Infrared chiral anomaly at finite temperature

- Ashok Das

References:

- *The thermal chiral anomaly in the Schwinger model*, A. Das and J. Frenkel, Phys. Lett. **704**, 85 (2011).
- *Infrared chiral anomaly at finite temperature*, A. Das and J. Frenkel, Phys. Lett. **B696**, 556 (2011).
- A. Das and A. da Silva, Phys. Rev. **D59**, 105011 (1999).

Outline of the talk

- General properties of anomalies
- Difference between massive and massless theories and the two point anomaly
- Higher point amplitudes
- Summing the thermal anomaly
- Thermal anomaly and the index
- “Adler-Bardeen” like theorem

General properties of anomalies

- When a classical conservation law, say, of the form

$$\partial_\mu J^\mu = \Delta,$$

is violated by quantum mechanical corrections, we say that there is an anomaly in the theory (the conservation law becomes anomalous).

- At zero temperature, we understand anomalies as arising because of the incompatibility of the ultraviolet regularization with some of the symmetries of the theory. (Example: Gauge invariance and chiral invariance)

- At finite temperature, however, there is no ultraviolet divergence and, consequently, it follows that there is no temperature dependent correction to the anomaly (in particular, the chiral anomaly).

- However, the infrared behavior becomes much more prominent at finite temperature and can even be divergent.

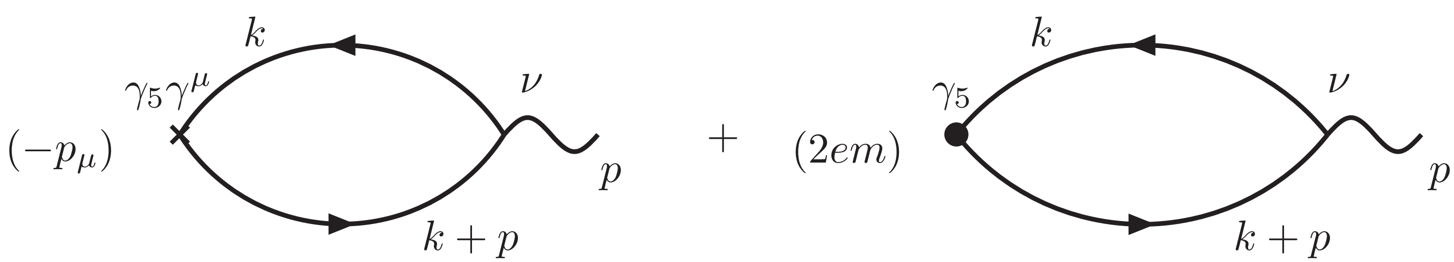
- We find that this can lead to temperature dependent corrections to the chiral anomaly (in massless theories) for background fields that do not vanish asymptotically. (Motivation: constant electric field backgrounds etc.)

- We call this the infrared chiral anomaly.

- We show this explicitly at one loop in the $1+1$ dimensional Schwinger model although it is likely to arise in infrared divergent graphs in higher dimensions.

Difference between massive and massless theories

- Since this is an infrared phenomenon, it arises in massless theories and not in massive theories. In the massive Schwinger model, for example, the temperature dependent corrections to the anomaly come from diagrams



$$
\int \mathrm{d}^2k \left[(k^2 - m^2)\epsilon^{\mu\nu}(k+p)_\mu - ((k+p)^2 - m^2)\epsilon^{\mu\nu}k_\mu\right] \delta(k^2 - m^2)
$$

$$
\times\, n_{\mathrm{F}}(|k^0|)\left(\frac{i}{(k+p)^2 - m^2 + i\epsilon} - \pi n_{\mathrm{F}}(|k^0 + p^0|)\delta((k+p)^2 - m^2)\right) A_\nu
$$

$$
= 0,
$$

independent of the asymptotic behavior of the background field $A_\nu(p)$.

- This is seen from the fact that the terms linear in delta functions vanish by delta function constraint or by anti-symmetry (both for massive and massless cases). We only have to analyze the terms which are quadratic in delta functions.

- We note that

$$
\begin{aligned}
&\delta(k^2 - m^2)\delta((k+p)^2 - m^2) \\
&= \frac{\theta(\omega_+(p_0))}{2\omega_+(p_0)}\delta(k_0 - \omega_+(p_0)) \\
&\times \left[\frac{1}{|J_{+,+}(p_0)|}\delta(k_1 - X_+(p_0)) + \frac{1}{|J_{+,-}(p_0)|}\delta(k_1 + X_+(p_0))\right] \\
&+ \frac{\theta(\omega_-(p_0))}{2\omega_-(p_0)}\delta(k_0 - \omega_-(p_0)) \\
&\times \left[\frac{1}{|J_{-,+}(p_0)|}\delta(k_1 - X_-(p_0)) + \frac{1}{|J_{-,-}(p_0)|}\delta(k_1 + X_-(p_0))\right] \\
&\quad + (k_0, p_0 \to -k_0, -p_0).
\end{aligned}
$$

It constrains the two variables of integration. The numerators of the integrand, with these constraints, make the integral vanish in this case.

- Here we have identified

$$
\begin{aligned}
\omega_{\pm}(p_0) &= \frac{1}{2}\left(-p_0 \pm |p_1|\sqrt{1-\frac{4m^2}{p^2}}\right),\\
X_{\pm}(p_0) &= \sqrt{\omega_{\pm}^2(p_0) - m^2},\\
J_{+,\pm}(p_0) &= \frac{\mp X_{+}(p_0)p^2 - 2m^2 p_1}{\omega_{+}^2(p_0)},\\
J_{-,\pm}(p_0) &= \frac{\mp X_{-}(p_0)p^2 - 2m^2 p_1}{\omega_{-}^2(p_0)}.
\end{aligned}
$$

- In a massless theory, on the other hand, we can factorize

$$\begin{aligned}&\delta(k^2)\delta((k+p)^2)\\&=\frac{1}{|k_-||p_+|}\delta(k_+)\delta(k_-+p_-)+\frac{1}{|k_+||p_-|}\delta(k_-)\delta(k_++p_+)\\&+\frac{1}{|k_-||k_-+p_-|}\delta(k_+)\delta(p_+)+\frac{1}{|k_+||k_++p_+|}\delta(k_-)\delta(p_-).\end{aligned}$$

- The first two terms constrain the two variables of integration and, therefore, lead to a vanishing value.

- On the other hand, the last two terms constrain the variables of integration only partially and lead to an anomaly

$$\left(p_+\delta(p_+)A_-(p)+p_-\delta(p_-)A_+(p)\right)I_2^{(\beta)},$$

where $I_2^{(\beta)}$ is a temperature dependent quantity.

- This would vanish for backgrounds that are well behaved and vanish asymptotically. However, note that if

$$A_{-}(p) \sim \frac{1}{p_{+}}, \quad \text{or,} \quad A_{+}(p) \sim \frac{1}{p_{-}},$$

this would lead to a finite contribution. This is the infrared origin of the correction alluded to earlier.

- In the high temperature limit, we can evaluate $I_2^{(\beta)}$ which takes the form

$$I_2^{(\beta)} = (ie)^2 T,$$

where T is the temperature.

- In fact, in the high temperature limit, the two point anomaly can be Fourier transformed to the coordinate space where it has the closed form (for arbitrary backgrounds)

$$\partial_\mu J^{\mu(\beta)}_{5,(2)}(x) = -\frac{(ie)^2 T}{\pi}\Big[\int \mathrm{d}y^+\,\mathrm{sgn}(x^+ - y^+)\left(E(y^+,\infty) - E(y^+,-\infty)\right) + \int \mathrm{d}y^-\,\mathrm{sgn}(x^- - y^-)\left(E(\infty,y^-) - E(-\infty,y^-)\right)\Big].$$

- For electric fields that are even functions of x^+, x^-, namely,

$$E(x^+,x^-) = E(-x^+,x^-) = E(x^+,-x^-),$$

these contributions vanish. A constant background electric field, therefore, would not lead to such a correction. However, an electric field that is asymptotically odd, for example,

$$E(x^+,x^-) = E_1 \mathrm{sgn}(x^+)\delta(x^-),\ \mathrm{or},\ E(x^+,x^-) = E_2 \mathrm{sgn}(x^-)\delta(x^+),$$

would lead to a nonvanishing contribution.

- For the first field configuration, for example, the temperature dependent infrared anomaly at the two point level is given by

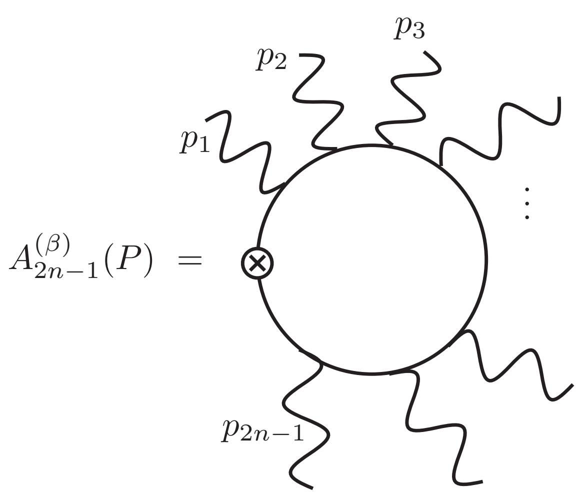
$$\partial_\mu J^{\mu(\beta)}_{5,(2)}(x) = -\frac{2E_1(ie)^2 T}{\pi}\,\mathrm{sgn}(x^-).$$

- This is the infrared anomaly that we are talking about since it has its origin in the large distance behavior of the electric field (in a massless theory) and is independent of the ultraviolet divergence structure of the amplitude.

- Since at zero temperature, we do not have products of (two) delta functions, such a contribution will not arise at zero temperature. It is genuinely a finite temperature phenomenon.

Higher point amplitudes

- The conventional zero temperature anomaly (in the Schwinger model) involves only the two point function because this is the only diagram in the theory with a UV divergence and the conventional anomaly is associated with the ultraviolet divergence of the diagram.



- In contrast, the infrared anomaly is not associated with the ultraviolet divergence of the amplitude and, therefore, can be present at any (even) order.

- The complete temperature dependent (one loop) effective action for the Schwinger model has already been calculated and has contribution at every even order $\Gamma_{\rm eff}^{(\beta)} = \sum_{n=0} \Gamma_{{\rm eff},(2n)}^{(\beta)}$.

- As a result, the axial current and the anomaly at every even order can be derived from the effective action,

$$A_{(2n-1)}^{(\beta)}(x) = \partial_\mu J_{5,(2n)}^{\mu(\beta)}(x) = \epsilon^{\mu\nu}\,\partial_\mu \frac{\delta\Gamma_{{\rm eff},(2n)}^{(\beta)}}{\delta A^\nu(x)}.$$

- At each even order the effective action as well as the axial-current/anomaly contains a temperature dependent factor

$I_{2n}^{(\beta)}$ which simplifies at high temperature to the form

$$I_{2n}^{(\beta)} = \pi^{2(n-1)}(2n-1)!(ie)^{2n}C_{2n}T,\ C_{2n} = \sum_{m=1}^{n}\frac{1}{2m-1},$$

and the anomaly in the momentum space has the form ($\bar{u}^{\mu} = \epsilon^{\mu\nu}p_{\nu}/p_1$ in the rest frame)

$$A_{2n-1}^{(\beta)}(P) = -\frac{(2ie)^{2n}C_{2n}}{2(2\pi)^2}\Bigg[\int d^2p_{2n-1}\left(\prod_{j=1}^{2(n-1)}\left(d^2p_j(\bar{u}\cdot A(p_j))\delta(p_{j,+})\right)\right)$$

$$\times\, p_{2n-1,+}\delta(p_{2n-1,+})(\bar{u}\cdot A(p_{2n-1}))\delta^2(P-\sum_{i=1}^{2n-1}p_i)$$

$$+\, p_{j,+}\rightarrow p_{j,-} + \text{permutations}\Bigg]T,$$

- In the coordinate space, this leads to

$$A^{(\beta)}_{(2n-1)}(x) = -(ie)^{2n}\pi^{2n-3}(2n-1)C_{2n}T \times \left[(I(x^+))^{2(n-1)}J(x^-) + (I(x^-))^{2(n-1)}J(x^+)\right],$$

where we have identified

$$I(x^\pm) = \int \mathrm{d}^2y\,\mathrm{sgn}(x^\pm - y^\pm)E(y),$$

$$J(x^+) = \int \mathrm{d}y^+\,\mathrm{sgn}(x^+ - y^+)(E(y^+,\infty) - E(y^+,-\infty)),$$

$$J(x^-) = \int \mathrm{d}y^-\,\mathrm{sgn}(x^- - y^-)(E(\infty,y^-) - E(-\infty,y^-)).$$

Summing the thermal anomaly

- The perturbative anomaly seems to have a divergent structure. For the field configuration of the first kind, for example, it has the form

  $$A^{(\beta)}_{2n-1}(x) = -(2E_1)^{2n-1}(ie)^{2n}\pi^{2n-3}(2n-1)C_{2n}\,|x^+|^{2(n-1)}\,\mathrm{sgn}(x^-)\,T.$$

  This is only an artifact of perturbation theory.

- Let us note that we can write (B_{2n}'s are Bernoulli's numbers, $C \simeq 0.577$ is the Euler constant)

  $$C_{2n} = \sum_{m=1}^{n} \frac{1}{2m-1} = \frac{1}{2}(C + \ln n) + \ln 2 + \frac{B_2}{8n^2} + \frac{7B_4}{64n^4} + \cdots .$$

  For $n = 1$, the first three terms give the exact result up to 2% accuracy, for $n = 2$ the result is accurate to 0.4% and so on.

- Furthermore, the polylogarithms are defined as

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = \int\limits_0^z \mathrm{d}t\, \frac{Li_{s-1}(t)}{t}, \quad Li_1(z) = -\ln(1-z),$$

which have at best only one singularity at $z = 1$ (branch cut or pole).

- In terms of these functions, the anomaly functional can be summed and has the form

$$A^{(\beta)}(x) = -\frac{(ie)^2 T}{\pi} \left(S_+ J(x^-) + S_- J(x^+)\right),$$

where (S_- is obtained with $z_+ \to z_- = -(e\pi I(x^-))^2$)

$$S_+ = \frac{1}{2}\Big[(C + 2\ln 2)\, \frac{1+z_+}{(1-z_+)^2} + \frac{1}{z_+} \left(\frac{\mathrm{d}Li_s(z_+)}{\mathrm{d}s}\Big|_{s=0} - 2\frac{\mathrm{d}Li_s(z_+)}{\mathrm{d}s}\Big|_{s=-1} \right)$$

$$+\frac{B_2}{4z_+}(2Li_1(z_+) - Li_2(z_+)) + \cdots \Big], \quad z_+ = -(e\pi I(x^+))^2.$$

- This shows that the complete anomaly functional is well behaved, the only singularities at $z_\pm = 1$ lie outside the physical region. It is well behaved even at infinity since $\frac{Li_s(z)}{z}$ is. The apparent divergence structure noted earlier is only an artifact of perturbation theory.

- The points $|z_\pm| = 1$ can be thought of as the boundary points between strong and weak coupling.

Thermal anomaly and the index

- The complete anomaly has a finite nontrivial value for classes of electric fields of the separable form

$$E(x^+, x^-) = f(x^+)g(x^-),$$

with $f(x^+)$ even (odd) and $g(x^-)$ odd (even). The particular models described earlier belong to this class.

- Because of the anomaly, the axial charge is no longer conserved and satisfies

$$Q_5^{(\beta)}(\infty) - Q_5^{(\beta)}(-\infty) = 2\int dx^+ dx^- \left(A^{(\beta)} + \frac{1}{2}(\partial_+ - \partial_-)J_0^{(\beta)}\right).$$

- The surface term cannot be neglected for fields that are nonvanishing asymptotically.

- The left hand side is normally identified with the index of the Dirac operator and conventionally can be identified with the integrated anomaly. (Here the right hand side has an extra contribution.)

- In the present case, rotation to Euclidean space as well as defining the theory on a compact manifold is not possible. So, it is not clear if the left hand side can still be identified with the index of the Dirac operator (or whether the index of the Dirac operator can even be defined meaningfully).

- If it can be identified, the right hand side will, in general, be a continuous function of the temperature, consistent with the expectation that fermion (number) distributions are continuous functions of temperature. Therefore, it will no longer be an integer. (Such a behavior already shows up in the case of the Witten index.)

- However, the left hand side will vanish for the special class of currents and electric fields that are CPT odd. (Note that classical currents are CPT odd.) The particular background fields discussed earlier belong to this class.

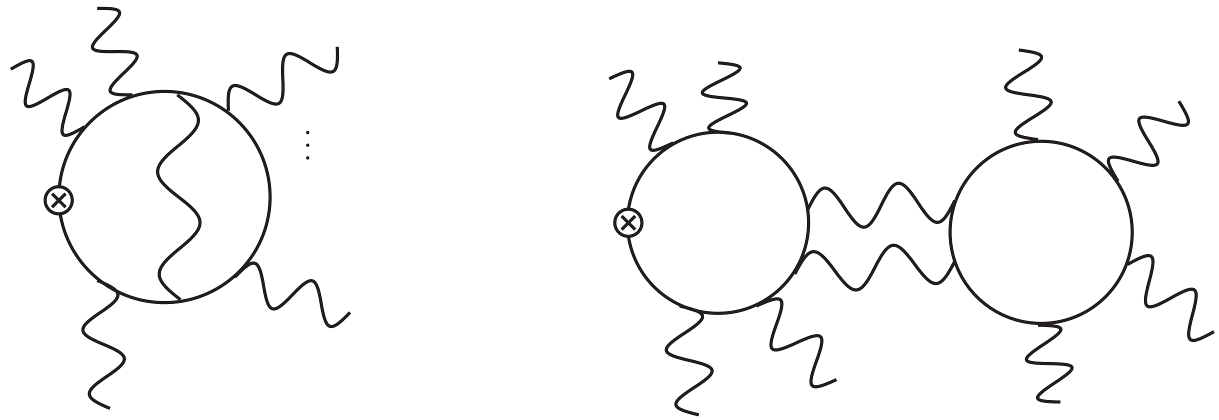
"Adler-Bardeen" like theorem

- The thermal anomaly has its contribution only at one loop (Adler-Bardeen like behavior). This is seen by noting that the photon propagator (in an arbitrary covariant gauge) at finite temperature has the form

$$D_{\mu\nu}(p) = -\left(\frac{i}{p^2 - m^2 + i\epsilon} + 2\pi n_B(|p_0|)\delta(p^2 - m^2)\right)\left(\eta_{\mu\nu} - \frac{\xi p_\mu p_\nu}{p^2}\right).$$

  where ξ is the gauge fixing parameter.

- Higher loop corrections to the anomaly can arise from diagrams of the forms



- Contracting any two photon lines leads to

$$\bar{u}^{\mu}(-p)D_{\mu\nu}(p)\bar{u}^{\nu}(p)\delta(p_{\pm}) \sim \frac{p^2}{(p_1)^2}\delta(p_{\pm}) = 0.$$

so that such diagrams cannot contribute to the anomaly.

- Physically, this can be understood from the fact that the photon field becomes massive beyond one loop and, therefore, falls off asymptotically. As a result, it cannot lead to an infrared anomaly beyond one loop.