Topological Effects of D-brane Charges in String Theory

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Introduction

- One of the fundamental objects in string theory are the D-branes to which the open string endpoints are attached.
- D-branes are charged under the RR fields of the massless spectrum of closed strings and constitute BPS states of string theory. These charges (RR-charges) satisfy Dirac's quantization condition and are sources of RR-fields [Polchinski 1995].
- RR fields are differential forms and hence one would expect that D-brane quantum charge is classified by integral cohomology.
- However the fact that D-branes leads Yang-Mills theories defined on their worldvolumes requires that RR-charge is classified by K-theory instead of cohomology. In addition to cohomology, K-theory has torsion groups.

- K-theory enters in defining the RR-charges in Type I and II superstring theories [Minasian and Moore 1997].
- Constructed upon some Sen's work on non-BPS states in string theory [Sen 1998] it was found that K-theory is the mathematical structure underlying D-brane RR charge [Witten 1998].
- A prescription specific for Type IIA theory was proposed in [Horava 1998].
- Equivariant K-theory $K_G(X)$ (Atiyah-Segal 1970) describes RR-charges in orbifold singularities [G.-C. 1998].

- Real K-theory $KR(X)$ (Atiyah 1966) describes RR-charges in a Type I D-branes or Type II orientifolds [Gukov 1999, Bergmann, Gimon and Sugimoto 2001].
- K-homology is the dual to K-theory in the same sense as homology is Poincaré dual to de Rham cohomology [Periwal 2000, Terashima 2001, Asakawa et al 2001, Reis and Szabo 2005].
- Kasparov KK-theory (Kasparov 1980) is a generalization of both, K-theory and K-homology, and it contains both theories as particular cases. Moreover, KK-theory is defined in general for noncommutative spaces (non-commutative C^*-algebras). Thus it is natural to classify charges in this framework [Asakawa et al 2002].

- In this talk a classification of D-branes charges in Type IIB Op^- orientifolds in terms of Real KK-groups is given. We classify D-branes intersecting orientifold planes from which are recovered some special limits as the spectrum for D-branes on top of Op^- orientifold arising as projections from Type IIB string theory and the bivariant classification of D-branes in Type I string theory [G.-C., Herrera, Itzá-Ortiz and Loaiza-Brito 2009].
- The gauge group and transformation properties of the low energy effective field theory living in the corresponding unstable D-brane system are computed by the use of Clifford algebras.

D-branes and K-theory

- In Type II superstring theories D-branes are constructed as solitons on unstable systems either formed by pairs of brane-antibranes or by single unstable D-branes.
- In particular, the (complex) K-theory group classifying a Dd-brane in Type IIB is given by $K(\mathbb{R}^{9-d})$ which renders the Dd-brane as a soliton constructed by the pair $D9$-$\overline{D9}$. One can instead consider a Dd-brane as a soliton constructed by an unstable system formed by lower dimensional Dq-branes ($q > d$).

- The groups classifying the corresponding vector bundles over transversal space to the Dd-brane worldvolume, in a nine-dimensional or q-dimensional unstable system, are isomorphic as expected from Bott periodicity and are given by $K^i(\mathbb{R}^{q-d}) \cong K^{i\pm 2}(\mathbb{R}^{q-d})$ respectively.
- One can see that in these theories we have

$$T(X) = \mu \sum_{i=1}^{q-d} X^i \gamma_i. \tag{1}$$

(Atiyah-Bott-Shapiro Construction).

D-branes and K-homology

In the context of Matrix theory it is possible to construct Dd-branes not from higher-dimensional unstable brane systems, but from infinitely many lower-dimensional D-branes. The idea was developed by Terashima [Terashima 2001] in order to construct commutative D-branes, which in turn are going to be classified by K-homology [Asakawa et al 2001] in the case where the lower dimensional D-branes are D-instantons.

The information is encoded in

$$T = \frac{2\pi\mu}{\alpha'^{1/2}} p,$$

$$\phi_0 = \frac{1}{2\pi\alpha'^{1/2}} x, \qquad \phi_i = 0, \quad (i = 1, \cdots, 9), \tag{2}$$

where the operators x and p are identified with the transversal coordinates and momentum of the non-BPS D(-1)-branes.

In general we can construct higher-dimensional Dd-branes in type IIB theory from an infinite number of $D(-1) - \overline{D(-1)}$ pairs, whose tachyon and scalar fields are

$$T = \mu \sum_{j=0}^{d} p_j \otimes \gamma^j,$$

$$\phi_i^{(1)} = \phi_i^{(2)} = \frac{1}{2\pi\alpha'^{1/2}} x^i, \quad (i = 0, \cdots, d),$$

with $\mu \to \infty$, and γ^j being the $2^{[\frac{d}{2}]} \times 2^{[\frac{d}{2}]}$ gamma matrices in d dimensions.

As in the *usual* case of tachyon condensation from higher dimensional non-BPS D-branes, the construction of D-branes from unstable D(-1)-branes leads to their classification in terms of the so called K-homology $K_n(X)$ (Topological K-homology of any locally compact space X classifies triples (M, E, ϕ) where M is a $Spin^c$-manifold without boundary, E is a complex bundle over M and $\phi : M \to X$ is an embedding.) which roughly speaking, is the dual to the K-theory group $K^n(X)$ in the sense that it has a natural pairing with the K-theory group. Instead of classifying vector bundles on the transverse space to a Dd-brane as in K-theory, K-homology classifies vector bundles on the worldvolume of the extended Dd-branes constructed from unstable $D(-1)$-branes.

Kasparov KK-Theory

It is then natural to combine the above two setups in order to construct a Dd-brane by a kind of combination of tachyon condensation from higher- and lower-dimensional D-branes. For branes in Type II theories, the extension was given by Asakawa, Sugimoto and Terashima [Asakawa et al 2001], together with a proposal to classify them.

In this scenario, a Dd-brane located in coordinates $x^0, \cdots, x^{q-s}, x^{q+1}, \cdots, x^{d+s}$ is constructed roughly speaking by tachyon condensation from an unstable Dq-brane located in coordinates $x^0, \cdots, x^q$ with a tachyon configuration given by

$$F = \mu \sum_{i=0}^{s} X^i \otimes \Gamma_j + \mu \sum_{j=q+1}^{d+q} p_j \otimes \Gamma^j.$$

The "part" of the Dd-brane localized inside the unstable Dq-brane is constructed by tachyon condensation as in Sen's descent relations, while the rest can be seen as constructed from unstable Dq-branes.

KK-theory is a bifunctor and constitutes a generalization of both K-theory and K-homology

$$KK(\mathcal{A},\mathcal{B}),$$

where $\mathcal{A}$ and $\mathcal{B}$ are in general two non-commutative C^*-algebras. For commutative C^*-algebras $\mathcal{A} = C^0(X) = X$ and $\mathcal{B} = C^0(Y) = Y$

$$KK(X, Y).$$

It has two limits

$$KK^{-n}(pt, Y) = K^{-n}(Y),$$

for some integer n, and

$$KK^{-n}(X, pt) = K_n(X).$$

As in the case of K-theory which is the set of equivalence classes of vector bundles, KK-theory is the set of equivalence classes of Kasparov (Connes) triplets $(\mathcal{H}, \phi, T)$. In pedestrian words:

- $\mathcal{H}$ is the set of all Chan-Paton degrees of freedom living on the worldvolume of the unstable Dq-brane
- ϕ and T are as usual the transversal position and tachyon fields.

In this sense, a zero class representing the vacuum is gathered by a tachyon field T which condensates trivially (i.e., without a kink solution in momentum or spatial configurations) implying that $T^2 = 1$ (T has been normalized) and T and ϕ depending on non-conjugate position and momentum, i.e. $[T, \phi] = 0$.

- For the case in which the tachyon condensates in a non-trivial way, it is said that the triplet is non-trivial, representing a Dd-brane configuration in which the tachyon field configuration is given by

$$F = \mu \sum_{i=0}^{s} X^i \otimes \Gamma_j + \mu \sum_{j=q+1}^{d+q} p_j \otimes \Gamma^j .$$

Hence, KK-theory is the set of triplets which are equivalent up to the addition of a zero-class triplet.

D-branes and Kasparov KK-Theory

Let us consider the simple case of a Dd-brane in Type IIB(A) string theory, constructed from unstable Dq-branes. In particular, for a configuration of a Dd-brane located in coordinates $x^0, \cdots, x^{q-s}$ $, x^{q+1}, \cdots, x^{d+s}$, the spaces X and Y are given by $\mathbb{R}^{d-q+s}$ and $\mathbb{R}^{9-q+s}$ from which the relevant KK-theory group is given by [Asakawa et al 2001]

$$KK^{0(-1)}(\mathbb{R}^{d-q+s}, \mathbb{R}^{9-q+s}) = K^{0(-1)}(\mathbb{R}^{9-d}). \tag{3}$$

It is important to stress out that it is possible to extract information of the system through the relation with complexified Clifford algebras $\mathbb{C}l^n$ given by

$$KK^{-n}(X, Y) = KK(C^0(X), C^0(Y) \otimes \mathbb{C}l^n), \tag{4}$$

where $C^0(X)$ ($C^0(Y)$) denotes the algebra of complex valued (real valued when dealing with orthogonal KK-groups) continuous functions on X (Y) vanishing at infinity. Such relation with Clifford algebras shall become very important in our description of Dd-branes in more general backgrounds.

The next natural step is to classify Dd-branes in Type I theory, i.e., in the presence of a negative RR charged orientifold nine-plane $O9^-$. This was done also by Asakawa, Sugimoto and Terashima [Asakawa et al 2002], where the authors proposed that the relevant group for such classification is the real Kasparov bifunctor, denoted as $KKO(X,Y)$, in which roughly speaking, all complex fields become real by the orientifold projection.
Let us consider a Dd-brane in an $O9^-$-plane background extended in the same coordinates as the previous case. In this situation the Kasparov KK-theory group turns out to be orthogonal (real) given by $KKO(\mathbb{R}^{d-q+s},\mathbb{R}^{9-q+s})$. Using a series of isomorphisms the above group reduces to

$$KKO^{q-1}(\mathbb{R}^{d+s-q},\mathbb{R}^{s}) = KO^{q-1}(\mathbb{R}^{q-d}) = KO(\mathbb{R}^{9-d}), \qquad (5)$$

as expected [Bergman 2000].

The relation with real Clifford algebras $Cl^{*,*}$ is given in a similar context as in Type II,

$$KKO^{q-1}(X,Y) = KKO(C^0(X), C^0(Y) \otimes Cl^{1,q}), \tag{6}$$

for which the tachyon configuration reads

$$F = u \sum_{\alpha=q-m+1}^{q} x_\alpha \otimes \Gamma^\alpha + u \sum_{\beta=q+1}^{d+m} (-i\partial_\beta) \otimes \Gamma^\beta, \tag{7}$$

where Γ^α and $-i\Gamma^\beta$ are in $M_n(\mathbb{R}) \otimes Cl^{1,q}_{odd}$ for some n, satisfying

$$\Gamma^{\alpha\dagger} = \Gamma^\alpha, \qquad (-i\Gamma^\beta)^\dagger = i\Gamma^\beta, \tag{8}$$
$$\{\Gamma^\alpha, \Gamma^{\alpha'}\} = 2\delta^{\alpha,\alpha'}, \qquad \{\Gamma^\beta, \Gamma^{\beta'}\} = 2\delta^{\beta,\beta'}, \qquad \{\Gamma^\alpha, \Gamma^\beta\} = 0.$$

D-branes in orientifold planes

An orientifold in Type II superstring theory is the loci of fixed points under the action $\mathcal{I}_{9-p}$ of a discrete symmetry group which reverses the transverse $(9-p)$ coordinates, and that of Ω that reverses the string worldsheet orientation. In other words, an orientifold Op^- is given by the plane

$$\mathbb{R}^{p+1} \times (\mathbb{R}^{9-p}/\Omega \cdot \mathcal{I}_{9-p} \cdot J),$$

where J is given by

$$J = \begin{cases} 1 & p = 0, 1 \ mod\ 4, \\ (-1)^{F_L} & p = 2, 3 \ mod\ 4, \end{cases}$$

where F_L is the left fermionic number operator.

There are at least two types of orientifolds with the underlying geometry Op^+ and Op^-. They carry a RR charge equal to $+2^{p-5}$ and -2^{p-5} respectively in D-brane charge units. (These two kinds of orientifolds appear as T-dual versions of the nine-dimensional orientifold planes $O9^+$ and $O9^-$ respectively.) The $O9^-$ orientifold is the usual Type I string theory with gauge group $SO(32)$ and the $O9^+$ orientifold is a string theory with gauge group $USp(32)$. The physical states in a orientifold plane are those of the quantized open string with eigenvalue 1 under the action of the generator $\widehat{\Omega}$ of the symmetry group, which acts on the Chan-Paton factors as

$$\widehat{\Omega} : |\Psi, ij\rangle \to (\gamma_{\widehat{\Omega}})_{ii'} |\widehat{\Omega} \cdot \psi, j'i'\rangle (\gamma_{\widehat{\Omega}}^{-1})_{j'j},$$

with $\gamma_{\widehat{\Omega}}$ being a unitary of a representation of the symmetry group. It can be shown that $\gamma_{\widehat{\Omega}}$ must satisfy

$$\gamma_{\widehat{\Omega}}^T = \gamma_{\widehat{\Omega}} \qquad \text{or} \qquad \gamma_{\widehat{\Omega}}^T = -\gamma_{\widehat{\Omega}}.$$

When $\gamma_{\widehat{\Omega}}^T = \gamma_{\widehat{\Omega}}$ we have symplectic Chan-Paton factors and when $\gamma_{\widehat{\Omega}}^T = -\gamma_{\widehat{\Omega}}$ we have orthogonal Chan-Paton factors. Hence, in the specific case of nine-dimensional orientifolds we have the $USp(32)$ string theory and the usual Type I string theory respectively. This means that 9-branes are quantized with symplectic Chan-Paton factors in $USp(32)$ string theory (and therefore in all of its T-dual string theories) and with orthogonal Chan-Paton factors in Type I string theory (and in all of its T-duals). A complete classification of D-branes and fluxes in orientifolds [G.-C. and Loaiza-Brito 2003].

Orientifolds and Real K-theory

We focus on Type IIB Op^- orientifolds with $G = \mathbb{Z}_2$, so that the spacetime is equipped with an involution τ (such that $\tau^2 = 1$) and the action of the world-sheet orientation reversal operator Ω i.e. $p = 0, 1$ *mod* 4.
It was shown by Gukov [Gukov 1999], that D_d-branes on top of these orientifold backgrounds i.e $d < p$ are classified by

$$KR(\mathbb{R}^{9-p,p-d}) \cong KO(\mathbf{S}^{2p-d-1}),$$

where

$$\mathbb{R}^{9-p,p-d} = (\mathbb{R}^{9-p}/\Omega \cdot \mathcal{I}_{9-p} \cdot J) \times \mathbb{R}^{p-d},$$

Dd-branes from unstable Dq-branes in orientifold backgrounds and K-theory

K-theory group classifies D-branes with respect to a general unstable Dq-brane system. Hence a K-theory group which classifies Dd-branes on top of an orientifold plane.
The general situation can be divided in two different configurations:

1. The Op-plane is immersed in the unstable Dq-brane, i.e., $q \geqslant p$.
2. The opposite situation in which $p \geqslant q$. We concentrate on those cases in which the Dd-brane is totally immersed in the orientifold plane. More general cases are taken into account in the KK-theory formalism.

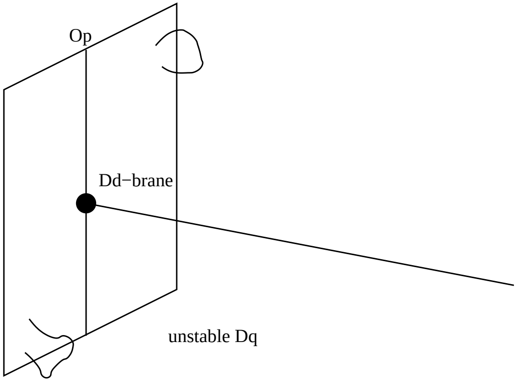
- **Case 1:** $q \geqslant p$. The important issue is to construct the transversal space to the Dd-brane. It is easy to see that such space is given by $\mathbb{R}^{(9-q)+(q-p),p-d}$, from which we can construct the associated K-theory group as $KR(\mathbb{R}^{(9-q)+(q-p),p-d})$. By using the following relations for KR

$$
\begin{aligned}
KR(\mathbb{R}^{0,m}) &= KO(\mathbf{S}^m), \\
KR(\mathbb{R}^{n,m}) = KR^{n,0}(\mathbb{R}^{0,m}) &= KR^{0,m}(\mathbb{R}^{n,0}), \\
KR^{n,m}(X) &= KR(X \times \mathbb{R}^{n,m}), \\
KR^{n,m}(X) = KR^{n-m}(X) &= KR^{n-m\pm 8}(X).
\end{aligned} \tag{9}
$$

We can rewrite the K-theory group as

$$KR^{1-q}(\mathbb{R}^{q-p,p-d}), \tag{10}$$

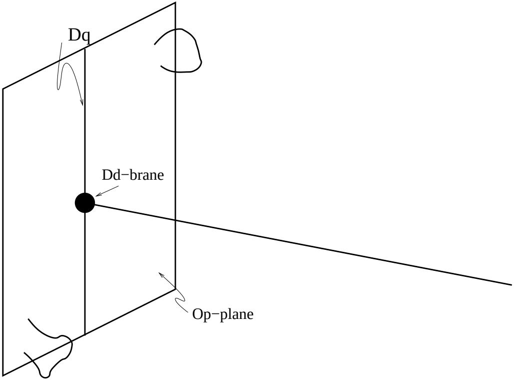
where $\mathbb{R}^{q-p,p-d}$ is the transverse space of the Dd-brane respect the unstable Dq-brane system.


Op
Dd−brane
unstable Dq

- **Case 2:** $p \geqslant q$. For the orientifold plane containing the unstable Dq-brane, the transversal space for the Dd-brane is $\mathbb{R}^{9-p,(p-q)+(q-d)}$, for which the corresponding K-theory group is

$$KR(\mathbb{R}^{9-p,p-d}) = KR^{9-2p+q}(\mathbb{R}^{0,q-d}), \tag{11}$$

where we have again used the isomorphisms for KR in the left hand side. Notice that the K-theory group written in such a way, allows us to identify the space $\mathbb{R}^{0,q-d}$ as the transversal one to the Dd-brane with respect to the Dq-brane, as in case 1.


Dq
Dd−brane
Op−plane

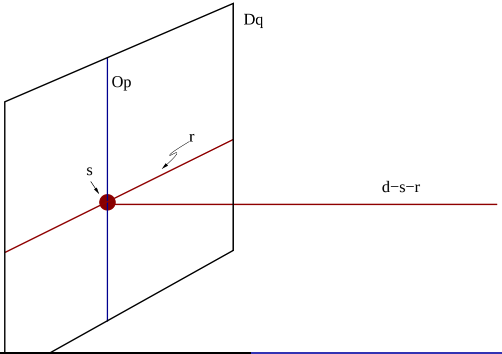
Now, let us check if the above two formulae are consistent with what we already know. Essentially we have two limits to check. First of all, if $q = p = 9$ we reproduce immediately the known formula which classifies Dd-branes in Type I theory, i.e., Gukov's formula $KO(\mathbb{R}^{9-d})$. The second limit to recover is Bergman's formula for Dd-branes in Type I theory, from unstable Dq-branes [Bergman 2000]. Hence in this case, $p = 9$ but different from q. In such a case, the related K-theory group reads

$$KR^{q-1}(\mathbb{R}^{0,q-d}) = KO^{q-1}(\mathbb{R}^{q-d}), \tag{12}$$

which indeed validates our proposal.

D-branes in orientifolds and Real KK-theory

Let us start by identifying the spaces X and Y'. There are two different configurations according to the relative values between q and p, i.e., whether the plane Op^{-} is immersed in the unstable Dq-brane ($q > p$) or viceversa ($p > q$) [G.-C., Herrera, Itzá-Ortiz and Loaiza-Brito 2009]



We fix our notation by claiming that the final Dd-brane is located in coordinates $x^0, \cdots, x^s, x^{p+1}, \cdots, x^{p+r}, x^{q+1}, \cdots, x^{q+d-s-r}$. Notice also that our assumption is that the subspace of the Dd-worldvolume of dimension $(s+r)$ is created by the usual tachyon condensation from the Dq-brane, while the subspace of dimension $(d-s-r)$ is gathered from tachyon condensation as in the K-matrix theory. Therefore, the transversal space Y' to the subspace of dimension $(r+s)$ is given by $\mathbb{R}^{(9-q)+(q-p-r),p-s}$, while the subspace X with dimension $(d-s-r)$ is $\mathbb{R}^{d-s-r,0}$. Hence it follows that the KKR-group classifying Dd-branes in this configuration is given by

$$KKR(\mathbb{R}^{d-s-r,0}, \mathbb{R}^{(9-q)+(q-p-r),p-s})$$

$$= KKR^{1-q}(\mathbb{R}^{d-s-r,0}, \mathbb{R}^{q-p-r,p-s}). \tag{13}$$

We can see that $Y = \mathbb{R}^{q-p-r,p-s}$.

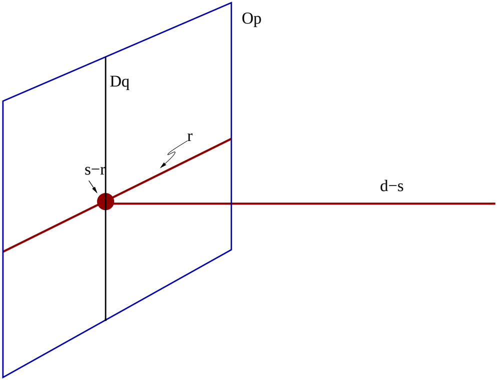


Figure: *Dq-brane dimensionally lower than the orientifold plane Op.*

Let us now focus in our second configuration, i.e., the case in which the Dq-brane is immersed in the orientifold plane Op^{-} as depicted in Figure ($p > q$). Notice that in this case, there are some transversal coordinates of the Dd-brane with respect to the Dq-brane which are extended also inside the orientifold plane. We consider the Dd-brane to be extended in coordinates $x^0, x^1, \cdots, x^{s-r}, x^{q+1}, \cdots, x^{q+r}, x^{p+1}, \cdots, x^{p+d-s}$, while the unstable D$q$ and the orientifold are extended in coordinates labeled by their dimensions.
Hence the transversal space Y' is $\mathbb{R}^{9-p,p-q+(q+r-s)}$, while the space X is given by $\mathbb{R}^{d-s,r}$ such that the relevant KKR-group is

$$KKR(\mathbb{R}^{d-s,r}, \mathbb{R}^{9-p,p-q+(q+r-s)}) = KKR^{9-2p+q}(\mathbb{R}^{d-s,r}, \mathbb{R}^{0,q+r-s}). \quad (14)$$

Since we are working with $p = 1, 5, 9$, this last group reduces to $KKR^{q-1}(\mathbb{R}^{d-s,r}, \mathbb{R}^{0,q+r-s})$. Notice that in this way, we can identify the second entrance in the bifunctor $\mathbb{R}^{0,q+r-s}$ as the Dd-brane's transversal space within the Dq-brane.

There are actually two special limits we want to consider. In Type I theory one has $p = 9$ and we should recover the results given in [Asakawa et al 2002]. Indeed, in this case ($p > q$) we get that $s = d$ and from (14)

$$\begin{aligned} KKR^{q-1}(\mathbb{R}^{d-s,r}, \mathbb{R}^{0,q+r-s}) &= KKR^{q-1}(pt, \mathbb{R}^{0,q-d}) \\ &= KR^{q-1}(\mathbb{R}^{0,q-d}) \\ &= KO^{q-1}(\mathbb{R}^{q-d}) \\ &= KKO^{q-1}(\mathbb{R}^{d+m-q}, \mathbb{R}^{m}), \quad (15) \end{aligned}$$

where $m = q - d + r$ is the codimension between the part of the Dd-brane inside of the unstable Dq-brane system.

The second limit we want to check is that of a Dd-brane located on top of an orientifold with $p \neq 9$ and for this we can take the case $q \geq p$. From the corresponding Figure this configuration is equivalent to set $d = s$ and $r = 0$ in (13). Hence we have that $X = pt$ and

$$\begin{aligned} KKR^{1-q}(\mathbb{R}^{d-s-r,0}, \mathbb{R}^{q-p-r,p-s}) &= KKR^{1-q}(pt, \mathbb{R}^{q-p,p-d}) \\ &= KR^{1-q}(\mathbb{R}^{q-p,p-d}). \end{aligned} \quad (16)$$

Extra information??
Notice that (14) can be written as $KKR(\mathbb{R}^{d-s,0}, \mathbb{R}^{9-p,p-s})$ which satisfies

$$KKR(\mathbb{R}^{d-s,0}, \mathbb{R}^{9-p,p-s}) = KO(\mathbb{R}^{2p-2s+d-1}), \quad (17)$$

from which we can see that specific values of q and r are not important. This means that it does not matter which unstable brane we select to construct a Dd-brane, but how many coordinates s of the Dd-brane are inside the orientifold plane.

Example 1

It is easy to check that the orthogonal KO-group in (17) classifies D-branes in a Type I T-dual version. To see this, consider for instance a D3-brane in coordinates (x^0, x^1, x^2, x^3), and an orientifold $O1^-$ located in coordinates x^0, x^1. By applying T-duality on coordinates x^2-x^9, one gets a D7-brane in Type I theory. Such a brane is classified by $KO(\mathbb{R}^2) = \mathbb{Z}_2$. Now let us check if Eq. (17) leads us to the same group. In this case, $q \geq p$ and the configuration is similar to that depicted in the corresponding Figure. It turns out that $s = p = 1$ and

1. $r = 2$ and $d - s - r = 0$, or
2. $r = 1$ and $d - s - r = 1$, or
3. $r = 0$ and $d - s - r = 2$.

For all cases, Eq. (17) gives the group $KO(\mathbb{R}^2)$ in agreement with T-duality. The same applies for all different configuration of D-branes and orientifold planes. The KO-theory groups from Eq. (17) classifies the T-dual version in Type I theory.
One can try to do the same for other type of orientifolds, like the ones with a negative square involution (positive RR charge) and for orientifolds in Type IIA. However, for such cases the related KK groups are not well known from the mathematical point of view. Hence, we can only establish some expected properties for such groups based on physical motivation.

Unstable non-BPS D-branes in orientifolds and KKR-theory

Our proposal for the classification of Dd-branes in Op^--plane background is given according whether $q > p$ or $p > q$.
For these cases one can write down

$$KKR^{1-q}(X,Y) = KKR_{q-1}(X,Y)$$

$$= \begin{cases} KKR(C^0(X;\mathbb{C}) \otimes {}^p\mathbb{C}l^{0,q-1}, C_0(Y;\mathbb{C})) & q-1>0, \\ KKR(C^0(X;\mathbb{C}) \otimes {}^p\mathbb{C}l^{1-q,0}, C^0(Y;\mathbb{C})) & 1-q>0, \end{cases} \quad (18)$$

for $p < q$ while for $p > q$, we have

$$KKR^{q-1}(X,Y) = KKR_{1-q}(X,Y)$$

$$= \begin{cases} KKR(C^0(X;\mathbb{C}), C^0(Y;\mathbb{C}) \otimes {}^p\mathbb{C}l^{1-q,0}) & 1-q>0, \\ KKR(C^0(X;\mathbb{C}), C^0(Y;\mathbb{C}) \otimes {}^p\mathbb{C}l^{0,q-1}) & q-1>0. \end{cases} \quad (19)$$

The Real involution and orientifolds

Let us now describe explicitly how the real involution induced by the orientifold Op^{-}-plane acts on the generators of the Clifford algebras entering in the definition of the KKR-groups.
The complexified Real Clifford algebra is defined as

$$ {}^{p}\mathbb{C}l^{n,m} = {}^{p}\left(Cl^{n,m} \otimes \mathbb{C}\right). \tag{20} $$

Since the involution acts as conjugation on the complex part we can write

$$ {}^{p}\left(Cl^{n,m} \otimes \mathbb{C}\right) = {}^{p}Cl^{n,m} \otimes \overline{\mathbb{C}}, \tag{21} $$

where $\overline{\mathbb{C}}$ denotes the field of complex numbers with Real involution defined by usual complex conjugation and ${}^{p}Cl^{n,m}$ denotes the Clifford algebra $Cl^{n,m}$ with some Real involution (again, this involution is determined by the orientifold plane on the generators of the algebra and extended by linearity). Thus, it suffices to study the involution in the real part ${}^{p}Cl^{n,m}$. Hence, we shall concentrate on how to fix the involution inhereted from the orientifold Op^{-}-plane on the generators of the real Clifford algebra.

According to Eqs. (18) and (19) the complex Clifford algebras with involution we use, are of the form ${}^{p}\mathbb{C}l^{n,0}$ or ${}^{p}\mathbb{C}l^{0,n}$. Hence we shall concentrate on the involution on their associated orthogonal (real) Clifford algebras, whose generators can be identified with spatial coordinates via the vector space isomorphism

$$Cl^{n,0} \cong \Lambda^{*}\mathbb{R}^{n} \cong Cl^{0,n}. \tag{22}$$

Let us consider the case in which $1-q<0$ such that the related Clifford algebra is ${}^{p}\mathbb{C}l^{0,q-1}$. By the above isomorphism we identify the generators e_i, $(i=1,\cdots,q-1)$ of the Clifford algebra $Cl^{0,q-1}$ with vectors of the little group $SO(q-1)$ of a Dq-brane.

Hence, the involution inhereted form the orientifold p-plane, denoted as $\mathcal{I}_{9-p}$ acts on the generators of the complex Clifford algebra as in the longitudinal coordinates x^i to the Dq-brane. Because of this, the involution depends on the relative value between p and q. Then, if $q < p$ we consider a Dq-brane inside the orientifold plane, as in the following configuration

	0	1	2	$\cdots$	q-1	q	q+1	$\cdots$	p-1	p	p+1	$\cdots$	8	9
Op^-	—	—	—	—	—	—	—	—	—	—	$\times$	$\times$	$\times$	$\times$
Dq	—	—	—	—	—	—	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

inducing a trivial involution on all the Clifford algebra generators

$$\mathcal{I}_{9-p}(e_i) = e_i \quad \text{for all } i. \tag{23}$$

On the other hand, if $q > p$, the unstable Dq-brane is located as

	0	1	2	$\cdots$	p-1	p	p+1	$\cdots$	q-1	q	q+1	$\cdots$	8	9
Op^-	$-$	$-$	$-$	$-$	$-$	$-$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
Dq	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$\times$	$\times$	$\times$	$\times$

and the involution is given by

$$\mathcal{I}_{9-p}(e_i) = \begin{cases} e_i & \text{for} \quad i = 1, ..., p, \\ -e_i & \text{for} \quad i = p+1, ..., q. \end{cases} \tag{24}$$

By Bott periodicity we have

$$KKR(X,\ {}^{p}\mathbb{C}l^{0,q-1}) = KKR(X,\ {}^{p}\mathbb{C}l^{9-q,0}). \tag{25}$$

In this way, we can use instead the $(9-q)$ generators e_i of ${}^{p}\mathbb{C}l^{9-q,0}$, with $i = q+1, \cdots, 9$ which are identified with the transversal coordinates to the Dq-brane. The involution is again dependent on the relative values between q and p. For $q < p$ we have

$$\mathcal{I}_{9-p}(e_i) = \begin{cases} e_i \quad \text{for} \quad i = 1, \cdots, p-q, \\ -e_i \quad \text{for} \quad i = p-q+1, ..., 9-q, \end{cases} \tag{26}$$

while for $q > p$ we have

$$\mathcal{I}_{9-p}(e_i) = -e_i \qquad \text{for all } i. \tag{27}$$

Therefore, one sees that for $q > 2$, we have at least two different ways to identity the Clifford algebra generators with spatial coordinates i.e. internal or transversal coordinates to the Dq-brane system. For each identification there are two choices for the involution on the Clifford generators, depending on the relative value of q and p. However we also see that for $q < p$ is simpler to establish the identification with internal coordinates to the Dq-brane, while for the case $p < q$ is the opposite. We shall adopt this identification henceforth.

Although the identifications are not so geometric for $q < 2$, we have similar involutions. For $q = -1$ the relevant Clifford algebra is ${}^{p}Cl^{2,0}$ and the involution acts on the generators as $\mathcal{I}_{9-p}(e_i) = e_i$ $(i = 1, 2)$. Similarly for $q = 0$, the Clifford algebra is ${}^{p}Cl^{1,0}$ and the involution acts also trivially on the generator.

Non-BPS D-branes in orientifold backgrounds

Now, we are going to get the representations of the tachyon, gauge and scalar fields from the corresponding Clifford algebras and we will show that they correspond to the properties of unstable non-BPS Dd-branes classified by the groups in Eqs. (13) and (14). Due to Bott periodicity in $KKR^n(X,Y) \sim KKR^{n\pm 8}(X,Y)$, all cases are considered within the range $-4 \leq n \leq 4$. However, in contrast with D-branes in Type I theory, the involution acts different for a Dq-brane than for a D$(q+8)$-brane. Notice as well that, although Eqs. (13) and (14) do not depend on p (in these kinds of non-BPS branes), the involution does.

Example 2

Consider for instance the case of a non-BPS D8-brane and an $O1^-$-plane in a configuration as follows

	0	1	2	3	4	5	6	7	8	9
$O1^-$	$-$	$-$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$D8$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$\times$.

The corresponding group is $KKR^{-7}(pt, pt) \sim KKR^1(pt, pt)$ with an associated Real Clifford algebra ${}^1\mathbb{C}l^{1,0}$. The action of the involution on the generator of ${}^1Cl^{1,0}$ is given by

$$\mathcal{I}_8(e_1) = -e_1. \qquad (28)$$

This determines the fixed point algebra for ${}^1\mathbb{C}l^{1,0}$ and hence, the corresponding representation for the tachyon, gauge and scalar fields. By imposing the condition $\mathcal{I}_8(a) = a$ for $a \in {}^1\mathbb{C}l^{1,0}$, one gets that

$$\left({}^1\mathbb{C}l^{1,0}\right)_{\text{fix}} = Cl^{0,1},$$

which fixes the tachyon T and the scalar field ϕ to be symmetric tensor representations $\square\square$ of the gauge group $O(\infty)$. These results correspond to the field content of an unstable non-BPS D2-brane in Type I theory. This is in agreement with formula (13) since for this case $p = s = 1$, $d = q = 8$ and $r = 7$, and the relevant KKR group is given by

$$KKR^{-7}(pt, pt) = KO(\mathbb{R}^7) = 0,$$

which indeed is the K-theory group which classifies D2-branes in Type I theory. One can as well check that under T-duality on transversal coordinates to the $O1^-$-plane, the unstable D8-brane transforms into a D2-brane in Type I theory. Notice that the involution does not change for $p = 5$, for which we get the same field content for a D8 in an $O5^-$-plane.

Example 3

Contrary to the case in Type I theory, the field content for a non-BPS D0-brane in an $O1^-$-plane

	0	1	2	3	4	5	6	7	8	9
$O1^-$	$-$	$-$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$D0$	$-$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

should not be the same than for a D8. This is obtained by realizing that for a D0-brane, although the Real Clifford algebra also is ${}^1\mathbb{C}l^{1,0}$, the involution on the single one generator e_1 is trivial, $\mathcal{I}_8(e_1) = e_1$. This implies that

$$\left({}^1\mathbb{C}l^{1,0} \right)_{\text{fix}} = Cl^{1,0}. \tag{29}$$

Therefore, the tachyon field T and the scalar field ϕ are antisymmetric $\yng(1,1)$ and symmetric $\yng(2)$ tensor representations, respectively, of the gauge group $O(\infty)$. This field content is precisely that of an unstable non-BPS D0-brane in Type I theory. This also is in agreement with formula (14) in which $r = s = q = d = 0$ and $p = 1$, implying

$$KKR^{-1}(pt, pt) = KO(\mathbb{R}^1) = \mathbb{Z}_2, \tag{30}$$

which classifies D8-branes in Type I theory. Indeed, the configuration of a D0-brane in an $O1^-$-plane is T-dual to a D8 in an $O9^-$-plane. For $p = 5$, the involution is the same and we get the same group.

Example 4

Another interesting situation presents for $q = 5$, i.e., D5-branes in $O1^-$ and $O5^-$-planes. The Real Clifford algebra is given by ${}^{p}\mathbb{C}l^{4,0}$ for $p = 1, 5$. In this case, the involution acts as $\mathcal{I}_4(e_i) = -e_i$ for $i = 2, 3, 4, 5$. As a consequence, the fixed point algebra is $Cl^{0,4}$. For this case, we can also take the Real Clifford algebra as ${}^{p}\mathbb{C}l^{4,0} = {}^{p}\mathbb{C}l^{0,4}$. However the involution acts trivially on the corresponding generators. The fixed point algebra is then $Cl^{4,0}$. It is easy to check that $Cl^{4,0} = Cl^{0,4}$. Hence, as it was shown in [Asakawa 2002], the tachyon and scalars fields transforms in the bifundamental and antisymmetric tensor representations of the gauge group $Sp(\infty) \times Sp(\infty)$. This is the field content of a pair D5-$\overline{\text{D5}}$ branes in Type I theory, which agrees with the result given by

$$KKR^{-4}(pt, pt) = KO(\mathbb{R}^4) = KSp(pt) = \mathbb{Z}. \qquad (31)$$

The complete set of Real Clifford algebras for all unstable non-BPS branes is summarized in Table 1.

	Dd	${}^p\mathbb{C}l^{n,m}$	$({}^pCl^n)_{\text{fix}}$	KKR^n	$KO^n(pt)$	T-dual in Type I
$p=1$ $p=5$	D(-1)	${}^1\mathbb{C}l^{2,0}$ ${}^5\mathbb{C}l^{2,0}$	$Cl^{2,0}$	KKR^{-2} KKR^{-10}	$KO^{-2}=\mathbb{Z}_2$ $KO^{-10}=\mathbb{Z}_2$	D7 D(-1)
	D0	${}^1\mathbb{C}l^{1,0}$ ${}^5\mathbb{C}l^{1,0}$	$Cl^{1,0}$	KKR^{-1} KKR^{-9}	$KO^{-1}=\mathbb{Z}_2$ $KO^{-9}=\mathbb{Z}_2$	D8 D0
	D1-$\overline{\text{D1}}$	${}^1\mathbb{C}l^{1,1}$ ${}^5\mathbb{C}l^{1,1}$	$Cl^{1,1}$	KKR^{0} KKR^{-8}	$KO^{0}=\mathbb{Z}$ $KO^{-8}=\mathbb{Z}$	D9-$\overline{\text{D9}}$ D1-$\overline{\text{D1}}$
	D2	${}^1\mathbb{C}l^{0,1}$ ${}^5\mathbb{C}l^{0,1}$	$Cl^{1,0}$ $Cl^{0,1}$	KKR^{-1} KKR^{-7}	$KO^{-1}=\mathbb{Z}_2$ $KO^{-7}=0$	D8 D2
	D3	${}^1\mathbb{C}l^{0,2}$ ${}^5\mathbb{C}l^{0,2}$	$Cl^{2,0}$ $Cl^{0,2}$	KKR^{-2} KKR^{-6}	$KO^{-2}=\mathbb{Z}_2$ $KO^{-6}=0$	D7 D3
	D4	${}^1\mathbb{C}l^{0,3}$ ${}^5\mathbb{C}l^{0,3}$	$Cl^{3,0}$ $Cl^{0,3}$	KKR^{-3} KKR^{-5}	$KO^{-3}=0$ $KO^{-5}=0$	D6 D4
	D5-$\overline{\text{D5}}$	${}^1\mathbb{C}l^{0,4}$ ${}^5\mathbb{C}l^{0,4}$	$Cl^{4,0}$	KKR^{-4}	$KO^{-4}=\mathbb{Z}$	D5+$\overline{\text{D5}}$
	D6	${}^1\mathbb{C}l^{3,0}$ ${}^5\mathbb{C}l^{3,0}$	$Cl^{0,3}$	KKR^{-5}	$KO^{-5}=0$	D4
	D7	${}^1\mathbb{C}l^{2,0}$ ${}^5\mathbb{C}l^{2,0}$	$Cl^{0,2}$	KKR^{-6}	$KO^{-6}=0$	D3

Dq	$Cl^{n,m}$	ϕ	T	Gauge group
D(-1) D7	$Cl^{2,0}$	*adj.*	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$U(\infty)$
D0 D8	$Cl^{1,0}$	$\square\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$O(\infty)$
D1 D9	$Cl^{1,1}$	$(1,\square\square),(\square\square,1)$	$(\square,\bar{\square})$	$O(\infty)\times O(\infty)$
D2	$Cl^{0,1}$	$\square\square$	$\square\square$	$O(\infty)$
D3	$Cl^{0,2}$	*adj.*	$\square\square$	$U(\infty)$
D4	$Cl^{0,3}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square\square$	$Sp(\infty)$
D5	$Cl^{4,0}$	$(1,\begin{smallmatrix}\square\\\square\end{smallmatrix}),(\begin{smallmatrix}\square\\\square\end{smallmatrix},1)$	$(\square,\bar{\square})$	$Sp(\infty)\times Sp(\infty)$
D6	$Cl^{3,0}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$Sp(\infty)$

Table: Field content of unstable non-BPS Dq-branes in Type I theory and the relevant real Clifford algebra.

Conclusions

- In this talk, we have extended to Type IIB orientifold Op^--orientifold backgrounds the KK-theory formalism proposed recently in the literature for Type IIB and Type I string theory respectively.
- In particular, for the orientifold case, we considered Op^--planes with $p = 1, 5, 9$, for which the presented formalism naturally incorporates stable D-branes intersecting the orientifold planes, generalizing in this sense the proposal in by (Gukov 1999).

- Specifically we propose that Dd-branes intersecting Op^{-}-planes are given by the groups in Eqs.(13) and (14)

$$KKR(\mathbb{R}^{d-s-r,0},\mathbb{R}^{(9-q)+(q-p-r),p-s})$$
$$= KKR^{1-q}(\mathbb{R}^{d-s-r,0},\mathbb{R}^{q-p-r,p-s}) \quad \text{for } p<q$$
$$KKR(\mathbb{R}^{d-s,r},\mathbb{R}^{9-p,p-q+(q+r-s)})$$
$$= KKR^{9-2p+q}(\mathbb{R}^{d-s,r},\mathbb{R}^{0,q+r-s}) \quad \text{for } p>q.$$

- In order to show that these groups correctly classify the corresponding configurations of D-branes and orientifolds, we also compute, by extensive use of the Clifford algebras and the structures defined on them, the gauge group and transformation properties of the effective fields living in the worldvolume of the unstable Dq-branes. The transformation properties of the tachyon and scalar fields of this unstable systems are read off from the *fixed point Clifford algebra*.