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Non-Abelian Tensor Multiplet in Four Dimensions

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This work is done in collaboration with Dr. H. Nishino. I will present the work as follows.

- STATEMENT OF THE PROBLEM
- THE SOLUTION
- SUPERSYMMETRY
- SUPERFIELD LANGUAGE
- DISCUSS RELEVANCE OF THE WORK

  a. STANDARD MODEL

  b. ON THE QUEST FOR THE UNIFICATION OF FUNDAMENTAL FORCES

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d-bein	$e_M{}^A$	$d(d-3)/2$
Gravitino	ψ_M	$2^\alpha(d-3)$
Vector	A_M	$(d-2)$
Spinor	χ	2^α
Scalar	ϕ	1

ANTISYMMETRIC TENSOR GAUGE FIELDS

A_{MNP}	$(d-4)(d-3)(d-2)/6$
A_{MN}	$(d-3)(d-2)/2$
A_M	$(d-2)$
A	1

Table 1: Degrees of freedom in d dimensions. For Dirac spinors $\alpha = d/2$ if d is even and $\alpha = (d-1)/2$ if d is odd. Divide by two for Majorana spinors, by two for Weyl and by four for Majorana-Weyl. Similarly divide by two for self-dual antisymmetric tensor field strengths.

1. The Problem

The basic problem with a non-Abelian tensor, when it has its own kinetic term, is easily seen as follows. Let I be the adjoint index of a non-Abelian group G, and let a non-Abelian vector field $A_\mu{}^I$ couple minimally to the antisymmetric tensor $B_{\mu\nu}{}^I$. Consider the most conventional field strength

$$G^{(0)}_{\mu\nu\rho}{}^I \equiv +3D_{[\mu}B_{\nu\rho]}{}^I \equiv +3(\partial_{[\mu}B_{\nu\rho]}{}^I + gf^{IJK}A_{[\mu}{}^J B_{\nu\rho]}{}^K) \quad , \tag{1.1}$$

where D_μ is the usual gauge-covariant derivative with the minimal coupling with the coupling constant g and the structure constant f^{IJK} of the group G. Consider a tentative action $I_0 \equiv \int d^4x\, \mathcal{L}_0$ with the lagrangian[2)]

$$\mathcal{L}_0 \equiv -\tfrac{1}{12}(G^{(0)}_{\mu\nu\rho}{}^I)^2 - \tfrac{1}{4}(F_{\mu\nu}{}^I)^2 \quad , \tag{1.2}$$

with $F_{\mu\nu}{}^I \equiv 2\partial_{[\mu}A_{\nu]}{}^I + gf^{IJK}A_\mu{}^J A_\nu{}^K$. Obviously, the B-field equation is[3)]

$$\frac{\delta\mathcal{L}_0}{\delta B_{\mu\nu}{}^I} = +\tfrac{1}{2}D_\rho G^{(0)\mu\nu\rho I} \doteq 0 \quad . \tag{1.3}$$

The problem is that the divergence of this B-field equation does *not* vanish:

$$0 \overset{?}{\doteq} D_\nu\left(\frac{\delta\mathcal{L}_0}{\delta B_{\mu\nu}{}^I}\right) = +\tfrac{1}{4}gf^{IJK}F_{\nu\rho}{}^J G^{(0)\mu\nu\rho K} \neq 0 \quad , \tag{1.4}$$

unless $F_{\mu\nu}{}^I$ or $G^{(0)}_{\mu\nu\rho}{}^I$ vanishes trivially. This inconsistency problem is already at the classical level before quantization. This is also one of the reasons, why topological formulations with vanishing field strength $F_{\mu\nu}{}^I \doteq 0$ such as [1] are easier to formulate for non-Abelian tensors.

An additional problem is related to the so-called local tensorial gauge transformation of the B-field:

$$\delta_\beta B_{\mu\nu}{}^I = +D_{[\mu}\beta_{\nu]}{}^I - D_{[\nu}\beta_{\mu]}{}^I \quad , \tag{1.5}$$

because the field strength $G_{\mu\nu}{}^I$ is *not* invariant under δ_β:

$$\delta_\beta G^{(0)}_{\mu\nu\rho}{}^I = +3gf^{IJK}F_{[\mu\nu}{}^J\beta_{\rho]}{}^K \neq 0 \quad . \tag{1.6}$$

This further implies the *non*-invariance of the action: $\delta_\beta I_0 \neq 0$. These two problems are mutually related, because the non-vanishing of (1.4) is also interpreted as the action non-invariance $\delta_\beta I_0 \neq 0$.

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2) We use the signature $(-,+,+,+)$ for four dimensions (4D) in this paper.

3) The symbol $\doteq$ stands for a field equation, to be distinguished from an algebraic identity. We also use the symbol $\overset{?}{\doteq}$ for an equality under question.

2. The Solution to Problem

The solution to the problem above is to introduce a non-trivial Chern-Simons (CS) term into the G-field strength:

$$G_{\mu\nu\rho}{}^I \equiv +3D_{[\mu}B_{\nu\rho]}{}^I \equiv +3(\partial_{[\mu}B_{\nu\rho]}{}^I + gf^{IJK}A_{[\mu}{}^J B_{\nu\rho]}{}^K) - 3f^{IJK}C_{[\mu}{}^J F_{\nu\rho]}{}^K$$
$$\equiv +G^{(0)}_{\mu\nu\rho}{}^I - 3f^{IJK}C_{[\mu}{}^J F_{\nu\rho]}{}^K \quad , \tag{2.1}$$

where $C_\mu{}^I$ is a 'compensator' vector field, also carrying the adjoint index. The field strength for C is defined by

$$H_{\mu\nu}{}^I \equiv +D_{[\mu}C_{\nu]}{}^I - D_{[\nu}C_{\mu]}{}^I + gB_{\mu\nu}{}^I \quad . \tag{2.2}$$

Now these field strengths G and H are *invariant* under the δ_β-transformation

$$\delta_\beta B_{\mu\nu}{}^I = +D_{[\mu}\beta_{\nu]}{}^I - D_{[\nu}\beta_{\mu]}{}^I \tag{2.3a}$$
$$\delta_\beta C_\mu{}^I = -g\beta_\mu{}^I \quad , \tag{2.3b}$$

which is the 'proper' gauge transformation for $B_{\mu\nu}{}^I$, and δ_γ-transformations

$$\delta_\gamma B_{\mu\nu}{}^I = -f^{IJK}F_{\mu\nu}{}^J\gamma^K \quad , \tag{2.4a}$$
$$\delta_\gamma C_\mu{}^I = D_\mu\gamma^I \quad . \tag{2.4b}$$

is the 'proper' gauge transformation for $C_\mu{}^I$.

The role played by the $C\wedge F$-term in (2.1) is to cancel the unwanted term in (1.6). The C-field itself should have its own 'gauge' transformation as the covariant gradient (2.4b). The contribution of $\delta_\gamma(2D_{[\mu}C_{\nu]}{}^I)$ in (2.2) is cancelled by the contribution of $\delta_\gamma(gB_{\mu\nu}{}^I)$, so that $\delta_\gamma H_{\mu\nu}{}^I = 0$.

In other words, we have the total invariances

$$\delta_\beta G_{\mu\nu\rho}{}^I = 0 \quad , \qquad \delta_\beta H_{\mu\nu}{}^I = 0 \quad , \tag{2.5a}$$
$$\delta_\gamma G_{\mu\nu\rho}{}^I = 0 \quad , \qquad \delta_\gamma H_{\mu\nu}{}^I = 0 \quad . \tag{2.5b}$$

Accordingly, we also have the consistency problem (1.4) solved. Consider the kinetic terms for the B, C and A-fields:

$$\mathcal{L}_1 \equiv -\frac{1}{12}(G_{\mu\nu\rho}{}^I)^2 - \frac{1}{4}(H_{\mu\nu}{}^I)^2 - \frac{1}{4}(F_{\mu\nu}{}^I)^2 \quad . \tag{2.6}$$

The total action is also invariant $\delta_\beta I_1 = \delta_\gamma I_1 = 0$. The new field equations for B and C-fields are

$$\frac{\delta \mathcal{L}_1}{\delta B_{\mu\nu}{}^I} = + \frac{1}{2} D_\rho G^{\mu\nu\rho\, I} - \frac{1}{2} g H^{\mu\nu\, I} \doteq 0 \ , \tag{2.7a}$$

$$\frac{\delta \mathcal{L}_1}{\delta C_\mu{}^I} = - D_\nu H^{\mu\nu\, I} + \tfrac{1}{2} f^{IJK} F_{\rho\sigma}{}^J G^{\mu\rho\sigma\, K} \doteq 0 \ , \tag{2.7b}$$

The divergence of the B-field equation vanishes now:

$$0 \stackrel{?}{=} D_\nu \left(\frac{\delta \mathcal{L}_1}{\delta B_{\mu\nu}{}^I} \right) = + \tfrac{1}{2} g \left(\frac{\delta \mathcal{L}_1}{\delta C_\mu{}^I} \right) \doteq 0 \ , \tag{2.8}$$

where the last equality holds because of the C-field equation. In other words, the unwanted FG-term in (1.4) is now cancelled by the contribution of the C-field equation. This has solved the previous problem (1.4).

Relevantly, the divergence of (2.10) also vanishes, as it should:

$$0 \stackrel{?}{=} D_\mu \left(\frac{\delta \mathcal{L}_1}{\delta C_\mu{}^I} \right) = + f^{IJK} F_{\mu\nu}{}^J \left(\frac{\delta \mathcal{L}_1}{\delta B_{\mu\nu}{}^K} \right) \doteq 0 \ , \tag{2.9}$$

without any inconsistency.

We emphasize repeatedly that these invariances have never been accomplished without the peculiar CS terms both in (2.1) and (2.2).

Recently, the long-standing problem with non-Abelian tensors [2] has been solved by de Wit, Samtleben, and Nicolai [3][4]. The original motivation in [3] was to generalize the tensor and vector field interactions in manifestly $E_{6(+6)}$-covariant formulation of five-dimensional (5D) maximal supergravity by gauging non-Abelian sub-groups. In [4], this work was further related to M-theory [5] by confirming the representation assignments under the duality group of the gauge charges. The underlying hierarchies of these tensor and vector gauge fields are presented with the consistency of general gaugings.

The hierarchy in [3][4] has been further applied to the conformal supergravity in 6D [6]. In ref. [6], the 'minimal tensor hierarchy' as a special case of the more general hierarchy in [3][4] has been discussed. This hierarchy consists of $A_\mu{}^r$ and two-form gauge potentials $B_{\mu\nu}{}^I$, with two labels r and I. Also introduced is the 3-form gauge potentials $C_{\mu\nu\rho\, r}$ with the index ${}_r$ is dual to ^{r} of $A_\mu{}^r$. The field strengths of vector and two-form gauge potentials are defined by [6]

$$\begin{aligned} \mathcal{F}_{\mu\nu}{}^r \ &\equiv 2\partial_{[\mu}A_{\nu]}{}^r + h_I{}^r B_{\mu\nu}{}^I \ , && \text{(1.1a)} \\ \mathcal{H}_{\mu\nu\rho}{}^I \ &\equiv 3D_{[\mu}B_{\nu\rho]}{}^I + 6d_{rs}{}^I A_{[\mu}{}^r\partial_\nu A_{\rho]}{}^s - 2f_{pq}{}^s d_{rs}{}^I A_{[\mu}{}^r A_\nu{}^p A_{\rho]}{}^q + g^{Ir}C_{\mu\nu\rho r} \ . && \text{(1.1b)} \end{aligned}$$

The prescription for tensor-vector system, which we will be based upon, is described with eq. (3.22) in [6]. To be more specific, we consider in the present paper the product of two identical gauge groups $G\times G$ [7], whose adjoint indices are respectively ${}_{r,\ s,\ \cdots}$ and ${}_{r',\ s',\ \cdots}$. Accordingly, we use the coefficients

$$\begin{aligned} f_{rs}{}^t \ &= \mathrm{f}_{rs}{}^t \ , \qquad f_{rs'}{}^{t'} = -f_{s'r}{}^{t'} = +\tfrac{1}{2}\mathrm{f}_{rs'}{}^{t'} \ , && \text{(1.2a)} \\ d^t_{rs'} \ &= d^t_{s'r} = -\tfrac{1}{2}\mathrm{f}_{rs'}{}^t \ , \qquad h^{r'}_s = \delta^{r'}_s \ , && \text{(1.2b)} \end{aligned}$$

where $\mathrm{f}_{rs}{}^t$ is the structure constant of a non-Abelian gauge group. We use the same field content arising by this prescription.

Since the outstanding paper [6] gives the extensive details of how to get our system from [3][4][7], there is nothing new to explain, except for our notational preparation. In our notation, the field strengths of the B and C-fields are respectively G and H defined by

$$\begin{aligned} G_{\mu\nu\rho}{}^I \ &\equiv +3D_{[\mu}B_{\nu\rho]}{}^I - 3f^{IJK}C_{[\mu}{}^J F_{\nu\rho]}{}^K \ , && \text{(1.3a)} \\ H_{\mu\nu}{}^I \ &\equiv +2D_{[\mu}C_{\nu]}{}^I + gB_{\mu\nu}{}^I \ . && \text{(1.3b)} \end{aligned}$$

The gauge transformations for B, C and A-fields are

$$\begin{aligned} \delta_\alpha(B_{\mu\nu}{}^I, C_\mu{}^I, A_\mu{}^I) \ &= (-f^{IJK}\alpha^J B_{\mu\nu}{}^K, \ -f^{IJK}\alpha^J C_\mu{}^K, \ +D_\mu\alpha^I) \ , && \text{(1.4a)} \\ \delta_\beta(B_{\mu\nu}{}^I, C_\mu{}^I, A_\mu{}^I) \ &= (+2D_{[\mu}\beta_{\nu]}{}^I, \ -g\beta_\mu{}^I, \ 0) \ , && \text{(1.4b)} \\ \delta_\gamma(B_{\mu\nu}{}^I, C_\mu{}^I, A_\mu{}^I) \ &= (-f^{IJK}F_{\mu\nu}{}^J\gamma^K, \ D_\mu\gamma^I, \ 0) \ . && \text{(1.4c)} \end{aligned}$$

As (1.3b) or (1.4b) shows, $C_\mu{}^I$ is a vectorial Stueckelberg field, absorbed into the longitudinal component of $B_{\mu\nu}{}^I$. Due to the general hierarchy [3][4], all field strengths are invariant:

$$\delta_\alpha(G_{\mu\nu\rho}{}^I, \ H_{\mu\nu}{}^I, \ F_{\mu\nu}{}^I) = -f^{IJK}\alpha^J(G_{\mu\nu\rho}{}^K, \ H_{\mu\nu}{}^K, \ F_{\mu\nu}{}^K) \ , \qquad \text{(1.5a)}$$

$$\delta_\beta(G_{\mu\nu\rho}{}^I, \ H_{\mu\nu}{}^I, \ F_{\mu\nu}{}^I) = 0 \ , \qquad \delta_\gamma(G_{\mu\nu\rho}{}^I, \ H_{\mu\nu}{}^I, \ F_{\mu\nu}{}^I) = 0 \ . \qquad \text{(1.5b)}$$

Since the hierarchy given in [3][4] guarantees the gauge invariance of all field strengths, the construction of purely bosonic lagrangian is straightforward. Consider the action $I_1 \equiv \int d^4x\, g^2 \mathcal{L}_1$ [4] with

$$\mathcal{L}_1 \equiv -\tfrac{1}{12}(G_{\mu\nu\rho}{}^I)^2 - \tfrac{1}{4}(H_{\mu\nu}{}^I)^2 - \tfrac{1}{4}(F_{\mu\nu}{}^I)^2 \quad . \tag{1.6}$$

The gauge invariances of all field strength also guarantee the consistency of the A, B and C-field equations, such as the divergence $D_\nu(\delta\mathcal{L}_1/\delta B_{\mu\nu}{}^I) \doteq 0$.[5] Since we will do similar confirmation for supersymmetric system later, we skip the details for the purely bosonic system.

The purpose of our present paper is to supersymmetrize this system. The rest of our paper is organized as follows. In section 2, we give the component formulation of $N=1$ tensor multiplet (TM). In section 3, we give the superspace re-formulation of component result. In section 4, we give the generalization to non-adjoint representation of $G = SO(N)$ case. In section 5, we give the supergravity coupling to non-Abelian TM, as supporting evidence for the consistency of the global case. Section 6 is for concluding remarks. Appendix A is devoted to purely bosonic systems of non-Abelian tensors with much simpler structures than has been presented in arbitrary space-time dimensions with arbitrary signature. An example of tensor-vector duality $G = F^*$ in $D = 2+4$ dimensions, and its dimensional reduction (DR) into the self-dual YM $F = F^*$ in $D = 2+2$ is also presented.

3. Component Formulation of N=1 TM

The supersymmetrization of the purely bosonic system (1.6) is rather straightforward, except for subtlety to be mentioned later. Our system has three multiplets: (i) A TM $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$, (ii) A compensating vector multiplet (CVM) $(C_\mu{}^I, \rho^I)$, and (iii) A Yang-Mills vector multiplet (YMVM) $(A_\mu{}^I, \lambda^I)$. Our total action $I \equiv \int d^4x\, g^2 \mathcal{L}$ has the lagrangian

$$\begin{aligned}\mathcal{L} = & -\tfrac{1}{12}(G_{\mu\nu\rho}{}^I)^2 + \tfrac{1}{2}(\overline{\chi}^I \not{D}\chi^I) - \tfrac{1}{2}(D_\mu\varphi^I)^2 - \tfrac{1}{2}g^2(\varphi^I)^2 - g(\overline{\chi}^I\rho^I) \\ & - \tfrac{1}{4}(H_{\mu\nu}{}^I)^2 + \tfrac{1}{2}(\overline{\rho}^I \not{D}\rho^I) - \tfrac{1}{4}(F_{\mu\nu}{}^I)^2 + \tfrac{1}{2}(\overline{\lambda}^I \not{D}\lambda^I) \\ & - \tfrac{1}{2}g f^{IJK}(\overline{\lambda}^I\chi^J)\varphi^K + \tfrac{1}{2}f^{IJK}(\overline{\lambda}^I\gamma^\mu\rho^J)D_\mu\varphi^K + \tfrac{1}{12}f^{IJK}(\overline{\lambda}^I\gamma^{\mu\nu\rho}\rho^J)G_{\mu\nu\rho}{}^K \\ & + \tfrac{1}{4}f^{IJK}(\overline{\rho}^I\gamma^{\mu\nu}\chi^J)F_{\mu\nu}{}^K - \tfrac{1}{4}f^{IJK}(\overline{\lambda}^I\gamma^{\mu\nu}\chi^J)H_{\mu\nu}{}^K - \tfrac{1}{2}f^{IJK}F_{\mu\nu}{}^I H^{\mu\nu\, J}\varphi^K \quad ,\end{aligned} \tag{2.1}$$

[4] The reason we need the factor g^2 in the action is due to the mass-dimension assignments of our fields.

[5] We use the symbol $\doteq$ for a field equation to be distinguished from an algebraic equation.

up to quartic-order terms $\mathcal{O}(\phi^4)$.

It is clear that the scalar φ^I has its mass g, while there is a mixture between χ^I and ρ^I, again with the asme mass g. As has been mentioned after (1.4), $C_\mu{}^I$ plays the role of Stueckelberg field [8], being absorbed into the longitudinal component of $B_{\mu\nu}{}^I$. Eventually, the kinetic term of the C-field becomes the mass term of $B_{\mu\nu}{}^I$. Accordingly, the degrees of freedom (DOF) for the massive TM fields are $B_{\mu\nu}{}^I$ (3), χ with ρ^I (4) and $\varphi^I(1)$, up to the adjoint index I.

Our action I is invariant under global $N=1$ supersymmetry

$$\delta_Q B_{\mu\nu}{}^I = +(\overline\epsilon\gamma_{\mu\nu}\chi^I) - 2f^{IJK}C_{[\mu|}{}^J(\delta_Q A_{|\nu]}{}^K) \quad , \tag{2.2a}$$

$$\begin{aligned}\delta_Q\chi^I = &+\tfrac{1}{6}(\gamma^{\mu\nu\rho}\epsilon)G_{\mu\nu\rho}{}^I - (\gamma^\mu\epsilon)D_\mu\varphi^I \\ &+\tfrac{1}{2}f^{IJK}\big[+\epsilon(\overline\lambda^J\rho^K) - (\gamma_5\gamma^\mu\epsilon)(\overline\lambda^J\gamma_5\gamma_\mu\rho^K) - (\gamma_5\epsilon)(\overline\lambda^J\gamma_5\rho^K)\big] \quad ,\end{aligned} \tag{2.2b}$$

$$\delta_Q\varphi^I = +(\overline\epsilon\chi^I) \quad , \tag{2.2c}$$

$$\delta_Q C_\mu{}^I = +(\overline\epsilon\gamma_\mu\rho^I) + f^{IJK}(\overline\epsilon\gamma_\mu\lambda^J)\varphi^K \quad , \tag{2.2d}$$

$$\begin{aligned}\delta_Q\rho^I = &+\tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)H_{\mu\nu}{}^I - g\epsilon\varphi^I - \tfrac{1}{2}f^{IJK}(\gamma^{\mu\nu}\epsilon)F_{\mu\nu}{}^J\varphi^K \\ &+\tfrac{1}{4}f^{IJK}\big[+\epsilon(\overline\lambda^J\chi^K) - (\gamma^\mu\epsilon)(\overline\lambda^J\gamma_\mu\chi^K) + \tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)(\overline\lambda^J\gamma_{\mu\nu}\chi^K) \\ &\qquad - (\gamma_5\gamma^\mu\epsilon)(\overline\lambda^J\gamma_5\gamma_\mu\chi^K) - (\gamma_5\epsilon)(\overline\lambda^J\gamma_5\chi^K)\big] \quad ,\end{aligned} \tag{2.2e}$$

$$\delta_Q A_\mu{}^I = +(\overline\epsilon\gamma_\mu\lambda^I) \quad , \tag{2.2f}$$

$$\delta_Q\lambda^I = +\tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)F_{\mu\nu}{}^I + \tfrac{1}{2}f^{IJK}(\gamma_5\epsilon)(\overline\rho^J\gamma_5\chi^K) \quad , \tag{2.2g}$$

up to cubic terms $\mathcal{O}(\phi^3)$ in fields. The fermionic quadratic terms in (2.2b), (2.2e) and (2.2g) are fixed in superspace formulation, as will be explained later. In the *conventional* dimensions with all the bosonic (or fermionic) fields with 1 (or 3/2) mass dimensions,[6)]these terms lead to non-renormalizability. For example, the l.h.s. of (2.2b) has dimension 3/2, while its r.h.s. for the $\epsilon(\overline\lambda\gamma\rho)$ term has $(-1/2)+(3/2)+(3/2)=5/2$. In other words, there is an implicit coupling constant ℓ with the dimension of length in front of fermionic quadratic terms. This feature is also related to the existence of Pauli-terms which are non-renormalizable, already at a *globally* supersymmetric system. These features are similar to supergravity [9], even though our system so far has only *global* supersymmetry.

6) Our bosonic (or fermionic) fields have dimensions 0 (or 1/2), in contrast to the conventional dimensions 1 (or 3/2).

The usual non-Abelian gauge transformation δ_α and our tensorial gauge transformation δ_β, and δ_γ-transformation are exactly the same as (1.4), while all the fermionic fields are transforming only under δ_α, as the B and C-fields do, so that there arises no problem with the δ_β and δ_γ-invariances of the field strengths as in (1.5). These immediately lead to the invariances of our action $\delta_\alpha I = 0$, $\delta_\beta I = 0$ and $\delta_\gamma I = 0$.

The Bianchi identities (BIds) for our field strengths G, H and F are:

$$D_{[\mu} G_{\nu\rho\sigma]}{}^I - \frac{3}{2} f^{IJK} F_{[\mu\nu}{}^J H_{\rho\sigma]}{}^K \equiv 0 \ , \tag{2.3a}$$

$$D_{[\mu} H_{\nu\rho]}{}^I - \frac{1}{3}\, g\, G_{\mu\nu\rho}{}^I \equiv 0 \ , \tag{2.3b}$$

$$D_{[\mu} F_{\nu\rho]}{}^I \equiv 0 \ . \tag{2.3c}$$

Relevantly, the non-trivial δ_Q-transformations of the field strengths are

$$\delta_Q G_{\mu\nu\rho}{}^I = + 3(\bar\epsilon \gamma_{[\mu\nu} D_{\rho]} \chi^I) + 3 f^{IJK} (\delta_Q A_{[\mu}{}^J) H_{\nu\rho]}{}^K - 3 f^{IJK} (\delta_Q C_{[\mu}{}^J) F_{\nu\rho]}{}^K \ , \tag{2.4a}$$

$$\delta_Q H_{\mu\nu}{}^I = - 2(\bar\epsilon \gamma_{[\mu} D_{\nu]} \rho^I) + g(\bar\epsilon \gamma_{\mu\nu} \chi^I) + 2 f^{IJK} D_{[\mu|} \left[(\delta_Q A_{|\nu]}{}^J) \varphi^K \right] \ , \tag{2.4b}$$

$$\delta_Q F_{\mu\nu}{}^I = - 2(\bar\epsilon \gamma_{[\mu} D_{\nu]} \lambda^I) \ , \tag{2.4c}$$

reflecting the presence of CS terms.

Note that our YMVM and CVM has *on-shell* DOF 2+2, while *off-shell* DOF 3+4, because we have *not* added the D-auxiliary field. On the other hand, our TM is in the *off-shell* formulation, because the total off-shell DOF is $4+4$, because the off-shell DOF of each field are $[(4-1)\cdot(4-2)]/2 = 3$ for $B_{\mu\nu}$, 4 for χ and 1 for φ.

The field equations for λ^I, χ^I, ρ^I, $A_\mu{}^I$, $B_{\mu\nu}{}^I$, φ^I and $C_\mu{}^I$ are respectively[7)]

$$\begin{aligned} &+ \not{D} \lambda^I - \tfrac{1}{2} g f^{IJK} \chi^J \varphi^K + \tfrac{1}{2} f^{IJK} (\gamma^\mu \rho^J) D_\mu \varphi^K \\ &\qquad\qquad - \tfrac{1}{4} f^{IJK} (\gamma^{\mu\nu} \chi^J) H_{\mu\nu}{}^K + \tfrac{1}{12} f^{IJK} (\gamma^{\mu\nu\rho} \rho^J) G_{\mu\nu\rho}{}^K \doteq 0 \ , \end{aligned} \tag{2.5a}$$

$$+ \not{D} \chi^I - g \rho^I + \tfrac{1}{2} g f^{IJK} \lambda^H \varphi^K - \tfrac{1}{4} f^{IJK} (\gamma^{\mu\nu} \lambda^J) H_{\mu\nu}{}^K + \tfrac{1}{4} f^{IJK} (\gamma^{\mu\nu} \rho^J) F_{\mu\nu}{}^K \doteq 0 \ , \tag{2.5b}$$

$$\begin{aligned} &+ \not{D} \rho^I - g \chi^I + \tfrac{1}{2} f^{IJK} (\gamma^\mu \lambda^J) D_\mu \varphi^K \\ &\qquad\qquad - \tfrac{1}{12} f^{IJK} (\gamma^{\mu\nu\rho} \lambda^J) G_{\mu\nu\rho}{}^K + \tfrac{1}{4} f^{IJK} (\gamma^{\mu\nu} \chi^J) F_{\mu\nu}{}^K \doteq 0 \ , \end{aligned} \tag{2.5c}$$

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7) These equations are fixed up to $\mathcal{O}(\phi^3)$-terms, due to the quartic fermion terms in the lagrangian.

$$+ D_\nu F_\mu{}^{\nu I} + g f^{IJK} \varphi^J D_\mu \varphi^K + \frac{1}{2} g f^{IJK} (\overline{\lambda}^J \gamma_\mu \lambda^K) + f^{IJK} H_{\mu\nu}{}^J D^\nu \varphi^K$$
$$- \frac{1}{2} f^{IJK} G_{\mu\rho\sigma}{}^J H^{\rho\sigma\, K} + \frac{1}{2} f^{IJK} (\overline{\chi}^J D_\mu \rho^K) + \frac{1}{2} f^{IJK} (\overline{\rho}^J D_\mu \chi^K) \doteq 0 \quad , \qquad (2.5\text{d})$$

$$+ D_\rho G^{\mu\nu\rho\, I} - g H^{\mu\nu\, I} - \frac{1}{2} f^{IJK} D_\rho (\overline{\lambda}^J \gamma^{\mu\nu\rho} \rho^K)$$
$$+ g f^{IJK} F^{\mu\nu\, J} \varphi^K - \frac{1}{2} g f^{IJK} (\overline{\lambda}^J \gamma^{\mu\nu} \chi^K) \doteq 0 \quad , \qquad (2.5\text{e})$$

$$+ D_\mu^2 \varphi^I - g f^{IJK} (\overline{\lambda}^J \chi^K) - g^2 \varphi^I - \frac{1}{2} f^{IJK} F_{\mu\nu}{}^J H^{\mu\nu\, K} \doteq 0 \quad , \qquad (2.5\text{f})$$

$$+ D_\nu H^{\mu\nu\, I} - \frac{1}{2} f^{IJK} F_{\rho\sigma}{}^J G^{\mu\rho\sigma\, K} - \frac{1}{2} f^{IJK} (\overline{\chi}^J D^\mu \lambda^K) - \frac{1}{2} f^{IJK} (\overline{\lambda}^J D^\mu \chi^K)$$
$$+ \frac{1}{2} g f^{IJK} (\overline{\lambda}^J \gamma^\mu \rho^K) - f^{IJK} F^{\mu\nu\, J} D_\nu \varphi^K \doteq 0 \quad . \qquad (2.5\text{g})$$

In the derivation of these field equations, we have also used other field equations, in order to simply their final expressions, as a conventional prescription.

In the above computation, we do not attempt to fix the $\mathcal{O}(\phi^3)$-terms in field equations, or equivalently the fermionic $\mathcal{O}(\phi^4)$-terms in the lagrangian. There are several remarks about these terms. First, our system is non-renormalizable as supergravity theory [9], as has been mentioned after eq. (2.2). Accordingly, the (fermion)^2-terms in the fermionic transformations such as (2.2b), (2.2e) and (2.2g) are accompanied by the implicit constant ℓ carrying the dimension of (legnth). In supergravity theory [9], this is the gravitational coupling κ. In our lagrangian, all the quartic-fermion terms carry ℓ^2, so that the lagrangian has the mass dimension +4. Accordingly, a typical Noether-term has the structure $\ell\, \Psi^2 \partial \Phi$, that produces the terms of the form $\ell^2 \epsilon \Psi^3 \partial \Phi$ *via* $\delta_Q \Psi \approx \ell \epsilon \Psi^2$. Here Ψ (or Φ) is a general fermionic (or bosonic) fundamental field. These $\ell^2 \epsilon \Psi^3 \partial \Phi$-terms are cancelled by the variation of the fermionic quartic terms $\ell^2 \Psi^4$, *via* $\delta_Q \Psi \approx \epsilon \partial \Phi$. In other words, the structure of these cancellations associated with quartic-fermion terms is parallel to supergravity [9], since ℓ is analogous to κ.

However, in our peculiar system, this cancellation mechanism may be *not* simply parallel to conventional supergravity [9]. For example, there may be $\ell^2 \Psi^2 \Phi \partial \Psi$-type terms in the action, while $\ell^2 \epsilon \Psi^2 \Phi$-type terms in the transformation rules may exist, because both of them yield $\ell^2 \epsilon \Psi^3 \partial \Phi$-type terms, canceling each other in $\delta_Q I$. At the present time, we do not know, if such terms arise, because the $\ell^2 \epsilon \Psi^2 \Phi$-type terms in transformations are at $\mathcal{O}(\phi^3)$, while $\ell^2 \Psi^2 \Phi \partial \Psi$-type terms in the action are at $\mathcal{O}(\phi^4)$. In fact, even in the superspace re-confirmation in the next section, we have fixed only the $\mathcal{O}(\phi^1)$ and $\mathcal{O}(\phi^2)$-terms in the transformation rules for fermions, such as (3.2d), (3.2e) and (3.2f), but *not* cubic terms $\mathcal{O}(\phi^3)$. Our consistent principle in this paper is to fix only $\mathcal{O}(\phi^1)$, $\mathcal{O}(\phi^2)$ and $\mathcal{O}(\phi^3)$-terms in the lagrangian, $\mathcal{O}(\phi^1)$ and $\mathcal{O}(\phi^2)$-terms in all transformation rules, while $\mathcal{O}(\phi^1)$ and $\mathcal{O}(\phi^2)$-terms in all field equations. However, we try to fix *neither* $\mathcal{O}(\phi^4)$-terms in the lagrangian, *nor* $\mathcal{O}(\phi^3)$-term in all transformation rules, *nor* $\mathcal{O}(\phi^3)$-terms in all field equations. We do *not* specify each field meant by ϕ is fermionic or bosonic in this paper, either.

Second, as an additional difference from supergravity [9], the fermionic quartic terms do *not* contain any gravitino. This implies that we can not use the conventional technique of 'supercovariantizing' fermionic field equations. Due to this feature, as well as the above-mentioned possible non-purely-fermionnic $\ell^2\Psi^2\Phi\partial\Psi$-type terms, the quartic terms $\mathcal{O}(\phi^4)$ at $\mathcal{O}(\ell^2)$ will be more involved than conventional supergravity [9] which are tedious. For these reasons, we do not attempt to fix them in this paper.

Third, according to the past experience in supergravity theory [9], it is understood that the series in terms of κ in a lagrangian will stop at a finite order, such as the quartic-fermion terms at $\mathcal{O}(\kappa^2)$ [9]. However, at the present time, we do *not* know, whether this is also the case with our globally supersymmetric system. This is because of the above-mentioned differences of our system from supergravity [9], and therefore the analogy with supergravity might be not valid in our system. Fourth, since we have already fixed the cubic terms in the lagrangian, they seem sufficient for non-trivial and consistent couplings as a supersymmetric system.

4. Superspace Reformulation of N=1 TM

As a reconfirmation of the total consistency of our system, we re-formulate our theory in terms of superspace language. Our basic superspace BIds for the superfield strengths $F_{AB}{}^I$, $G_{ABC}{}^I$ and $H_{AB}{}^I$ are[8)]

$$+\frac{1}{6}\nabla_{[A}G_{BCD)}{}^I - \frac{1}{4}T_{[AB|}{}^E G_{E|CD)} - \frac{1}{4}f^{IJK}F_{[AB}{}^J H_{CD)}{}^K \equiv 0 \ , \tag{3.1a}$$

$$+\frac{1}{2}\nabla_{[A}H_{BC)}{}^I - \frac{1}{2}T_{[AB|}{}^D H_{D|C)}{}^I - g\,G_{ABC}{}^I \equiv 0 \ , \tag{3.1b}$$

$$+\frac{1}{2}\nabla_{[A}F_{BC)}{}^I - \frac{1}{2}T_{[AB|}{}^D F_{D|C)}{}^I \equiv 0 \ . \tag{3.1b}$$

8) Only in this superspace section, we use the indices $A = (a,\alpha)$, $B = (b,\beta)$, $\cdots$ for superspace coordinates, where $a,\ b,\ \cdots = 0,1,2,3$ (or $\alpha,\ \beta,\ \cdots = 1,2,3,4$) are for bosonic (or fermionic) coordinates. In superspace, the (anti)symmetrization convention, *e.g.*, $X_{[AB)} \equiv X_{AB} - (-1)^{AB}X_{BA}$ is different from our component notation.

These BIds are the superspace generalizations of the component BIds (2.3), with the super-torsion terms added for local Lorentz indices, as usual in superspace.

Our basic superspace constraints at mass dimensions $0 \le d \le 1$ are

$$T_{\alpha\beta}{}^c = +2(\gamma^c)_{\alpha\beta} \quad , \qquad G_{\alpha\beta c}{}^I = +2(\gamma_c)_{\alpha\beta}\,\varphi^I \quad , \tag{3.2a}$$

$$G_{\alpha bc}{}^I = -(\gamma_{bc}\chi^I)_\alpha \quad , \qquad H_{\alpha b}{}^I = -(\gamma_b\rho^I)_\alpha - f^{IJK}(\gamma_b\lambda^J)_\alpha\,\varphi^K \quad , \tag{3.2b}$$

$$F_{\alpha b}{}^I = -(\gamma_b\lambda^I)_\alpha \quad , \qquad \nabla_\alpha\varphi^I = -\chi_\alpha{}^I \quad , \tag{3.2c}$$

$$\begin{aligned} \nabla_\alpha\chi_\beta{}^I = & -\frac{1}{6}(\gamma^{cde})_{\alpha\beta}G_{cde}{}^I - (\gamma^c)_{\alpha\beta}\nabla_c\varphi^I \\ & - \tfrac{1}{2}f^{IJK}\Big[+C_{\alpha\beta}(\overline\lambda{}^J\rho^K) - (\gamma_5\gamma^c)_{\alpha\beta}(\overline\lambda{}^J\gamma_5\gamma_c\rho^K) - (\gamma_5)_{\alpha\beta}(\overline\lambda{}^J\gamma_5\rho^K)\Big] \quad , \end{aligned} \tag{3.2d}$$

$$\begin{aligned} \nabla_\alpha\rho_\beta{}^I = & +\tfrac{1}{2}(\gamma^{cd})_{\alpha\beta}H_{cd}{}^I + g\,C_{\alpha\beta}\,\varphi^I - \tfrac{1}{2}f^{IJK}(\gamma^{cd})_{\alpha\beta}F_{cd}{}^J\varphi^K \\ & - \tfrac{1}{4}f^{IJK}\Big[+C_{\alpha\beta}(\overline\lambda{}^J\chi^K) + (\gamma^c)_{\alpha\beta}\,(\overline\lambda{}^J\gamma_c\chi^K) - \tfrac{1}{2}(\gamma^{cd})_{\alpha\beta}(\overline\lambda{}^J\gamma_{cd}\chi^K) \\ & \qquad - (\gamma_5\gamma^c)_{\alpha\beta}(\overline\lambda{}^J\gamma_5\gamma_c\chi^K) - (\gamma_5)_{\alpha\beta}(\overline\lambda{}^J\gamma_5\chi^K) \quad , \end{aligned} \tag{3.2e}$$

$$\nabla_\alpha\lambda_\beta{}^I = +\tfrac{1}{2}(\gamma^{cd})_{\alpha\beta}F_{cd}{}^I - \tfrac{1}{2}(\gamma_5)_{\alpha\beta}\,f^{IJK}(\overline\rho{}^J\gamma_5\chi^K) \quad . \tag{3.2f}$$

All other components, such as $G_{\alpha\beta\gamma}{}^I$, $T_{\alpha\beta}{}^\gamma$, $T_{ab}{}^c$, $H_{\alpha\beta}{}^I$ *etc.* at $d \le 1$ are zero. Note that $(\text{fermion})^2$-terms in (3.2d) through (3.2f) have been determined in superspace by satisfying BIds at $d = 1$. Note that these results are valid up to $\mathcal{O}(\phi^3)$-terms, which we do not attempt to fix these terms in this paper. However, all the $\mathcal{O}(\phi^2)$-terms have been included, as has been also mentioned at the end of last section.

There are also useful relationships obtained from $d = +3/2$ BIds:

$$\nabla_\alpha G_{bcd} = -\tfrac{1}{2}(\gamma_{[bc}\nabla_{d]}\chi^I)_\alpha - \tfrac{1}{2}f^{IJK}(\gamma_{[b|}\lambda^J)_\alpha H_{|cd]}{}^K + \tfrac{1}{2}f^{IJK}(\gamma_{[b|}\rho^J)_\alpha F_{|cd]}{}^K \quad , \tag{3.3a}$$

$$\nabla_\alpha H_{bc}{}^I = +(\gamma_{[b}\nabla_{c]}\rho^I)_\alpha - g(\gamma_{bc}\chi^I)_\alpha - f^{IJK}\nabla_{[b}\Big[(\gamma_{c]}\lambda^J)_\alpha\varphi^K\Big] \quad , \tag{3.3b}$$

$$\nabla_\alpha F_{bc}{}^I = +(\gamma_{[b}\nabla_{c]}\lambda^I)_\alpha \quad , \tag{3.3c}$$

up to $\mathcal{O}(\phi^3)$-terms. Note the existence of the $\mathcal{O}(\phi^2)$-terms in (3.3a) and (3.3b), reflecting the corresponding terms in the component results (2.4a) and (2.4b).

As usual, the satisfaction of all the BIds in superspace by the constraints (3.2) and (3.3) is straightforward to perform, from the dimension $d = 0$ to $d = 3/2$, as usual. In particular, the $(\text{Fermions})^2$-terms in (3.2d) through (3.2f) are the results of our superspace re-formulation.

The fermionic λ and ρ-field equations (2.5a) and (2.5c) are obtained as usual by computing $\{\nabla_\alpha, \nabla_\beta\}\,\lambda^{\beta I}$ and $\{\nabla_\alpha, \nabla_\beta\}\,\rho^{\beta I}$, while the χ-field equation is shown to be consistent with the component lagrangian. As has been mentioned, since the TM is *off-shell* multiplet, we can *not* get the χ-field equation (2.5b) in superspace directly, but we can show that (2.5b) is consistent in superspace. The bosonic field equations (2.5d) - (2.5g) are obtained by applying another fermionic derivative on the fermionic field equations (2.5a) - (2.5c).

5. Generalization to Non-Adjoint Representations of G = SO(N)

We have so far considered the case for the TM and CVM both carrying only the adjoint representation. We can generalize this result to other more general representations, such as an arbitrary real representation of a $SO(N)$-type gauge group.[9)]

To be more specific, we consider the TM $(B_{\mu\nu}{}^i, \chi^i, \varphi^i)$ and the CVM $(C_\mu{}^i, \rho^i)$, where the index i is for any real representation of a gauge group $G = SO(N)$. Let $(T^I)^{jk}$ be the generator of the group G. Then our action $I' \equiv \int d^4x\, \mathcal{L}'$ has the lagrangian[10)]

$$\begin{aligned}\mathcal{L}' = & -\tfrac{1}{12}(G_{\mu\nu\rho}{}^i)^2 + \tfrac{1}{2}(\overline{\chi}^i \not{D}\chi^i) - \tfrac{1}{2}(D_\mu\varphi^i)^2 - \tfrac{1}{2}g^2(\varphi^i)^2 - g(\overline{\rho}^i\chi^i) \\ & - \tfrac{1}{4}(H_{\mu\nu}{}^i)^2 + \tfrac{1}{2}(\overline{\rho}^i \not{D}\rho^i) - \tfrac{1}{4}(F_{\mu\nu}{}^I)^2 + \tfrac{1}{2}(\overline{\lambda}^I \not{D}\lambda^I) \\ & - \tfrac{1}{2}g(T^I)^{jk}(\overline{\lambda}^I\chi^j)\,\varphi^k + \tfrac{1}{2}(T^I)^{jk}(\overline{\lambda}^I\gamma^\mu\rho^j)D_\mu\varphi^k + \tfrac{1}{12}(T^I)^{jk}(\overline{\lambda}^I\gamma^{\mu\nu\rho}\rho^j)\,G_{\mu\nu\rho}{}^k \\ & + \tfrac{1}{4}(T^I)^{jk}(\overline{\rho}^j\gamma^{\mu\nu}\chi^k)F_{\mu\nu}{}^I - \tfrac{1}{4}(T^I)^{jk}(\overline{\lambda}^I\gamma^{\mu\nu}\chi^j)H_{\mu\nu}{}^k - \tfrac{1}{2}(T^I)^{jk}F_{\mu\nu}{}^I H^{\mu\nu\,j}\varphi^k \quad ,\end{aligned} \tag{4.1}$$

up to quartic terms $\mathcal{O}(\phi^4)$. Our action I' is invariant under global $N=1$ supersymmetry

$$\delta_Q B_{\mu\nu}{}^i = +(\overline{\epsilon}\gamma_{\mu\nu}\chi^i) - 2(T^J)^{ik}C_{[\mu|}{}^k(\delta_Q A_{|\nu]}{}^J) \quad , \tag{4.2a}$$

$$\begin{aligned}\delta_Q\chi^i = & +\tfrac{1}{6}(\gamma^{\mu\nu\rho}\epsilon)G_{\mu\nu\rho}{}^i - (\gamma^\mu\epsilon)D_\mu\varphi^i \\ & - \tfrac{1}{2}(T^J)^{ik}\Big[+\epsilon(\overline{\lambda}^J\chi^k) - (\gamma_5\gamma^\mu\epsilon)(\overline{\lambda}^J\gamma_5\gamma_\mu\chi^k) - (\gamma_5\epsilon)(\overline{\lambda}^J\gamma_5\chi^k)\Big] \quad ,\end{aligned} \tag{4.2b}$$

$$\delta_Q\varphi^i = +(\overline{\epsilon}\chi^i) \quad , \tag{4.2c}$$

$$\delta_Q C_\mu{}^i = +(\overline{\epsilon}\gamma_\mu\rho^i) - (T^J)^{ik}(\overline{\epsilon}\gamma_\mu\lambda^J)\varphi^k \quad , \tag{4.2d}$$

$$\begin{aligned}\delta_Q\rho^i = & +\tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)H_{\mu\nu}{}^i - g\epsilon\varphi^i + \tfrac{1}{2}(T^J)^{ik}(\gamma^{\mu\nu}\epsilon)F_{\mu\nu}{}^J\varphi^k \\ & - \tfrac{1}{4}(T^J)^{ik}\Big[+\epsilon(\overline{\lambda}^J\chi^k) - (\gamma^\mu\epsilon)(\overline{\lambda}^J\gamma_\mu\chi^k) + \tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)(\overline{\lambda}^J\gamma_{\mu\nu}\chi^k)\end{aligned}$$

---

9) We can also consider the complex representation for $SU(N)$-type gauge groups.

10) Since the metric for the gauge group $G = SO(N)$ is positive definite, we do *not* distinguish the upper or lower indices for $i,\ j,\ \cdots = 1,\ 2,\ \cdots,\ \dim R$, where $\mathbf{R}$ is a real representation of G.

$$- (\gamma_5 \gamma^\mu \epsilon)(\overline{\lambda}^J \gamma_5 \gamma_\mu \chi^k) - (\gamma_5 \epsilon)(\overline{\lambda}^J \gamma_5 \chi^k) \Big] \ , \tag{4.2e}$$

$$\delta_Q A_\mu{}^I = + (\overline{\epsilon} \gamma_\mu \lambda^I) \ , \tag{4.2f}$$

$$\delta_Q \lambda^I = + \tfrac{1}{2} (\gamma^{\mu\nu} \epsilon) F_{\mu\nu}{}^I - \tfrac{1}{2} (T^I)^{jk} (\gamma_5 \epsilon)(\overline{\rho}^j \gamma_5 \chi^k) \ . \tag{4.2g}$$

The essential point is that all the cubic-order terms contain one component field $A_\mu{}^I$ or λ^I with the index I, and the remaining two component fields out of either TM or CVM carry the indices j and k. So the cancellation structure is parallel to the adjoint-representation case, *e.g.*, with the structure constant f^{IJK} replaced by the matrix $-(T^J)^{ik}$ in $D_\mu \chi^I = \partial_\mu \chi^I + g f^{IJK} A_\mu{}^J \chi^K \implies D_\mu \chi^i = \partial_\mu \chi^i - g (T^J)^{ik} A_\mu{}^J \chi^k$. Accordingly, the Stueckelberg mechanism [8] works in a parallel fashion, because $C_\mu{}^i$ is absorbed into the longitudinal component of $B_{\mu\nu}{}^i$, both in the same representation $\mathbf{R}$.

6. Coupling to $N=1$ Supergravity

Once we have established the $N=1$ global system of non-Abelian TM with non-trivial and consistent interactions, the next natural step is to make $N=1$ supersymmetry *local*, coupling to $N=1$ supergravity.

This coupling is rather straightforward, because most of the basic structure is parallel to the usual matter coupling to supergravity, except for certain couplings to be mentioned later. Our result for the lagrangian $\widetilde{\mathcal{L}}$ of our action is $\widetilde{I} \equiv \int d^4 x \, \widetilde{\mathcal{L}}$:

$$\begin{aligned} e^{-1} \widetilde{\mathcal{L}} = & - \tfrac{1}{4} R(\omega) - \left[\overline{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu(\omega) \psi_\rho \right] - \tfrac{1}{12} (G_{\mu\nu\rho}{}^I)^2 + \tfrac{1}{2} [\overline{\chi}^I \not{D}(\omega) \chi^I] - \tfrac{1}{2} (D_\mu \varphi^I)^2 \\ & - \tfrac{1}{4} (F_{\mu\nu}{}^I)^2 + \tfrac{1}{2} [\overline{\lambda}^I \not{D} \lambda^I] - \tfrac{1}{4} (H_{\mu\nu}{}^I)^2 + \tfrac{1}{2} [\overline{\rho}^I \not{D}(\omega) \rho^I] - g (\overline{\chi}^I \rho^I) - \tfrac{1}{2} g^2 (\varphi^I)^2 \\ & - \tfrac{1}{2} g f^{IJK} (\overline{\lambda}^I \chi^J) \varphi^K - \tfrac{1}{4} f^{IJK} (\overline{\lambda}^I \gamma^{\mu\nu} \chi^J) H_{\mu\nu}{}^K \\ & + \tfrac{1}{12} f^{IJK} (\overline{\lambda}^I \gamma^{\mu\nu\rho} \rho^J) G_{\mu\nu\rho}{}^K + \tfrac{1}{4} f^{IJK} (\overline{\rho}^I \gamma^\mu \chi^J) F_{\mu\nu}{}^K \\ & - \tfrac{1}{2} f^{IJK} F_{\mu\nu}{}^I H^{\mu\nu\, J} \varphi^K + \tfrac{1}{2} f^{IJK} (\overline{\lambda}^I \gamma^{\mu\nu} \rho^J) D_\mu \varphi^K \\ & + (\overline{\psi}_\mu \gamma^\nu \gamma^\mu \chi^I) D_\nu \varphi^I + \tfrac{1}{6} (\overline{\psi}_\mu \gamma^{\rho\sigma\tau} \gamma^\mu \chi^I) G_{\rho\sigma\tau}{}^I \\ & - \tfrac{1}{2} (\overline{\psi}_\mu \gamma^{\rho\sigma} \gamma^\mu \lambda^I) F_{\rho\sigma}{}^I - \tfrac{1}{2} (\overline{\psi}_\mu \gamma^{\rho\sigma} \gamma^\mu \rho^I) H_{\rho\sigma}{}^I - g (\overline{\psi}_\mu \gamma^\mu \rho^I) \varphi^I \ , \end{aligned} \tag{5.1}$$

up to $\mathcal{O}(\phi^4)$ terms.

Our action $\widetilde{I}$ is now invariant under local $N=1$ supersymmetry

$$\delta_Q e_\mu{}^m = - 2 (\overline{\epsilon} \gamma^m \psi_\mu) \ , \tag{5.2a}$$

$$
\begin{aligned}
\delta_Q \psi_\mu &= + D_\mu(\widehat\omega)\epsilon - \tfrac{1}{6}(\gamma_\mu{}^{\rho\sigma\tau}\epsilon)\widehat G_{\rho\sigma\tau}{}^I\varphi^I \quad , & (5.2\text{b})\\
\delta_Q B_{\mu\nu}{}^I &= + (\overline\epsilon\gamma_{\mu\nu}\chi^I) - 2f^{IJK}C_{[\mu|}{}^J(\delta_Q A_{|\nu]}{}^K) - 4(\overline\epsilon\gamma_{[\mu}\psi_{\nu]})\varphi^I \quad , & (5.2\text{c})\\
\delta_Q \chi^I &= + \tfrac{1}{6}(\gamma^{\mu\nu\rho}\epsilon)\widehat G_{\mu\nu\rho}{}^I - (\gamma^\mu\epsilon)\widehat D_\mu\varphi^I \\
&\quad + \tfrac{1}{2}f^{IJK}\big[+ \epsilon(\overline\lambda{}^J\rho^K) - (\gamma_5\gamma^\mu\epsilon)(\overline\lambda{}^J\gamma_5\gamma_\mu\rho^K) - (\gamma_5\epsilon)(\overline\lambda{}^J\gamma_5\rho^K)\big] \quad , & (5.2\text{d})\\
\delta_Q \varphi^I &= + (\overline\epsilon\chi^I) \quad , & (5.2\text{e})\\
\delta_Q C_\mu{}^I &= + (\overline\epsilon\gamma_\mu\rho^I) + f^{IJK}(\overline\epsilon\gamma_\mu\lambda^J)\varphi^K \quad , & (5.2\text{f})\\
\delta_Q \rho^I &= + \tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)\widehat H_{\mu\nu}{}^I - g\,\epsilon\,\varphi^I - \tfrac{1}{2}f^{IJK}(\gamma^{\mu\nu}\epsilon)\widehat F_{\mu\nu}{}^J\varphi^K \\
&\quad + \tfrac{1}{4}f^{IJK}\big[+ \epsilon(\overline\lambda{}^J\chi^K) - (\gamma^\mu\epsilon)(\overline\lambda{}^J\gamma_\mu\chi^K) + \tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)(\overline\lambda{}^J\gamma_{\mu\nu}\chi^K) \\
&\qquad\qquad - (\gamma_5\gamma^\mu\epsilon)(\overline\lambda{}^J\gamma_5\gamma_\mu\chi^K) - (\gamma_5\epsilon)(\overline\lambda{}^J\gamma_5\chi^K)\big] \quad , & (5.2\text{g})\\
\delta_Q A_\mu{}^I &= + (\overline\epsilon\gamma_\mu\lambda^I) \quad , & (5.2\text{h})\\
\delta_Q \lambda^I &= + \tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)\widehat F_{\mu\nu}{}^I + \tfrac{1}{2}f^{IJK}(\gamma_5\epsilon)(\overline\rho{}^J\gamma_5\chi^K) \quad , & (5.2\text{i})
\end{aligned}
$$

up to $\mathcal{O}(\phi^3)$ terms. The supercovariant field strengths are defined as usual in supergravity [9] by

$$
\begin{aligned}
\widehat F_{\mu\nu}{}^I &\equiv + 2\partial_{[\mu}A_{\nu]}{}^I + gf^{IJK}A_\mu{}^J A_\nu{}^K - 2(\overline\psi_{[\mu}\gamma_{\nu]}\lambda^I) = F_{\mu\nu}{}^I - 2(\overline\psi_{[\mu}\gamma_{\nu]}\lambda^I) \quad , & (5.3\text{a})\\
\widehat G_{\mu\nu\rho}{}^I &\equiv + 3D_{[\mu}B_{\nu\rho]}{}^I - 3f^{IJK}C_{[\mu}{}^J F_{\nu\rho]}{}^K - 3(\overline\psi_{[\mu}\gamma_{\nu\rho]}\chi^I) + 6(\overline\psi_{[\mu|}\gamma_{|\nu|}\psi_{|\rho]})\varphi^I \\
&= + G_{\mu\nu\rho}{}^I - 3(\overline\psi_{[\mu}\gamma_{\nu\rho]}\chi^I) + 6(\overline\psi_{[\mu|}\gamma_{|\nu|}\psi_{|\rho]})\varphi^I \quad , & (5.3\text{b})\\
\widehat H_{\mu\nu}{}^I &\equiv + 2D_{[\mu}C_{\nu]}{}^I + gB_{\mu\nu}{}^I - 2(\overline\psi_{[\mu}\gamma_{\nu]}\rho^I) = H_{\mu\nu}{}^I - 2(\overline\psi_{[\mu}\gamma_{\nu]}\rho^I) \quad , & (5.3\text{c})\\
\widehat D_\mu\varphi^I &\equiv + D_\mu\varphi^I - (\overline\psi_\mu\chi^I) \quad . & (5.3\text{d})
\end{aligned}
$$

Certain remarks are in order. First, the last term in (5.1) of the type $g(\overline\psi\gamma\rho)\varphi$ is related to the φ-linear term in $\delta_Q\rho$ in (5.2g). Second, the $\delta_Q B_{\mu\nu}$ contains the $(\overline\epsilon\gamma\psi)\varphi$-term. This is consistent with $G_{\alpha\beta c}{}^I = +2(\gamma_c)_{\alpha\beta}\,\varphi^I$ in (3.2a) in superspace. Third, for the $g\psi\rho\chi$-terms, we need non-trivial Fierz rearrangement. To be more specific, there are three contributions to this sector: (i) $g(\overline\psi\gamma\rho)\varphi$, (ii) $ge(\overline\chi\rho)$, and (iii) $(\overline\psi\gamma\gamma\rho)H$-terms. This rearrangement is highly non-trivial, showing the consistency of our total system.

As the couplings to supergravity in (5.1) show, our original *globally* supersymmetric system shares certain feature with supergravity, such as fermionic bilinear terms. Because such terms are common in supergravity [9], but *not* in conventional global supersymmetry.

Our original *global* system already possessed the feature of *local* $N = 1$ supersymmetry. As has been mentioned after (2.2), the conventional dimensional analysis tells that such terms imply non-renormalizability. In other words, our *globally* supersymmetric system already had a hidden gravitational constant κ providing negative mass dimension. In a sense, this feature resembles σ-models with non-renormalizable couplings, sharing certain features with gravity interactions.

7. Possible Application to Standard Model

A possible application to the standard model can be described as follows. The $SU(3) \times SU(2) \times U(1)$ gauge-field kinetic terms are

$$\mathcal{L}_{\rm KT}^{\rm Standard} = - \frac{1}{4} \, {\rm tr}\, (G_{\mu\nu})^2 - \frac{1}{4} (F_{\mu\nu}{}^I)^2 - \frac{1}{4} (Y_{\mu\nu})^2 \quad , \tag{3.1}$$

where $G_{\mu\nu}$, $F_{\mu\nu}{}^I$ and $Y_{\mu\nu}$ are respectively the field strengths of the gauge fields of $SU(3)$, $SU(2)$ and $U(1)$. We put the explicit adjoint indices ${\scriptstyle I,~J,~\cdots}$ for $SU(2)$ gauge group. Forgetting about supersymmetrization, the new fields we need are the non-Abelian tensor $B_{\mu\nu}{}^I$ and the extra compensator vector $C_\mu{}^I$ with their field strengths already defined:

$$G_{\mu\nu\rho}{}^I \equiv 3 D_{[\mu} B_{\nu\rho]}{}^I \equiv 3 (\partial_{[\mu} B_{\nu\rho]}{}^I + g f^{IJK} A_{[\mu}{}^J B_{\nu\rho]}{}^K) - 3 f^{IJK} C_{[\mu}{}^J F_{\nu\rho]}{}^K \quad , \tag{2.1}$$

$$H_{\mu\nu}{}^I \equiv 2 D_{[\mu} C_{\nu]}{}^I + g B_{\mu\nu}{}^I \quad . \tag{2.2}$$

The kinetic fields for B and C are

$$\mathcal{L}_{\rm KT}^{B\,\&\,C} \equiv - \frac{1}{12} (G_{\mu\nu\rho}{}^I)^2 - \frac{1}{4} (H_{\mu\nu}{}^I)^2 \quad . \tag{3.2}$$

The total action $I_{\rm KT}^{B\,\&\,C}$ is invariant under δ_β and δ_γ-transformations, because the G and H-field strengths are invariant under δ_β and δ_γ-transformations:

$$\delta_\beta G_{\mu\nu\rho}{}^I = 0 \quad , \qquad \delta_\beta H_{\mu\nu}{}^I = 0 \quad , \tag{2.5a}$$

$$\delta_\gamma G_{\mu\nu\rho}{}^I = 0 \quad , \qquad \delta_\gamma H_{\mu\nu}{}^I = 0 \quad . \tag{2.5b}$$

As (2.2) shows, the C-field is the compensator field absorbed into the longitudinal component of $B_{\mu\nu}{}^I$, making the latter massive. In fact, the KT of C is nothing but the mass term of B after this absorption. The resulting mass is g for $B_{\mu\nu}{}^I$, because $\widetilde{H}_{\mu\nu}{}^I = g\widetilde{B}_{\mu\nu}{}^I$, after the field re-definition $\widetilde{B}_{\mu\nu}{}^I \equiv B_{\mu\nu}{}^I + 2g^{-1}D_{[\mu}C_{\nu]}{}^I$.

The typical interactions with the non-Abelian groups $SU(3)$ or $SU(3)$ are found already in the field strength $G_{\mu\nu\rho}{}^I$ in (3.1). Namely, its last term $C \wedge F$ gives already non-trivial interaction between the new field C and the field strength F.

8. Unification Quest

Recently, the long-standing problem with non-Abelian tensors [10] has been solved by de Wit, Samtleben, and Nicolai [11][12]. The original motivation in [11] was to generalize the tensor and vector field interactions in manifestly $E_{6(+6)}$-covariant formulation of five-dimensional (5D) maximal supergravity by gauging non-Abelian sub-groups. In [12], this work was further related to M-theory [13] by confirming the representation assignments under the duality group of the gauge charges. The underlying hierarchies of these tensor and vector gauge fields are presented with the consistency of general gaugings.

The hierarchy in [11][12] has been further applied to the conformal supergravity in 6D [14]. In ref. [14], the 'minimal tensor hierarchy' as a special case of the more general hierarchy in [11][12] has been discussed. This hierarchy consists of $A_\mu{}^r$ and two-form gauge potentials $B_{\mu\nu}{}^I$, with two labels r and I. Also introduced is the 3-form gauge potentials $C_{\mu\nu\rho\, r}$ with the index $_r$ is dual to ^{r} of $A_\mu{}^r$. The field strengths of vector and two-form gauge potentials are defined by [14]

$$\mathcal{F}_{\mu\nu}{}^r \equiv 2\partial_{[\mu}A_{\nu]}{}^r + h_I{}^r B_{\mu\nu}{}^I \quad , \tag{1.1a}$$

$$\mathcal{H}_{\mu\nu\rho}{}^I \equiv 3D_{[\mu}B_{\nu\rho]}{}^I + 6d_{rs}{}^I A_{[\mu}{}^r\partial_\nu A_{\rho]}{}^s - 2f_{pq}{}^s d_{rs}{}^I A_{[\mu}{}^r A_\nu{}^p A_{\rho]}{}^q + g^{Ir}C_{\mu\nu\rho r} \quad . \tag{1.1b}$$

The prescription for tensor-vector system, which we will be based upon, is described with eq. (3.22) in [14].

6. Concluding Remarks

In this paper, we have carried out the $N = 1$ supersymmetrization in 4D of a non-Abelian tensor with consistent couplings, as a special case [7] of the minimal tensor hierarchy discussed in [14], which is further a special case of more general hierarchy in [11][12]. We have given both the component and superspace formulations of our system, providing the non-trivial consistency of our system. Our CVM $(C_\mu{}^I, \rho^I)$ plays the role of a Stueckelberg [8] compensator multiplet, being absorbed into the TM $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$, making the latter massive.

We have also generalized the adjoint-representation case to the general real representation for $G = SO(N)$. The action invariance works in a fashion parallel to the former. We foresee no obstruction against generalizing these result further to the complex representation of, *e.g.*, $G = SU(N)$ group. Finally, we have also coupled the global $N = 1$ system to $N = 1$ supergravity up to quartic terms. This has provided a non-trivial confirmation for the total consistency of the non-Abelian TM.

Our formulation has solved problems in supersymmetric gauge field theories, and has given a new system, based on a very simple field content. First, we have established the supersymmetric generalization of the non-Abelian tensor $B_{\mu\nu}{}^I$ with consistent couplings in explicit lagrangians. Second, we have solved the common problem with a vector field $C_\mu{}^I$ carrying an adjoint index, which is *not* the gauge field of the gauge group G itself. The solution turned out to be the introduction of an extra vector $C_\mu{}^I$ playing a role of Stueckelberg compensator, eventually absorbed into the longitudinal components of the non-Abelian tensor $B_{\mu\nu}{}^I$. In other words, the former is collaborating with the latter in a Stueckelberg mechanism [8], avoiding the common consistency problem of couplings. In fact, our coupling constant g coincides with the mass of the TM. This implies that the consistent couplings for the non-Abelian TM and its mass *via* the Stueckelberg mechanism [8] are closely related to each other. Third, the adjoint index on the non-gauge vector field $C_\mu{}^I$ is further generalized to an arbitrary real representation index of $G = SO(N)$. Fourth, most importantly, we have carried out the supersymmetrization of such a Stueckelberg mechanism for a non-Abelian tensor. Fifth, even though our *algebra* with δ_α, δ_β and δ_γ is indeed a special case of the hierarchy in [11], we have given explicit lagrangians with the *physically propagating vector field* $C_\mu{}^I$ that has not been presented before.

It has been known that certain problem exists in the quantization of Stueckelberg model [8] for non-Abelian gauge groups [15]. The common problem is that the longitudinal components of the gauge field do not decouple from the physical Hilbert space, upsetting the renormalizability and unitarity of the system [15]. For this issue, we clarify our standpoints as follows: First of all, our theory is *not* renormalizable from the outset, due to Pauli couplings. Our theory makes stronger sense, when couplings to supergravity are also taken into account, as we have done in section 5. Moreover, there are certain theories in 4D, such as non-linear sigma models which are *not* renormalizable, but are *not* excluded from the outset. So we do *not* go into the renormalizability issue in this paper. Second, thanks to $N = 1$ supersymmetry, our system has good chance to have a better quantum behavior, compared with non-supersymmetric systems.

As will be shown in Appendix A, the purely bosonic part of our system can be generalized to arbitrary space-time dimensions with arbitrary signatures. The key ingredient is the tensor $B_{\mu_1\cdots\mu_{p+1}}{}^I$ and a Stueckelberg-type [8] compensator $C_{\mu_1\cdots\mu_p}{}^I$.

The potential importance of the result in this paper is $N = 1$ supersymmetry that has better quantum behavior compared with non-supersymmetric cases. We have presented a new *supersymmetric* physical system with Stueckelberg mechanism that solves both the problem with non-Abelian tensor, and the problem with extra vector fields in the non-singlet representation of a non-Abelian gauge group.

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Appendix A: Higher-Dimensional Application of Purely Bosonic System

In this appendix, we generalize the purely bosonic part of our system in 4D into arbitrary space-time dimensions with arbitrary signatures. We also apply it to the case of tensor-vector duality in 6D, and perform a DR to 4D. Our field content is $(A_\mu{}^I, B_{[n-1]}{}^I, C_{[n-2]}{}^I)$.[11]

We generalize the definitions of field strengths (2.1a) and (2.1b) to arbitrary space-time dimension D as

$$G_{\mu_1\cdots\mu_n}{}^I \equiv +nD_{[\mu_1}B_{\mu_2\cdots\mu_n]}{}^I - \frac{n(n-1)}{2} f^{IJK}C_{[\mu_1\cdots\mu_{n-2}}{}^J F_{\mu_{n-1}\mu_n]}{}^K \quad , \tag{A.1a}$$

$$H_{\mu_1\cdots\mu_{n-1}}{}^I \equiv +(n-1)D_{[\mu_1}C_{\mu_2\cdots\mu_{n-1}]}{}^I + gB_{\mu_1\cdots\mu_{n-1}}{}^I \quad . \tag{A.1b}$$

[11] We use the symbols like $[n]$ for totally antisymmetric indices $\mu_1\mu_2\cdots\mu_n$ in order to save space.

The YM field strength F is the same as in (1.2). The BIds for these field strengths are

$$D_{[\mu}F_{\nu\rho]}{}^I \equiv 0 \quad , \tag{A.2a}$$

$$D_{[\mu_1}G_{\mu_2\cdots\mu_{n+1}]}{}^I \equiv +\frac{n}{2}\, f^{IJK}F_{[\mu_1\mu_2|}{}^J H_{|\mu_3\cdots\mu_{n+1}]}{}^K \quad , \tag{A.2b}$$

$$D_{[\mu_1}H_{\mu_2\cdots\mu_n]}{}^I \equiv +\frac{1}{n}\, g\, G_{\mu_1\cdots\mu_n}{}^I \quad . \tag{A.2c}$$

The α, β and γ-transformations for $A_\mu{}^I$, $B_{[n-1]}{}^I$ and $C_{[n-2]}{}^I$ are the generalizations of our 4D case:

$$\delta_\alpha(A_\mu{}^I,\ B_{[n-1]}{}^I,\ C_{[n-2]}{}^I) = (D_\mu\alpha^I,\ -gf^{IJK}\alpha^J B_{[n-1]}{}^K,\ -gf^{IJK}\alpha^J C_{[n-2]}{}^K) \quad , \tag{A.3a}$$

$$\delta_\alpha(F_{\mu\nu}{}^I,\ G_{[n]}{}^I,\ H_{[n-1]}{}^I) = -gf^{IJK}\alpha^J(F_{\mu\nu}{}^K,\ G_{[n]}{}^K,\ H_{[n-1]}{}^K) \quad , \tag{A.3b}$$

$$\delta_\beta B_{\mu_1\cdots\mu_{n-1}}{}^I = +(n-1)D_{[\mu_1}\beta_{\mu_2\cdots\mu_{n-1}]}{}^I \quad , \qquad \delta_\beta A_\mu{}^I = 0 \quad , \tag{A.3c}$$

$$\delta_\beta C_{\mu_1\cdots\mu_{n-2}}{}^I = -g\beta_{\mu_1\cdots\mu_{n-2}}{}^I \quad , \tag{A.3d}$$

$$\delta_\beta(F_{\mu\nu}{}^I,\ G_{[n-1]}{}^I,\ H_{[n-2]}{}^I) = 0 \quad , \tag{A.3e}$$

$$\delta_\gamma C_{\mu_1\cdots\mu_{n-2}}{}^I = +(n-2)D_{[\mu_1}\gamma_{\mu_2\cdots\mu_{n-2}]}{}^I \quad , \qquad \delta_\gamma A_\mu{}^I = 0 \quad , \tag{A.3f}$$

$$\delta_\gamma B_{\mu_1\cdots\mu_{n-1}}{}^I = +\frac{(n-1)(n-2)}{2}\, f^{IJK}\,\gamma_{[\mu_1\cdots\mu_{n-3}|}{}^J F_{|\mu_{n-2}\,\mu_{n-1}]}{}^K \quad , \tag{A.3g}$$

$$\delta_\gamma(F_{\mu\nu}{}^I,\ G_{[n-1]}{}^I,\ H_{[n-2]}{}^I) = 0 \quad . \tag{A.3h}$$

Eq. (A.3d) shows that the C-field is a Stueckelberg field absorbed into the longitudinal components of the B-field.

A typical action $I \equiv \int d^Dx\, \mathcal{L}$ is given by the lagrangian

$$\mathcal{L} = -\frac{1}{2(n!)}\,(G_{[n]}{}^I)^2 - \frac{1}{2\cdot(n-1)!}\,(H_{[n-1]}{}^I)^2 - \frac{1}{4}\,(F_{\mu\nu}{}^I)^2 \quad , \tag{A.4}$$

yielding the B and C-field equations

$$\frac{\delta\mathcal{L}}{\delta B_{[n-1]}{}^I} = \frac{1}{(n-1)!}\left(D_\mu G^{\mu[n-1]\,I} - gH^{[n-1]\,I}\right) \doteq 0 \quad , \tag{A.5a}$$

$$\frac{\delta\mathcal{L}}{\delta C_{[n-2]}{}^I} = \frac{1}{(n-2)!}\left(D_\nu H^{\nu[n-2]\,I} + \frac{1}{2}\, f^{IJK}F_{\rho\sigma}{}^J G^{[n-2]\rho\sigma\,K}\right) \doteq 0 \quad . \tag{A.5b}$$

As in the 4D case, it is straightforward to show the consistency

$$0 \stackrel{?}{=} D_\mu\left(\frac{\delta\mathcal{L}}{\delta B_{\mu[n-2]}{}^I}\right) \equiv -\frac{1}{n-1}\, g\left(\frac{\delta\mathcal{L}}{\delta C_{[n-2]}{}^I}\right) \doteq 0 \quad , \tag{A.6a}$$

$$0 \stackrel{?}{=} D_\mu\left(\frac{\delta\mathcal{L}}{\delta C_{\mu[n-3]}{}^I}\right) \equiv +\frac{n-1}{2}\, f^{IJK}F_{\rho\sigma}{}^J\left(\frac{\delta\mathcal{L}}{\delta B_{[n-3]\rho\sigma}{}^K}\right) \doteq 0 \quad (Q.E.D.) \tag{A.6b}$$

We next apply our result to $6D$ with the signature $(-,-,+,+,+,+)$, and consider the duality condition

$$F_{\mu\nu}{}^I \overset{*}{=} + \frac{1}{24}\, \epsilon_{\mu\nu}{}^{\rho\sigma\tau\lambda}\, G_{\rho\sigma\tau\lambda}{}^I \quad , \qquad G_{\mu\nu\rho\sigma}{}^I \overset{*}{=} + \frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma}{}^{\tau\lambda}\, F_{\tau\lambda}{}^I \quad . \tag{A.7}$$

This duality looks similar to eq. (3.6) in [14], but the existence of the physical scalar field ϕ^I in the latter makes the fundamental difference.

We have to first confirm the consistency of (A.7) with the G and H-BIds. First, the rotation of the 2nd equation in (A.7) gives

$$\begin{aligned} 0 \overset{?}{=} &+ \epsilon^{\mu\nu\rho\sigma\tau\lambda} D_\nu \left(G_{\rho\sigma\tau\lambda}{}^I - \frac{1}{2}\, \epsilon_{\rho\sigma\tau\lambda}{}^{\omega\psi} F_{\omega\psi}{}^I \right) \equiv +\epsilon^{\mu\nu\rho\sigma\tau\lambda} \left(2 f^{IJK} F_{\nu\rho}{}^J H_{\sigma\tau\lambda}{}^K \right) - 24 D_\nu F^{\mu\nu\, I} \\ &= -24 \left(D_\nu F^{\mu\nu\, I} - \frac{1}{12}\, \epsilon^{\mu\nu\rho\sigma\tau\lambda} f^{IJK} F_{\nu\rho}{}^J H_{\sigma\tau\lambda}{}^K \right) \quad . \end{aligned} \tag{A.8}$$

In the second identity in (A.8), we have used the G-BId (A.2b). The first term in the last line is the kinetic term of $A_\mu{}^I$, so that its last term is its source term. Second, in order to see if eq. (A.8) has consistent solutions, we can confirm the conservation of the source term, by applying D_μ on (A.8) based on H-BId (A.2c) and (A.7), but we skip the details here.

We next show that the usual self-duality relationship in $D = 2 + 2$

$$F_{\mu\nu}{}^I \overset{*}{=} + \frac{1}{2}\, \epsilon_{\mu\nu}{}^{\rho\sigma}\, F_{\rho\sigma}{}^I \tag{A.9}$$

is embedded into (A.7). To this end, we use *hat* symbols both on fields and indices in 6D, while *no hats* on 4D quantities from now on. We also use $\hat\mu,\, \hat\nu,\, \cdots = 1,\, 2,\, 3,\, 4,\, 5,\, 6$ and $\mu,\, \nu,\, \cdots = 1,\, 2,\, 3,\, 4$, while $\alpha,\, \beta,\, \cdots = 5,\, 6$. Our basic ansätze for the DR are

$$\widehat G_{\hat\mu\hat\nu\hat\rho\hat\sigma}{}^I \overset{*}{=} + \widehat F_{[\hat\mu\hat\nu}{}^I \widehat P_{\hat\rho\hat\sigma]} \quad , \qquad \widehat P_{\hat\mu\hat\nu} \equiv + \hat\partial_{\hat\mu} \widehat X_{\hat\nu} - \hat\partial_{\hat\nu} \widehat X_{\hat\mu} \quad , \qquad \widehat H_{\hat\mu\hat\nu\hat\rho}{}^I \overset{*}{=} + \frac{1}{2}\, g \widehat F_{[\hat\mu\hat\nu}{}^I \widehat X_{\hat\rho]} \quad , \tag{A.10a}$$

$$\widehat P_{\hat\mu\hat\nu} = \epsilon_{\alpha\beta} \quad (\text{for } \hat\mu = \alpha,\, \hat\nu = \beta) \quad , \qquad \widehat F_{\hat\mu\hat\nu}{}^I = \widehat F_{\mu\nu}{}^I = F_{\mu\nu}{}^I \quad (\text{for } \hat\mu = \mu,\, \hat\nu = \nu) \quad , \tag{A.10b}$$

$$\hat\epsilon^{\,\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau\hat\lambda} = \hat\epsilon^{\,\mu\nu\rho\sigma\alpha\beta} = \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta} \quad (\text{for } [\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau\hat\lambda] = [\mu\nu\rho\sigma\alpha\beta]) \quad . \tag{A.10c}$$

Other components, such as $\widehat P_{\mu\beta}$ are all zero. We can confirm that (A.10) are consistent with the BIds (A.2b) and (A.2c). It is easy to show that the $[\alpha\beta]$ and $[\mu\alpha]$-components of the first equation in (A.7) are satisfied, while the $[\mu\nu]$-component gives directly the 4D self-duality (A.9). Thus the 4D self-duality $F \overset{*}{=} \widetilde F$ is indeed embedded in the 6D duality (A.7).

We next generalize the 6D result to the $D = 2m+2$ with the signature $(-,-,\overbrace{+,\cdots,+}^{2m})$. The duality condition (A.7) is generalized to

$$\widehat F_{\hat\mu\hat\nu}{}^I \overset{*}{=} + \frac{1}{(2m)!}\, \hat\epsilon_{\hat\mu\hat\nu}{}^{\hat\rho_1\cdots\hat\rho_{2m}}\, \widehat G_{\hat\rho_1\cdots\hat\rho_{2m}}{}^I \quad , \qquad \widehat G_{\hat\rho_1\cdots\hat\rho_{2m}}{}^I \overset{*}{=} + \frac{1}{2}\, \hat\epsilon_{\hat\rho_1\cdots\hat\rho_{2m}}{}^{\hat\mu\hat\nu}\, \widehat F_{\hat\mu\hat\nu}{}^I \quad . \tag{A.11}$$

As in the 6D case, we can first confirm the consistency with BIds. We can next confirm the current conservation, whose details are skipped here.

The previous ansätze for 6D case in (A.10) are generalized to

$$\widehat{G}_{\hat\mu_1\cdots\hat\mu_{2m}}{}^I \overset{*}{=} + c\widehat{F}_{[\hat\mu_1\hat\mu_2|}{}^I \widehat{P}^{(1)}_{|\hat\mu_3\hat\mu_4|}\cdots \widehat{P}^{(m-1)}_{|\hat\mu_{2m-1}\,\hat\mu_{2m}]} \quad , \qquad \widehat{P}^{(k)}_{\hat\mu\hat\nu} \equiv \partial_{\hat\mu}\widehat{X}^{(k)}_{\hat\nu} - \partial_{\hat\nu}\widehat{X}^{(k)}_{\hat\mu} \quad , \tag{A.12a}$$

$$\widehat{H}_{\hat\mu_1\cdots\hat\mu_{2m-1}}{}^I \overset{*}{=} + \frac{1}{m}\, cg\widehat{F}_{[\hat\mu_1\hat\mu_2|}{}^I \widehat{P}^{(1)}_{|\hat\mu_3\hat\mu_4|}\cdots \widehat{P}^{(m-2)}_{|\hat\mu_{2m-3}\,\hat\mu_{2m-2}|}\widehat{X}_{|\hat\mu_{2m-1}]} \quad , \tag{A.12b}$$

$$\widehat{P}^{(k)}_{\hat\mu\hat\nu} = \widehat{P}^{(k)}_{2k+3,\ 2k+4} = -\widehat{P}^{(k)}_{2k+4,\ 2k+3} = \epsilon^{(k)}_{2k+3,\ 2k+4} = -\epsilon^{(k)}_{2k+4,\ 2k+3} = +1$$

$$\text{(for } \ \hat\mu = 2k{+}3,\ \hat\nu = 2k{+}4;\ k = 1,\ \cdots,\ m{-}1) \ , \tag{A.12c}$$

$$\widehat{F}_{\hat\mu\hat\nu}{}^I = F_{\mu\nu}{}^I \qquad \text{(for } \ \hat\mu = \mu,\ \hat\nu = \nu) \quad , \tag{A.12d}$$

$$\widehat{\epsilon}^{\hat\mu_1\cdots\hat\mu_{2m+2}} = \epsilon^{\mu\nu\rho\sigma}\,\epsilon^{\alpha_1\cdots\alpha_{2m-2}} = \epsilon^{\mu\nu\rho\sigma}\,\epsilon^{[\alpha_1\alpha_2|}_{(1)}\cdots\,\epsilon^{|\alpha_{2m-3}\alpha_{2m-2}]}_{(m-1)}$$

$$\text{(for } \ [\hat\mu_1\cdots\hat\mu_{2m+2}] = [\mu\nu\rho\sigma\alpha_1\cdots\alpha_{2m-2}]\big) \quad . \tag{A.12e}$$

where c is a constant to be fixed later.

As before, we can also confirm the G and H-BIds for (A.11). The constant c in (A.12a) is fixed by getting the 4D self-duality in the $[\mu\nu]$-component of the first equation in (A.11):

$$\begin{aligned} F_{\mu\nu}{}^I \overset{*}{=} &+ \frac{1}{(2m)!}\,\widehat{\epsilon}_{\mu\nu}{}^{\hat\rho_1\cdots\hat\rho_{2m}}\,\widehat{G}_{\hat\rho_1\cdots\hat\rho_{2m}}{}^I = + \frac{\binom{2m}{2}}{(2m)!}\,\widehat{\epsilon}_{\mu\nu}{}^{\rho\sigma\alpha_1\cdots\alpha_{2m-2}}\,\widehat{G}_{\rho\sigma\alpha_1\cdots\alpha_{2m-2}}{}^I \\ = &+ \frac{1}{2}\,c\left[\frac{1}{(m{-}1)!{\cdot}(2m{-}3)!!}\right]^2 \epsilon_{\mu\nu}{}^{\rho\sigma}\,F_{\rho\sigma}{}^I \quad . \end{aligned} \tag{A.13}$$

For this to agree with $F \overset{*}{=} \widetilde{F}$, we get $c = [\,(m-1)!\cdot(2m-3)!!\,]^2$. The remaining components $[\alpha\beta]$ and $[\mu\alpha]$ are trivially satisfied.

The above mechanism for $D = 2m+2$ is further generalized to $D = 2m+1$ with the signature $(-,-,\overbrace{+,+,\cdots,+}^{2m-1})$ with the duality condition

$$\widehat{F}_{\hat\mu\hat\nu}{}^I \overset{*}{=} + \frac{1}{(2m{-}1)!}\,\widehat{\epsilon}_{\hat\mu\hat\nu}{}^{\hat\rho_1\cdots\hat\rho_{2m-1}}\,\widehat{G}_{\hat\rho_1\cdots\hat\rho_{2m-1}}{}^I \quad , \qquad \widehat{G}_{\hat\rho_1\cdots\hat\rho_{2m-1}}{}^I \overset{*}{=} + \frac{1}{2}\,\widehat{\epsilon}_{\hat\rho_1\cdots\hat\rho_{2m-1}}{}^{\hat\mu\hat\nu}\,\widehat{F}_{\hat\mu\hat\nu}{}^I \quad . \tag{A.14}$$

The confirmation of G and H-BIds is just parallel to the $D = 2m+2$ case. The ansätze for DR is

$$\widehat{G}_{\hat\mu_1\cdots\hat\mu_{2m-1}}{}^I \overset{*}{=} + \frac{2c'}{3}\,\widehat{F}_{[\hat\mu_1\hat\mu_2|}{}^I \widehat{P}^{(1)}_{|\hat\mu_3\hat\mu_4|}\cdots \widehat{P}^{(m-3)}_{|\hat\mu_{2m-5}\,\hat\mu_{2m-4}|}\widehat{Q}_{|\hat\mu_{2m-3}\hat\mu_{2m-2}\hat\mu_{2m-1}]} \quad , \tag{A.15a}$$

$$\widehat{H}_{\hat\mu_1\cdots\hat\mu_{2m-2}}{}^I \overset{*}{=} + \frac{2c'g}{2m{-}1}\,\widehat{F}_{[\hat\mu_1\hat\mu_2|}{}^I \widehat{P}^{(1)}_{|\hat\mu_3\hat\mu_4|}\cdots \widehat{P}^{(m-3)}_{|\hat\mu_{2m-5}\,\hat\mu_{2m-4}|}\widehat{Y}_{|\hat\mu_{2m-3}\hat\mu_{2m-2}]} \quad , \tag{A.15b}$$

$$\widehat{P}^{(k)}_{\hat\mu\hat\nu} \equiv \partial_{\hat\mu}\widehat{X}^{(k)}_{\hat\nu} - \partial_{\hat\nu}\widehat{X}^{(k)}_{\hat\mu} \quad , \qquad \widehat{Q}_{\hat\mu\hat\nu\hat\rho} \equiv +\partial_{\hat\mu}\widehat{Y}_{\hat\nu\hat\rho} + \partial_{\hat\nu}\widehat{Y}_{\hat\rho\hat\mu} + \partial_{\hat\rho}\widehat{Y}_{\hat\mu\hat\nu} \quad , \tag{A.15c}$$

$$\widehat{P}^{(k)}_{\hat{\mu}\hat{\nu}} = \widehat{P}^{(k)}_{2k+3,\ 2k+4} = -\widehat{P}^{(k)}_{2k+4,\ 2k+3} = \epsilon^{(k)}_{2k+3,\ 2k+4} = -\epsilon^{(k)}_{2k+4,\ 2k+3} = +1 \quad , \tag{A.15d}$$

$$\widehat{Q}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \widehat{Q}_{2m-3,2m-2,2m-1} = \epsilon_{2m-3,2m-2,2m-1} = +1 \quad \big(\text{for } {\scriptstyle [\hat{\mu}\hat{\nu}\hat{\rho}] \,=\, [2m-3,2m-2,2m-1]}\big) \quad , \tag{A.15e}$$

$$\widehat{F}_{\hat{\mu}\hat{\nu}}{}^I = F_{\mu\nu}{}^I \qquad \big(\text{for } {\scriptstyle \hat{\mu} \,=\, \mu,\ \hat{\nu} \,=\, \nu}\big) \quad , \tag{A.15f}$$

$$\widehat{\epsilon}^{\,\hat{\mu}_1\cdots\hat{\mu}_{2m+1}} = \epsilon^{\mu\nu\rho\sigma}\,\epsilon^{\alpha_1\cdots\alpha_{2m-3}} = \epsilon^{\mu\nu\rho\sigma}\,\epsilon_{(1)}^{[\alpha_1\alpha_2|}\cdots\,\epsilon_{(m-3)}^{|\alpha_{2m-7}\alpha_{2m-6}|}\epsilon^{|\alpha_{2m-5}\alpha_{2m-4}\alpha_{2m-3}]} \quad . \tag{A.15g}$$

The totally antisymmetric constant tensor $\epsilon^{\alpha\beta\gamma}$ is for the last three coordinates in $D = 2m+1$. The satisfaction of the duality (A.14) fixes the constant $c' = [(m-3)!\cdot(2m-7)!!]^2$.

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