right) \qquad (R_\xi \text{ gauge})$$

in a simple Abelian model with ϕ a charged scalar, one can derive, using Ward identities, the so-called Nielsen identity for the effective potential:

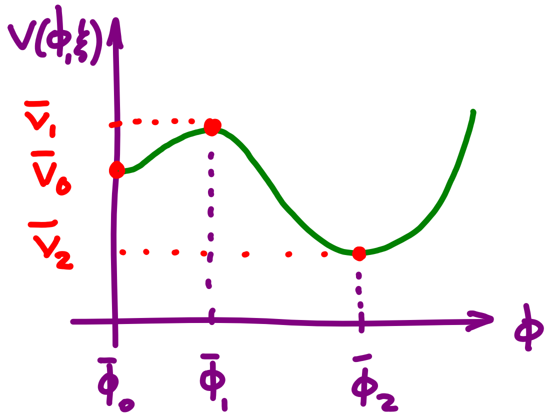
$$\xi \frac{\partial V(\phi)}{\partial \xi} + C(\phi, \xi) \frac{\partial V}{\partial \phi} = 0$$

where $C(\phi, \xi)$ is a calculable constant of order $\hbar$.

From this identity we immediately conclude that $\partial V / \partial \xi = 0$ precisely at the minimum of the potential, determined by $\partial V/\partial\phi = 0$. Therefore, the value of the potential at its minimum is a gauge-independent quantity. In more detail, notice that we cannot say the same about the value of ϕ at which that minimum appears. The minimization condition leads to ξ-dependent solutions:

$$\frac{\partial V(\phi,\xi)}{\partial\phi} = 0 \qquad \text{at} \qquad \phi = \bar{\Phi}_i(\xi)$$

I allow here the possibility of having several stationary points (labelled by i) corresponding to different minima and saddle points / maxima in between them:



The $\bar{\Phi}_i(\xi)$ are functions of the gauge-parameter in general, as indicated. But the Nielsen identity implies that the $\bar{V}_i \equiv V(\bar{\Phi}_i(\xi),\xi)$ are ξ-independent:

$$\frac{d\bar{V}_i}{d\xi} = \frac{\partial V}{\partial\phi}\cdot\frac{d\bar{\Phi}_i}{d\xi} + \frac{\partial V}{\partial\xi}$$

$$= \left(\frac{d\bar{\Phi}_i}{d\xi} - \frac{\xi}{C(\bar{\Phi}_i,\xi)}\right)\underbrace{\frac{\partial V}{\partial\phi}}_{0 \text{ at } \bar{\Phi}_i(\xi)} = 0$$

This result is crucial to make sense of the potential as a tool to study symmetry breaking. Eg. it allows us to determine that $\bar{V}_2 < \bar{V}_0$ in the example above even if the shape of the potential is gauge-dependent.

We can interpret the Nielsen identity as

$$\frac{dV}{d\xi} = \frac{\partial V}{\partial \xi} + \frac{\partial V}{\partial \phi}\frac{d\phi}{d\xi} = 0$$

i.e. assuming that ϕ has an implicit ξ-dependence given by

$$\frac{d\phi}{d\xi} = C(\phi, \xi)$$

A change of ξ can then be compensated by a change in ϕ according to the equation above and all physical quantities should not change when this is done. It can be checked that this is indeed the case.

1.9 Imaginary Part of Veff

As we have seen in previous discussions, for some regions of field space $\partial^2 V/\partial\phi^2 < 0$, corresponding to an unstable background field configuration. We will see in this section how such instability is reflected in the appearance of an imaginary part in the perturbative potential for such field regions.

Remember the one-loop contribution of particle α to the effective potential:

$$\delta_\alpha V_1(\phi) = \frac{N_\alpha}{64\pi^2} M_\alpha^4 \left(\log \frac{M_\alpha^2}{Q^2} - c_\alpha\right)$$

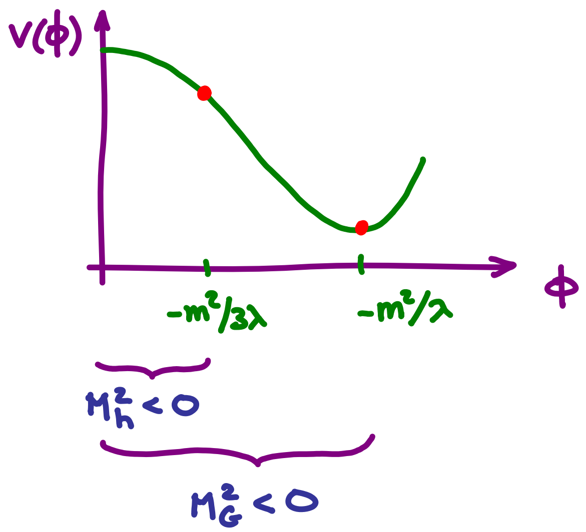
where $M_\alpha^2(\phi)$ is the mass squared of α in the ϕ-backg.
If $M_\alpha^2(\phi) < 0$ for some ϕ, this induces an imaginary part:

$$\delta_\alpha^I V_1(\phi) = \frac{N_\alpha}{64\pi^2} M_\alpha^4 \cdot (-i\pi)$$

where the sign is determined by $\log M_\alpha^2 \to \log(M_\alpha^2 - i\epsilon)$.
In the Standard Model we have such negative masses. As discussed in page 1.30, the Higgs and Goldstones have

$$M_h^2 = m^2 + 3\lambda\phi^2 < 0 \quad \text{for} \quad \phi^2 < -m^2/3\lambda$$
$$M_G^2 = m^2 + \lambda\phi^2 < 0 \quad \text{for} \quad \phi^2 < -m^2/\lambda$$

That is:



To better understand how this is related to an instability it is convenient to use the picture of V_{eff} as the sum of zero-point energies of harmonic oscillators (see page. 1.26)

$$\delta_\alpha V_1(\phi) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\hbar}{2} \underbrace{\omega_{k,\alpha}}_{\sqrt{\vec{k}^2 + M_\alpha^2(\phi)}}$$

per each α d.o.f. Consider now a case for which we have $M_\alpha^2(\phi)<0$. We see that this contribution $\delta_\alpha V_1$ is real for short wavelength modes with $\vec{k}^2>|M_\alpha^2(\phi)|$ and pure imaginary for long wavelength modes with $\vec{k}^2<|M_\alpha^2(\phi)|$. In fact, the integral in k over these long wavelength modes reproduces the imaginary part of $V_1(\phi)$ obtained above directly from the logarithmic expression. These long wavelength modes are obviously unstable as their evolution is governed by an upside-down harmonic potential with negative ω_k^2.

We can learn about the behaviour of the unstable modes from an elementary analysis of such inverted potential in QM:

$$\hat{H}=\frac{1}{2}\dot{q}^2-\frac{1}{2}\bar{\omega}^2q^2$$

(with $\bar{\omega}^2>0$). We can prepare an initial state $|\psi\rangle$ initially localized near $q=0$, e.g. satisfying

$$\langle\psi|q^2|\psi\rangle\leq a/\omega$$

for $a\sim O(1)$, and then check its time evolution at $t\to\infty$. We can impose the above constraint as an equality using Lagrange multipliers (and also $\langle\psi|\psi\rangle=1$) as we already did in the discussion of the energy interpretation of V_{eff}.

Proceeding in this way we arrive at

$$(\hat{H}+\frac{1}{2}\lambda_1\bar{\omega}^2q^2)|\psi\rangle=\lambda_2\bar{\omega}|\psi\rangle$$

which is a normal harmonic oscillator if $\lambda_1>1$. One finds inmediately:

$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right) \Rightarrow \lambda_2 = \sqrt{\lambda_1 - 1}\left(n+\tfrac{1}{2}\right)$$

$$\langle\psi|q^2|\psi\rangle = \frac{n+1/2}{\sqrt{\lambda_1-1}\,\bar\omega} = \frac{a}{\bar\omega} \Rightarrow \lambda_1 = 1 + \left(n+\tfrac{1}{2}\right)^2/a^2$$

Taking $|\psi\rangle$ as the ground state $(n=0)$ one gets

$$\psi(q) = \left(\frac{\bar\omega}{2a\pi}\right)^{1/4} e^{-\bar\omega q^2/4a}$$

with

$$\langle\psi|\hat H|\psi\rangle = \frac{\bar\omega}{4}\left(\frac{1}{2a} - 2a\right)$$

We will take this $|\psi\rangle$ as our initial state at $t=0$ and let it evolve with t according to the inverted harmonic oscillator hamiltonian $\hat H$. The solution can be found analytically and is

$$\psi(q,t) = \left(\frac{\bar\omega\, s_{2\phi}}{2\pi}\right)^{1/4} \frac{\exp\left[-\frac{\bar\omega q^2}{2}\tan(\phi - i\bar\omega t)\right]}{\left[\cos(\phi - i\bar\omega t)\right]^{1/2}}$$

with $\tan\phi = \frac{1}{2a}$. With this we can then calculate the "persistence" probability:

$$|\langle\psi(0)|\psi(t)\rangle|^2 = \left[1 + \frac{\sinh^2(\bar\omega t)}{\sin^2(2\phi)}\right]^{-1/2} \xrightarrow[t\to\infty]{} 2 s_{2\phi}\, e^{-\bar\omega t}$$

showing that the decay rate of this state is $\bar\omega/2$. The same conclusion is reached by examining the exponential growth of $\langle\psi(t)|q^2|\psi(t)\rangle \sim e^{2\bar\omega t}$.

This is analogous to the usual situation in which some stable state of definite energy, $\psi(t) \sim e^{-iEt}$, is destabilized throug interactions that induce a negative imaginary part of its energy

$E = E_R - i\Gamma/2$, leading to $\psi(t) \sim e^{-iE_R t} e^{-\Gamma t/2}$

$$|\langle\psi(0)|\psi(t)\rangle|^2 = e^{-\Gamma t}$$

Going back to QFT, the functional describing the field state will be a product of the wave-functions of different modes. The "persistence" probability will pick up a decay exponential for each unstable mode (the rest are stable and do not influence this quantity) with the total decay rate being the sum of the decay rates over all such modes. The end result will be

$$\Gamma = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\, \bar{\omega}_k\, \Theta(|M_\phi^2| - \vec{k}^2)$$

which precisely corresponds to the imaginary part of V_{eff}. Notice that this result is nicely independent of a, the parameter that controls the shape of $\psi(0)$. The real part of the energy contribution from the unstable modes does depend on a, however. It can be argued that the natural choice for a is $a = 1/2$, as this choice gives the largest $|\langle\psi(0)|\psi(t)\rangle|^2$ (or equivalently minimizes $\langle\psi(t)|q^2|\psi(t)\rangle$). With that choice one has $\langle\psi(0)|\hat{H}|\psi(0)\rangle = 0$ further agreeing with the result of the perturbative potential.

This instability of the background field configuration for values of the field between the two minima $\phi_\pm$ resolves itself by its decay (through the unstable long wavelength modes) into an state with a non-homogeneous value of ϕ. In some regions of space the field will be at ϕ_+, while in different regions it will be at ϕ_-. the average energy density will be $V(\phi_\pm)$, lower than the initial homogeneous field configuration. The energy stored in the surfaces separating those different + and − patches scales only as $(\text{length})^2$ and therefore it is negligible in comparison with the total energy, that scales with the volume. Although such field configuration has the minimum energy predicted by the true effective potential we have just seen that the perturbative V_{eff} corresponding to the homogeneous field configuration carries interesting physics information, and it is the quantity we will be interested in.

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See in particular §4.2.