# Effective Field Theories in Flavor Physics and as Indirect Probes Beyond the Standard Model: Exercises

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1. **Yukawa couplings, CKM matrix, and unitarity triangles**

   a) Show that flavor non-diagonal kinetic terms in the Standard Model Lagrangian can always be diagonalized and brought into standard form by field redefinitions. To this end, study the Lagrangian

   $$\mathcal{L}_{\text{kinetic}} = \bar{Q}_L\, Z_Q\, i\not{D}\, Q_L + \bar{u}_R\, Z_u\, i\not{D}\, u_R + \bar{d}_R\, Z_d\, i\not{D}\, d_R\,,$$

   where all fields are 3-component vectors in generation space, and $Z_A$ are non-negative, hermitian $3\times 3$ matrices.

   b) Show that an arbitrary complex matrix $Y$ can be diagonalized by a biunitary transformation:

   $$W^\dagger\, Y\, U = \lambda\,,$$

   where $U$, $W$ are unitary matrices, and $\lambda$ is a real, diagonal matrix with non-negative eigenvalues. (*Hint:* Consider the matrices $YY^\dagger$ and $Y^\dagger Y$.)

   c) Derive the number of mixing angles and physical (i.e., observable) phases of the CKM matrix for the Standard Model with $N$ fermion generations.

   d) Show that the Jarlskog determinant $J$ defined as

   $$\text{Im}\left(V_{ij} V_{kl} V_{il}^* V_{kj}^*\right) = J \sum_{m,n} \epsilon_{ikm}\, \epsilon_{jln} \qquad (i\ne k,\, j\ne l)$$

   is invariant under phase redefinitions of the quark fields, and calculate its value in terms of the Wolfenstein parameters to leading nontrivial order in $\lambda$.

   e) Show that all unitarity triangles have the same area $J/2$.

2. **Exclusive $B$-meson decays to CP eigenstates**

   The time-dependent rate for the decay of a $\bar{B}^0$ meson into a CP eigenstate $f$ is

   $$\Gamma_{\bar{B}^0\to f}(t) \propto 1 + \frac{|A|^2 - |\bar{A}|^2}{|A|^2 + |\bar{A}|^2}\,\cos(\Delta m\, t) - \frac{2\,\text{Im}(A^* \bar{A}\, e^{2i\beta})}{|A|^2 + |\bar{A}|^2}\,\sin(\Delta m\, t)\,,$$

   where $A$ and $\bar{A}$ are the decay amplitudes for $\bar{B}^0\to f$ and $B^0\to f$, respectively, and $2\beta$ is the weak phase of the $B^0$–$\bar{B}^0$ mixing amplitude. The rate for the decay of a $B^0$ meson into the same state $f$ is given by the analogous expression with $A\leftrightarrow\bar{A}$ and $\beta\to-\beta$.

   a) Show that the time-dependent CP asymmetry has the general form

   $$A_{\text{CP}}(t) \equiv \frac{\Gamma_{\bar{B}^0\to f}(t) - \Gamma_{B^0\to f}(t)}{\Gamma_{\bar{B}^0\to f}(t) + \Gamma_{B^0\to f}(t)} = \mathcal{C}\cos(\Delta m\, t) - \mathcal{S}\sin(\Delta m\, t)\,,$$

   where the coefficient $\mathcal{C}$ is given by the direct CP asymmetry of the decay $\bar{B}^0\to f$.

b) The general expression for the decay rate $A$ in the Standard Model is

$$A = A_1\, e^{i\delta_1} + A_2\, e^{i\delta_2}\, e^{-i\gamma} \propto 1 + r\, e^{i\delta}\, e^{-i\gamma}\,,$$

where $r = A_2/A_1$ and $\delta = \delta_2 - \delta_1$. Here $A_i$ are real amplitudes, and $\delta_i$ are CP-conserving strong rescattering phases. Obtain an exact expression for the coefficient $\mathcal{S}$ in terms of $r$, $\delta$, $\beta$, and $\gamma$.

c) In rare, penguin-dominated hadronic $B$ decays the coefficient $r$ is numerically very small. Expand your result from part b) to first order in $r$ and simplify the answer as much as possible. Using the experimental values $\gamma \approx 68^\circ$ and $\beta \approx 22^\circ$, as well as the QCD prediction that the strong phase shift $\delta$ is small in the heavy-quark limit, show that the measured coefficient $\mathcal{S}$ is always larger than $\sin 2\beta$.

3. **Matching of Wilson coefficients in the effective weak Hamiltonian**

Assume that, in addition to its standard interactions, the $Z^0$ boson has a small flavor-changing coupling to left-handed $b$ and $s$ quarks:

$$\mathcal{L}_Z = \frac{g_2}{\cos\theta_W} Z^\mu \left\{ \sum_f \bar f\, \gamma_\mu \left( T_f^3\, \frac{1-\gamma_5}{2} - Q_f \sin^2\theta_W \right) f + \left( \varepsilon_{bs}\, \bar s\, \gamma_\mu\, \frac{1-\gamma_5}{2}\, b + \text{h.c.} \right) \right\},$$

where $|\varepsilon_{bs}| \ll 1$. The sum in the first term is over all Standard Model fermions. $T_f^3$ is the third component of weak isospin, $Q_f$ the electric charge in units of $e$, $g_2$ the SU(2) gauge coupling, and $\theta_W$ the weak mixing angle.

Calculate the contributions to the Wilson coefficients $C_{3-10}$ in the effective weak Hamiltonian for $b \to s\bar q q$ transitions arising from tree-level $Z$-boson exchange, working to first order in $\varepsilon_{bs}$. Recall that $m_Z \cos\theta_W = m_W$ and $G_F/\sqrt{2} = g_2^2/8m_W^2$. Use the fact that $T_f^3 = 0$ for right-handed quarks, while $T_f^3 = Q_f - Y$ with $Y = 1/6$ for left-handed quarks.

4. **Effective Lagrangian for light-by-light scattering**

In the Standard Model, the elastic scattering of two photons $\gamma\gamma \to \gamma\gamma$ (light-by-light scattering) is forbidden at tree level, because the photon is a neutral particle. This process can, however, by induced at one-loop order via the exchange of virtual particles carrying electric charge.

a) Draw the one-loop diagrams contributing to the $\gamma\gamma \to \gamma\gamma$ process in the Standard Model.

b) At very low energy ($E_\gamma \ll m_e$), the light-by-light scattering amplitude can be obtained from an effective low-energy Lagrangian. Construct the leading (lowest-dimensional) gauge-invariant operators that can appear in this Lagrangian. What is the mass dimension of these operators?

c) Estimate the corresponding contributions to the scattering amplitude by dimensional analysis, taking into account factors of gauge couplings, loop factors, powers of external momenta, and masses of heavy virtual particles. Which loop graphs give the dominant contributions at very low energy?