1. THE EFFECTIVE ACTION AND THE EFFECTIVE POTENTIAL

1. The effective action and the effective potential

In Quantum Field Theory (QFT), the effective action and potential play a central role in the study of symmetries (and their breaking) at the quantum level. (In fact "quantum effective action/potential" would be better names.)

The effective action, Γ, is closely related to the Ward identities that encapsulate the symmetries obeyed by the theory Green's functions and also plays an important role in proving the renormalizability of non-Abelian QFTs.

The effective potential, V, very closely related to the effective action, Γ, is the quantity that determines the vacuum of the theory as the lowest energy state and is crucial in the study of spontaneous symmetry breaking, the most famous example being the Higgs breaking of the SM electroweak gauge symmetry.

These objects, Γ and V, are naturally formulated in the most transparent and elegant way in the path integral language.

I will remind you a few basic QFT results required to state/understand what Γ and V are and how they can be calculated in perturbation theory.

1.1 Review of basic QFT results needed

For illustration of the main points I will use first a simple QFT example with a scalar field ϕ.

The starting point will be the generating functional for the Green functions of ϕ:

$$Z[J] = N \int \mathcal{D}\phi \; e^{iS[\phi] + i\int d^4x\, J(x)\phi(x)}$$

where

* $S[\phi]$ is the classical action functional, $S[\phi] = \int d^4x\, \mathcal{L}(\phi)$. $\frac{1}{2}(\partial\phi)^2 - V(\phi)$
* $J(x)$ is an external source we use to probe the theory.

  E.g. it can be used to create/annihilate ϕ-particles and prepare initial/final states in an scattering experiment. It is linearly coupled to ϕ so that $Z[J]$ generates the correlation functions of ϕ:

  Time-ordered Green function from path-integral definition

$$\int_{B.C.} \mathcal{D}\phi \, e^{iS} \phi(x_1)\ldots\phi(x_n) = \langle \phi(\vec{x}',t')|\, T[\phi_H(x_1)\ldots\phi_H(x_n)]\,|\phi(\vec{x},t)\rangle$$

fields in Heisenberg picture

Boundary conditions (initial state $|\phi(\vec{x},t)\rangle$, final state $\langle\phi(\vec{x}',t')|$) we will choose in particular vacuum-to-vacuum transitions. Expanding in powers of J

$$Z[J] = \langle 0, \text{out} | 0, \text{in}\rangle_J$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n \, J(x_1)\ldots J(x_n) \underbrace{\langle 0|T[\phi(x_1)\ldots\phi(x_n)]|0\rangle}_{G^{(n)}(x_1,\ldots,x_n)}$$

Physical vacuum

Or, equivalently, the Green functions can be obtained from $Z[J]$ by functional derivatives wrt the source $J(x)$.

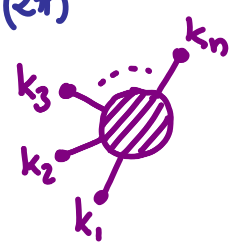
* The normalization constant is $N^{-1} = \int \mathcal{D}\phi \, e^{iS[\phi]} \Rightarrow Z[0] = 1$

Diagrammatic Expansion of $Z[J]$:

$Z[J]$ contains all the physics information of the theory, through the Green functions $G^{(n)}(x_1,\dots x_n)$, which describe the non-linear response of the system to the external source $J(x)$. In particular, $Z[J]$ contains the full S-matrix, but it's more than that: it generates off-shell Green functions:

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n \, J(x_1)\dots J(x_n)\, G^{(n)}(x_1,\dots,x_n)$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \dots \frac{d^4k_n}{(2\pi)^4} \tilde{J}(-k_1)\dots\tilde{J}(-k_n) \underbrace{\tilde{G}^{(n)}(k_1,\dots k_n)}_{\text{sum of all Feynman diags (with n external source insertions)}}$$



$\Leftarrow$ sum of all Feynman diags (with n external source insertions)

(Feynman rule from $i\int d^4x \, J(x)\phi(x) \Rightarrow$ $\times$— $iJ(x)$)

Note that the k_i are only constrained by momentum conservation $\delta^4(k_1+k_2+\dots+k_n)$ in $\tilde{G}^{(n)}$, but are otherwise arbitrary. This off-shell formulation is quite useful to deal with renormalization or unitarity in relativistic QFT.

So we have that

$$Z[J] = \sum(\text{vacuum diagrams in presence of } J)$$

and then $\quad = \exp\left[\sum(\text{connected v. diagrams})\right]$ as usual,

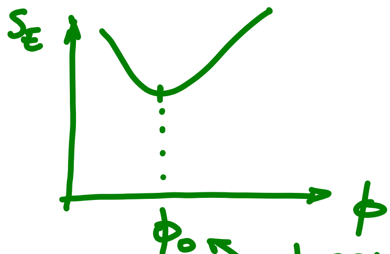
with the exponential taking into account all combinatoric factors for those diagrams composed of disconnected pieces. (Also, the normalization factor N in $Z[J]$, removes automatically all pure vacuum diagrams without J insertions).

Semiclassical Expansion

The path integral approach sheds light on the nature of the classical limit $\hbar \to 0$, explaining how for $S \gg \hbar$ quantum amplitudes are dominated by the classical solution, which corresponds to a stationary point of the classical action and leads to phase alignment.

$$\int \mathcal{D}\phi \; e^{-S_E[\phi]/\hbar}$$

Euclidean action



The semiclassical expansion is a systematic expansion in powers of $\hbar$ around that classical ($\hbar = 0$) limit.

Expanding $S_E[\phi]$ around the minimum ϕ_0:

$$S_E[\phi] = S_E[\phi_0] + \frac{1}{2}\int d^4x_1 d^4x_2 \left.\frac{\delta^2 S_E}{\delta\phi(x_1)\,\delta\phi(x_2)}\right|_{\phi_0} \delta\phi(x_1)\,\delta\phi(x_2) + \dots$$

where $\delta\phi = \phi(x) - \phi_0(x)$, and plugging this

$$\int \mathcal{D}\phi \; e^{-S_E[\phi]/\hbar} =$$

$$= \int \mathcal{D}\phi \; \underbrace{e^{-S_E[\phi_0]/\hbar}}_{\text{classical term}} \; \underbrace{\exp\left[-\frac{1}{2\hbar}\int d^4x_1 d^4x_2 \, \delta\phi(x_1) \left.\frac{\delta^2 S_E}{\delta\phi(x_1)\,\delta\phi(x_2)}\right|_{\phi_0} \delta\phi(x_2)\right]}_{\text{exp of quadratic term}}$$

$$\times \underbrace{\left[1 - \frac{1}{3!\,\hbar}\int \left.\frac{\delta^3 S_E}{\delta\phi\,\delta\phi\,\delta\phi}\right|_{\phi_0} \delta\phi\,\delta\phi\,\delta\phi + \dots\right]}_{\text{expansion of higher order terms}}$$

so that one has integrals of polynomials $\times$ Gaussian.

Using the well known result

$$\int dx_1 \dots dx_N \, e^{-\frac{1}{2}(\vec{x}, A\vec{x})} = \prod_{n=1}^{N} \int dx_n \, e^{-\frac{1}{2}\lambda_n x_n^2} =$$

$$= \prod_{n=1}^{N} \left(\frac{2\pi}{\lambda_n}\right)^{1/2} = (2\pi)^{N/2} (\mathrm{Det}\, A)^{-1/2}$$

we get

$$\int \mathcal{D}\phi \, e^{-S_E[\phi]/\hbar} \propto \underbrace{\left[\mathrm{Det}\left(\frac{1}{\hbar} \frac{\delta^2 S_E}{\delta\phi\delta\phi}\Big|_{\phi_0}\right)\right]^{-1/2}}_{\text{determined by the "harmonic" fluctuations around the classical trajectory } \phi_0} e^{-S_E[\phi_0]/\hbar} \left[1 + \underset{\uparrow}{O(\hbar)}\right]$$

determined by the "anharmonic" (interaction) fluctuations

[• To see this is indeed a series in $\hbar$ rescale $\delta\phi \to \sqrt{\hbar}\,\delta\phi$.

Odd terms in the polynomial in $\delta\phi$ integrate to zero. Even terms (of order $\delta\phi^{2n}$, $n=2,\dots$) lead to

$$\int \mathcal{D}\phi \, e^{-S''\delta\phi^2} \frac{1}{\hbar} (\sqrt{\hbar})^{2n} (\delta\phi)^{2n} \sim \hbar^{n-1} \ ]$$

As is well known, this semiclassical expansion in powers of $\hbar$ is the same as the Feynman weak coupling expansion. E.g. for

$$\frac{S}{\hbar} = \frac{1}{\hbar} \int d^4x \left[\frac{1}{2}\phi(-\Box - m^2)\phi - \frac{1}{4!}\lambda\phi^4 + J\phi\right]$$

$$\Big\downarrow \ \phi = \phi'/\sqrt{\lambda}$$

$$\frac{S}{\hbar} = \frac{1}{\lambda\hbar} \int d^4x \left[\frac{1}{2}\phi'(-\Box - m^2)\phi' - \frac{1}{4!}\phi'^4 + \sqrt{\lambda} J\phi\right]$$

For a particular Green function (corresponding to some power of J in $\int \mathcal{D}\phi \, e^{iS[\phi]/\hbar}$) expanding in $\hbar$ or λ is the same: the action depends only on their product.

So the Feynman diagram expansion is an expansion in small quantum fluctuations around the classical trajectory.

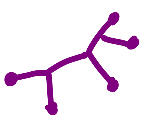
This result is general. When we evaluate $\int \mathcal{D}\phi\, e^{iS[\phi]/\hbar}$ diagrammatically, propagators are given by the inverse of the quadratic part of S, and therefore scale like $\hbar$. Vertices are proportional to S, and scale like $\hbar^{-1}$. So, a diagram with P propagators and V vertices scales like $\hbar^{P-V}$. Connected graphs with L loops satisfy the identity* $P - V + 1 = L$, and scales like $\hbar^{L-1}$. Therefore, tree graphs are order $\hbar^{-1}$ and each loop introduces an additional factor of $\hbar$.

Looking back at the diagrammatic expansion of $Z[J]$ we have

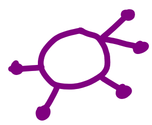
$$
\begin{aligned}
Z[J] &= \exp\left[\Sigma\,(\text{connected diagrams})\right] \\
&= \exp\Big[\underbrace{iS[\phi_0]}_{\text{tree}} + \underbrace{A\hbar}_{1\text{-loop}} + \underbrace{B\hbar^2}_{2\text{-loop}} + \dots\Big]/\hbar \\
&= e^{iS[\phi_0]/\hbar}\, e^{A}\, \left[1 + O(\hbar)\right]
\end{aligned}
$$

2-loops & higher: anharmonic corrs. to semiclassical exp.

Sum of all tree-graphs in the presence of source J



Sum of 1-loop graphs



gives the logarithm of the determinant of Gaussian fluctuations around the classical solution

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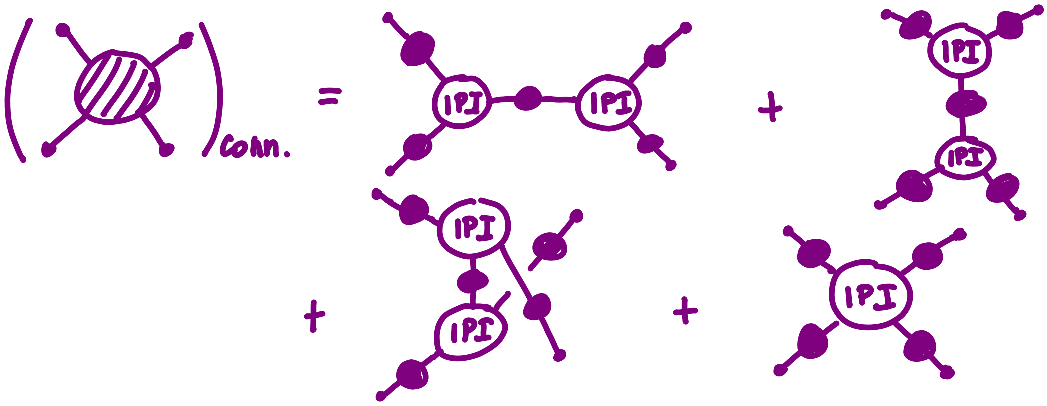
* Clear in momentum integrals: $\underbrace{\prod \int d^4p_i}_{P}\ \underbrace{\prod \delta(\Sigma p)}_{-V} = \underbrace{\delta(\Sigma_t p)}_{-1}\ \underbrace{\prod \int d^4p_j}_{L}$

1.2 Relation between the Effective Action and $Z[J]$

In order to study at the quantum level the symmetries of a QFT it is most convenient to focus on the symmetry constraints (the Ward identities) obeyed by the 1PI (one-particle irreducible) Green functions, which are the building blocks of the theory, in the following sense:

The key observation is that the sum of all connected diagrams can be obtained by constructing all "tree diagrams" with the exact connected 2-point function as propagator and the complete 1PI Green functions as vertices. E.g.



All loop corrections, no matter how complicated, can be reduced to this "tree-level" form in terms of such building blocks: 1PI vertices and complete propagator.

For this reason it would be useful to construct a generating functional of 1PI Green functions, just as $Z[J]$ is the generating functional of the complete Green functions:

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n \, J(x_1) \dots J(x_n) \underbrace{G^{(n)}(x_1, \dots, x_n)}_{\text{complete Green function}}$$

Such 1PI generating functional is the effective action, Γ.

Remembering that $Z[J] = \exp[\Sigma \text{ connected diags}]$ and writing

$$Z[J] = e^{iW[J]} \Rightarrow$$

$$iW[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n \, J(x_1)\dots J(x_n) \underbrace{G_c^{(n)}(x_1,\dots,x_n)}_{\text{Connected Green function}}$$

In the same way, we can write

$$\Gamma[\bar\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, \bar\phi(x_1)\dots\bar\phi(x_n) \underbrace{\Gamma^{(n)}(x_1,\dots,x_n)}_{\text{Connected 1PI Green function}^*}$$

where, for the time being, $\bar\phi$ is just a dummy variable of integration.

We want to find out what's the relation between $\Gamma[\bar\phi]$ and $W[J]$.

As the tree-level diagrams built with full propagator and 1PI vertices correspond to an effective field theory with action

$$S_{eff}[\bar\phi] = \Gamma[\bar\phi]$$

(which is complicated and non-local) and the tree-level approximation corresponds to the classical limit, we have

$$\int \mathcal{D}\bar\phi \; e^{(i\Gamma[\bar\phi] + i\int J\bar\phi)/\hbar} = \exp\Big[\underset{\substack{\uparrow \\ \text{Sum of connected tree-diagrams} \\ \text{of theory with action } \Gamma[\bar\phi]}}{iW[J]/\hbar} \; (1 + O(\hbar))\Big]$$

In the $\hbar \to 0$ limit this path-integral is dominated by the classical trajectory that extremizes the action $\Gamma[\bar\phi] + \int J\bar\phi$, so

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* For $n=2$, we define $\Gamma^{(2)}$ as the inverse of the exact propagator:

$\tilde\Gamma^{(2)}(p) = \dfrac{-1}{\tilde G_c^{(2)}(p)}$, which follows (see below) from

$$\frac{\delta^2\Gamma[\phi]}{\delta\phi(x)\,\delta\phi(y)} = -\frac{\delta J(x)}{\delta\phi(y)}$$

$$\frac{\delta^2 W[J]}{\delta J(x)\,\delta J(y)} = \frac{\delta\phi(x)}{\delta J(y)}$$

$$W[J] = \Gamma[\bar{\Phi}] + \int d^4x \, J(x) \bar{\Phi}(x) \Big|_{\text{stationary } \bar{\Phi}} \qquad (1)$$

found by solving the

"quantum" E.O.M. for $\bar{\Phi}$: $\dfrac{\delta \Gamma[\bar{\Phi}]}{\delta \bar{\Phi}(x)} = -J(x)$

So, $W[J]$ and $\Gamma[\bar{\Phi}]$ are related by a Legendre transformation. This is sometimes used as the starting point to define Γ, but then the definition looks cryptic and mysterious. Derived in this alternative way the result above follows nicely from the properties of Γ as generating functional of 1PI Green functions. To find $\Gamma[\bar{\Phi}]$ for an arbitrary $\bar{\Phi}$ we need to find the J that gives such $\bar{\Phi}$. The "EOM" for J is found by taking the functional derivative of (1) wrt $J(x)$:

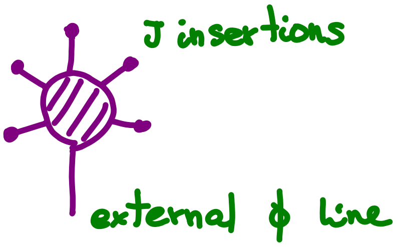
$$\frac{\delta W[J]}{\delta J(x)} = \bar{\Phi}(x) \qquad (2)$$

and we can invert (1) to obtain

$$\Gamma[\bar{\Phi}] = W[J] - \int d^4x \, J(x) \bar{\Phi}(x) \Big|_{\substack{\text{stationary } J \\ \text{as in (2)}}} . \qquad (3)$$

as the definition of the effective action $\Gamma[\bar{\Phi}]$.

Eq. (2) also tells us what $\bar{\Phi}$ is, as $\delta W / \delta J$ is the connected one-point function $\langle 0 | \phi(x) | 0 \rangle_J$ in the presence of the source.



That is, to compute $\Gamma[\bar{\Phi}]$, we choose the external source J, coupled to $\phi(x)$, so that the expectation value of $\phi(x)$ in the physical vacuum is the specified function $\bar{\phi}(x)$. Then, with that source present, we calculate $\Gamma[\bar{\phi}] = W[J] - \int d^4x \, J(x)\bar{\phi}(x)$.

1.3 Symmetries and Ward Identities

We can then address how the symmetries of the classical action S translate into symmetries of the quantum effective action Γ, or, what is equivalent, into the symmetries of 1PI Green functions.

Suppose the classical action is left invariant by an infinitesimal transformation of the fields of the form

$$\delta\phi(x) = \underset{\substack{\uparrow \\ \text{linear function of } \phi\text{'s.}}}{A(\phi)}\, \underset{\nwarrow\,\text{infinitesimal}}{\delta\omega(x)}$$

which also leaves invariant the functional measure $\mathcal{D}\phi$ (it has a trivial Jacobian). Shift $\phi \to \phi + \delta\phi$ in the functional integral defining $Z[J]$

$$\int \mathcal{D}\phi \; \exp\left[iS[\phi] + i\int d^4x \, J(x)\,\phi(x)\right] \overset{\phi\to\phi+\delta\phi}{=}$$

$$= \int \mathcal{D}\phi \; \exp\left[i\,S[\phi+\delta\phi] + i\int J(\phi+\delta\phi)\right]$$

invariant

$$= \int \mathcal{D}\phi \; \exp\left[iS[\phi] + i\int J\phi\right]\left\{1 + i\int \delta\phi\left(\frac{\delta S}{\delta\phi} + J\right) + \ldots\right\}$$

so that

$$0 = \int \mathcal{D}\phi \; e^{iS + i\int J\phi} \int d^4x \; \delta\phi(x)\left(\frac{\delta S}{\delta\phi} + J(x)\right)$$

Choosing again $J(x)$ so as to get $\langle\phi\rangle_J = \bar{\phi}$, we have $J = -\dfrac{\delta\Gamma}{\delta\bar{\phi}}$, and, substituting above, we obtain

$$\langle \delta S \rangle_J = \int d^4x \, \frac{\delta \Gamma}{\delta \bar{\phi}(x)} \langle \delta \phi(x) \rangle_J$$

As $\delta\phi$ is linear in ϕ:

$$\langle \delta\phi(x) \rangle_J = \langle A(\phi)\,\delta\omega(x) \rangle_J = A(\underbrace{\langle \phi \rangle_J}_{= \bar{\phi}})\,\delta\omega(x) = \delta\bar{\phi}$$

$$\Rightarrow \qquad \langle \delta S \rangle_J = \langle \delta \Gamma \rangle_J = 0$$

$$\Rightarrow \qquad \delta S[\bar{\phi}] = \delta \Gamma[\bar{\phi}] = 0$$

so that the effective action is also invariant. This is crucial for renormalizability so that all counterterms needed can be introduced respecting the symmetries of the classical action and all infinities can be absorbed. In this respect, it is also crucial that the regularization method used also respects the symmetries. The Green function identities implied by the symmetries of the action (Ward or Slavnov-Taylor identities) can be elegantly derived using this path-integral approach and expanding in powers of the source $J(x)$.

1.4 The Effective Potential

The non-local effective action can be Taylor-expanded, with nonlocalities giving higher order field derivatives. In our scalar field QFT example we have

$$\Gamma[\bar{\phi}] = \int d^4x \left[\Gamma_0(\bar{\phi}) + \Gamma_2(\bar{\phi})(\partial_\mu \bar{\phi})^2 + \Gamma_4(\bar{\phi})(\partial_\mu \bar{\phi})^4 + \ldots \right]$$

or, in a more suggestive notation,

$$\Gamma[\bar{\phi}] = \int d^4x \left[\frac{1}{2} Z(\bar{\phi})(\partial_\mu \bar{\phi})^2 - V_{eff}(\bar{\phi}) + \text{higher der.} \right]$$

and from ouprevious discussion on the semiclassical expansion, we know that

$$Z(\bar{\Phi}) = 1 + O(\hbar) \qquad V_{eff}(\bar{\Phi}) = V_{tree}(\bar{\Phi}) + O(\hbar)$$

and $V_{eff}(\bar{\Phi})$ is the quantum version of the classical potential. In other words, if we specialize to a position-independent field ϕ_0, the (quantum) effective potential is given by:

$$I[\phi_0] = -\underbrace{\int d^4x}_{\substack{\downarrow \\ \text{4-dim spacetime} \\ \text{volume } V\cdot T}} \underbrace{V_{eff}(\phi_0)}_{\substack{\text{ordinary function} \\ \searrow \text{a density}}}$$

The vacuum of the theory, i.e. the vacuum expectation value of ϕ in the absence of external sources ($J \to 0$) is determined by the minimum of V_{eff}. Remember the relations

$$\frac{\delta W[J]}{\delta J(x)} = \bar{\Phi}(x) = \langle 0|\phi(x)|0\rangle_J \quad , \quad \frac{\delta I[\bar{\Phi}]}{\delta \bar{\Phi}(x)} = -J(x)$$

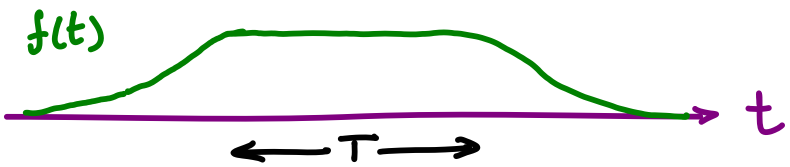
$$J = 0 \quad \Downarrow \quad \bar{\Phi}(x) = \phi_0$$

$$\underbrace{\phi_0 = \langle \phi \rangle}_{\substack{\phi_0 \text{ is sometimes called} \\ \text{classical field } (\phi_c) \\ \text{(in the sense of coherent homogeneous} \\ \text{field, but it's the minimum of} \\ \text{the quantum potential).}}} \qquad \& \qquad \underbrace{\frac{dV_{eff}(\phi_0)}{d\phi_0} = 0}_{\substack{\text{Minimization} \\ \text{condition} \\ \text{determines value} \\ \text{of } \phi_0 = v}}$$

Quantum (or radiative) corrections can play a very important role in modifying/triggering symmetry breaking. We will discuss how to include them shortly.

1.5 Energy Interpretation of V_{eff}

To further understand the physical meaning of the effective potential, consider the following thought experiment using our external source $J(x)$. We will examine how the vacuum responds to a constant homogeneous value of $J(x)$, although we turn $J(x)$ off slowly at $t \to \pm\infty$: $J(x) = f(t)\,\rho(\vec{x})$, with



We also choose $J(x)$ in such a way that it produces an specified value of the time-independent classical field $\phi_c(\vec{x})$:

$$\langle \phi(x) \rangle_J = \phi_c(\vec{x})$$

We end up with the same vacuum state we started with, up to a phase that the system picks up while staying in its ground state during time T, with the source present. That is :

$$\langle 0, out \,|\, 0, in \rangle_J = \exp\left(-i\, E[J]\, T\right)$$

where $E[J]$ is such energy. For sufficiently long T and for slow enough turning on and off of the source, we can neglect any phase introduced in that adiabatic turn on/off. Remembering the definition of $W[J]$, this means

$$W[J] = -T \cdot E[J]$$

The vacuum in the presence of J, $|0\rangle_J$ should therefore be an eigenstate of the Hamiltonian, supplemented by the source term:

$$\left(H - \int d^3\vec{x}\, \rho(\vec{x})\, \phi(\vec{x})\right)|0\rangle_J = E[J]\,|0\rangle_J \qquad (E)$$

and $E[J]$ must be the lowest energy eigenvalue, for the specified $\rho(\vec{x})$. We can show $|0\rangle_J$ is the state that minimizes the energy, given the two constraints ${}_J\langle 0|0\rangle_J = 1$ and ${}_J\langle 0|\phi(x)|0\rangle_J = \phi_c(\vec{x})$. This is a constrained minimization problem and can be stated using Lagrange multipliers. The quantity to be minimized is

$$\langle\psi|H|\psi\rangle - \underbrace{\lambda_1}_{\text{Lagrange multipliers}}\left(\langle\psi|\psi\rangle - 1\right) - \int d^3\vec{x}\, \lambda_2(\vec{x})\left[\langle\psi|\Phi|\psi\rangle - \phi_c(\vec{x})\right]$$

And the state $|\psi\rangle_{min}$ corresponding to that minimum must satisfy:

$$H|\psi\rangle_{min} - \lambda_1|\psi\rangle_{min} - \int d^3\vec{x}\, \lambda_2(\vec{x})\, \phi(\vec{x})\, |\psi\rangle_{min} = 0$$

But this is precisely eq. (E) above with

$$|\psi\rangle_{min} = |0\rangle_J\,; \quad \lambda_1 = E[J]\,; \qquad \lambda_2(\vec{x}) = J(x) = \rho(\vec{x})$$

and we have

$$ {}_J\langle 0|H|0\rangle_J = \underbrace{E[J]}_{-\frac{1}{T}W[J]} + \underbrace{\int d^3\vec{x}}_{\frac{1}{T}\int d^4x}\, \rho(\vec{x})\, \phi_c(\vec{x})$$

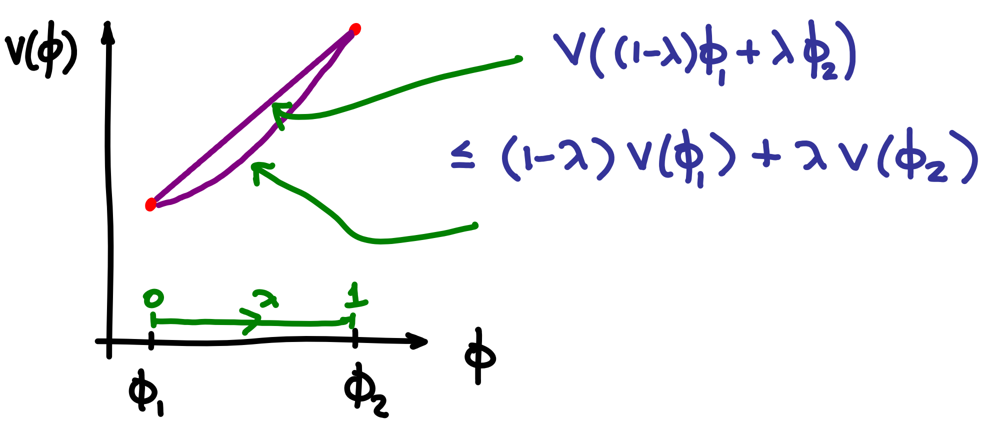
$$= \frac{-1}{T}\left[W[J] - \int d^4x\, J(x)\,\phi_c(x)\right] = -\frac{1}{T}\,\Gamma[\phi_c]$$

or $V(\phi) = {}_J\langle 0|H|0\rangle_J / V_3$.

We conclude that $V(\phi_c)$ is the minimum value of the energy density expectation value for all states that give $\langle \phi \rangle = \phi_c(x)$. In particular, for $J=0$, the vacuum state will correspond to a minimum of the effective potential.

Convexity of the Potential

From our definition $V(\phi) = -\Gamma[\phi]/V_4$ we can relate derivatives of the potential to derivatives of Γ, that is, 1PI Green functions (at zero external momentum). In particular $\partial^2 V/\partial\phi^2$ is directly related to $\tilde{\Gamma}^{(2)}$, the 2-point 1PI Green function. As we saw in page 1.7, footnote, $\tilde{\Gamma}^{(2)}$ is related to the inverse of the full ϕ propagator and tracking the signs we conclude that $\partial^2 V/\partial\phi^2$ and $\tilde{G}^{(2)} = \delta^2 W/\delta J \delta J$ have the same sign. It's easy to see, using the Euclidean formulation of the path integral, that $\partial^2 W/\partial J^2$ is positive and so one should have $\partial^2 V/\partial\phi^2 > 0$. In other words, the potential is convex:



This is clearly at odds with our experience with effective potentials, which e.g. in the SM, often have non-convex shape: