Cabibbo-Kobayashi-Maskawa matrix: parameterizations and rephasing invariants

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Abstract

Motivated by the rephasing invariance of the CKM observables we consider the general phase invariant monomials built out of the CKM matrix elements and their conjugates. We show, that there exist 30 fundamental phase invariant monomials and 18 of them are a product of 4 CKM matrix elements and 12 are a product of 6 CKM matrix elements. In our Main Theorem we show that all rephasing invariant monomials can be expressed as a product of at most 5 factors: 4 of them are fundamental phase invariant monomials and the fifth factor consists of powers of squares of absolute values of the CKM matrix elements.

The discussion of the rephasing invariants depends on the number of generations and we will discuss these invariants for 3 generations.

Introduction

The Yukawa Coupling and the CKM Matrix

Within the Standard Model and its extensions the Lagrangian for the Yukawa interactions associated with quarks has the generic structure

The Lagrangian for the Yukawa interactions

$$\mathscr{L}_{\mathsf{Y}} = y_u \bar{q}_L \Phi_1 \left(q_u\right)_R + y_d \bar{q}_L \Phi_2 \left(q_d\right)_R + h.c.,$$

where Φ_1 and Φ_2 are the two Higgs doublets. The factors y_u and y_d are 3×3 matrices containing the coupling constants for up quarks q_u and down quarks q_d, respectively. In the case of the Standard Model $\Phi_2 = \Phi$ and $\Phi_1 = \varepsilon\Phi^*$ (ε is the 2×2 antisymmetric tensor), the left handed terms $(q_{u,d})_L$ are doublets and the right handed terms $(q_{u,d})_R$ are singlets.

The Yukawa Coupling and the CKM Matrix

The Yukawa couplings can be diagonalized through biunitary transformations

$$y_u = U_L^\dagger Y_u U_R, \qquad y_d = D_L^\dagger Y_d D_R,$$

here U_L, U_R, D_L, D_R are unitary matrices, Y_u and Y_d are diagonal matrices.

The CKM matrix for the electroweak charged currents is the result of the product

There are two important bases. One where the masses are diagonal, called the **mass basis**, and the other where the $W^\pm$ interactions are diagonal, called the **interaction basis**. The CKM matrix is the matrix that rotates between these two bases.

The CKM matrix

$$V = U_L D_L^\dagger,$$

The CKM matrix V_{CKM}

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix},$$

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0,$$

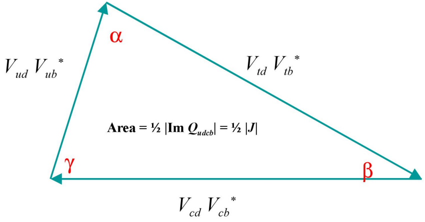

Figure 1: Sketch of the unitarity triangle.

The unitarity of the CKM matrix imposes $\sum_i V_{ij}V_{ik}^* = \delta_{jk}$ and $\sum_j V_{ij}V_{kj}^* = \delta_{ik}$. The six vanishing combinations can be represented as triangles in a complex plane and the areas of all triangles are the same, half of the Jarlskog invariant, J, which is a phase-convention-independent measure of CP violation, defined by $\mathrm{Im}\left(V_{ij}V_{kl}V_{ij}^*V_{kl}^*\right) = J\sum_{m,n}\varepsilon_{ikm}\varepsilon_{jln}$.

Rephasing invariants

The quark fields are defined up to a phase, therefore it is possible to introduce the states $u_\alpha = e^{i\varphi_\alpha} u'_\alpha$ and $d_k = e^{i\phi_k} d'_k$ so that the elements of the CKM matrix become

$$V'_{\alpha k} = e^{i(\phi_k - \varphi_\alpha)} V_{\alpha k},$$

The rephasing invariants are defined in the following way

Rephasing Invariants

$$\begin{aligned}
W_{\alpha i} &\equiv |V_{\alpha i}|^2, && (1a)\\
Q_{\alpha i \beta j} &\equiv V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}, \quad \alpha \neq \beta \quad i \neq j && (1b)\\
K_{\alpha\beta\gamma;ijk} &\equiv \begin{cases} V_{\alpha i} V_{\beta j} V_{\gamma k} V^*_{\alpha j} V^*_{\beta k} V^*_{\gamma i} \\ \\ V_{\alpha i} V_{\beta j} V_{\gamma k} V^*_{\alpha k} V^*_{\beta i} V^*_{\gamma j} \end{cases} \quad \alpha \neq \beta \neq \gamma \quad \text{y} \quad i \neq j \neq k && (1c)\\
&\vdots
\end{aligned}$$

In order to avoid $Q_{\alpha i \beta j} = W_{\alpha i} W_{\beta j}$ the conditions $\alpha \neq \beta$ and $i \neq j$ must be satisfied.

Rephasing invariants

Rephasing Invariants

$$
\begin{aligned}
W_{\alpha i} &\equiv |V_{\alpha i}|^2, \\
Q_{\alpha i\beta j} &\equiv V_{\alpha i}V_{\beta j}V^*_{\alpha j}V^*_{\beta i}, \quad \alpha\neq\beta \quad i\neq j \\
K_{\alpha\beta\gamma;ijk} &\equiv \left\{ \begin{array}{l} V_{\alpha i}V_{\beta j}V_{\gamma k}V^*_{\alpha j}V^*_{\beta k}V^*_{\gamma i} \\ \\ V_{\alpha i}V_{\beta j}V_{\gamma k}V^*_{\alpha k}V^*_{\beta i}V^*_{\gamma j} \end{array} \right. \quad \alpha\neq\beta\neq\gamma \quad \text{y} \quad i\neq j\neq k \\
&\vdots
\end{aligned}
$$

It is easy to show that the rephasing invariant defined by

$$
V_{11}V_{13}^2V_{21}^2V_{22}V_{32}^2V_{33}\left(V_{12}^3V_{23}^3V_{31}^3\right)^*.
$$

cannot be factored into the square of the CKM matrix elements and other rephasing invariants.

The Jarlskog's invariant

The most important property of $Q_{\alpha i\beta j}$ that follows from the unitarity of the CKM Matrix is:

The Jarlskog's invariant

$$\mathrm{Im}(Q_{\alpha i\beta j}) = \pm J, \qquad \alpha \neq \beta \qquad \text{and} \qquad i \neq j,$$

where J is the Jarlskog's invariant [1].

[1] C. Jarlskog, Phys. Rev. Lett. 55 (1985) 1039.

Rephasing invariants and the Jarlskog's invariant

Is there a relation between the rephasing invariants and the Jarlskog's invariant?

Rephasing invariants and the Jarlskog's invariant

$$\begin{aligned} \operatorname{Im}(W_{\alpha i}) &\equiv 0, \\ \operatorname{Im}(Q_{\alpha i\beta j}) &\equiv J\sum_{m,n}\varepsilon_{\alpha\beta m}\varepsilon_{ijn}, \\ \operatorname{Im}(K_{\alpha\beta\gamma;ijk}) &\equiv ???\ldots \\ &\vdots \end{aligned}$$

Why are the rephasing invariants important in the standard model?

All the observables that contain the CKM matrix are invariant under rephasing of the quarks fields [a]. This is the reason why the rephasing functions of the elements of the CKM matrix play an important role in the Standard Model.

[a] The only functions of V that can be measured are precisely the rephasing invariants.

A new form to construct rephasing invariants

A new form to construct rephasing invariants

We denote by $P(m,n)$ the most general monomial constructed from the CKM matrix elements and its conjugates:

$$\mathrm{P}(m,n) = \prod_{ij}(V_{ij})^{m_{ij}} \prod_{kj}(V^*_{kl})^{n_{ij}}$$

where m and n are the 3×3 matrices with integer non negative matrix elements [2] and $[m]_{ij} = m_{ij}$, $[n]_{kl} = n_{kl}$. The mapping between the monomial $P(m,n)$ and the matrices m and n is one to one.

The products $\mathrm{P}(m,n)$ fulfills the following properties

- $\mathrm{P}(m_1,n_1)\cdot\mathrm{P}(m_2,n_2) = \mathrm{P}(m_1+m_2,n_1+n_2)$.
- $\mathrm{P}(m,n)^* = \mathrm{P}(n,m)$.
- $\mathrm{P}(m,m) = \prod_{ij}(|V_{ij}|^2)^{m_{ij}}$.

From the definition of $\mathrm{P}(m,n)$ it is clear that it is not necessarily a rephasing invariant (Examples 1 and 2).

[2] The condition that the elements of the matrices m and n are integers may be relaxed, but the CKM observables are monomials that contain only integer powers.

Example 1.- If the monomial $\mathrm{P}(m_1,n_1)$ is equal to $V_{11}V_{22}V_{12}^*V_{21}^*$,

$$\mathrm{P}(m_1,n_1) = V_{11}V_{22}V_{12}^*V_{21}^*, \tag{4}$$

the matrices m_1, n_1 are:

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad n_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Example 2.- If

$$\mathrm{P}(m_2,n_2) = V_{11}^2V_{22}V_{33}V_{12}^*V_{21}^*V_{11}^*, \tag{6}$$

we have

$$m_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad n_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

The example 1 $\mathrm{P}(m_1,n_1)$ is a rephasing invariant and the example 2 $\mathrm{P}(m_2,n_2)$ is not.

If we want to know the conditions under which $P(m,n)$ represents a rephasing invariant we make the following phase transformation of the CKM matrix:

$$V_{CKM} \rightarrow \mathrm{diag}(\mathrm{e}^{i\phi_1},1,1)V_{CKM}, \tag{8}$$

then the monomial $P(m,n)$ is transformed in the following way

$$P(m,n) \rightarrow \mathrm{e}^{i\phi_1(m_{11}+m_{12}+m_{13}-n_{11}-n_{12}-n_{13})}P(m,n),$$

so we see that $P(m,n)$ is invariant under the transformation in Eq. (8) only if

$$m_{11}+m_{12}+m_{13}=n_{11}+n_{12}+n_{13},$$

i.e., if the sum of the elements of the first row of the matrices m and n are equal. From this follows the next theorem.

Theorem 1

The monomial $P(m,n)$ is rephasing invariant if the sums of the elements of the corresponding rows and columns of the matrices m and n are equal. It means that for the rephasing invariant monomial $P(m,n)$ the matrices m and n fulfill the following conditions

$$\sum_{j=1}^{3} m_{ij} = \sum_{j=1}^{3} n_{ij}, \quad \sum_{j=1}^{3} m_{ji} = \sum_{j=1}^{3} n_{ji}, \quad i=1,2,3. \tag{9}$$

Pure rephasing invariants

Definition 1

The rephasing invariant monomial of the CKM matrix which cannot be factored out into the product of the absolute values of the elements of the CKM matrix and other invariant is called the *pure rephasing invariant* (PRI).

The PRIs can be represented by two matrices m and n but it can also be represented by

$$\mathrm{B}(p) = \prod_{p_{ij}\geq 0} (V_{ij})^{p_{ij}} \prod_{p_{kl}\leq 0} (V^*_{kl})^{-p_{kl}}$$

where p is a 3×3 matrix with the following properties:

Properties of p

- The matrix elements of p are integers (positive, negative or 0).
- The sum of the elements of p in each row and column is equal to 0.
- A permutation of the rows and columns of the p matrix is reversible and the resulting matrix is also the p matrix of pure rephasing invariant.

Pure rephasing invariants

$$\mathrm{B}(p) = \prod_{p_{ij} \geq 0} (V_{ij})^{p_{ij}} \prod_{p_{kl} \leq 0} (V^*_{kl})^{-p_{kl}}$$

It is easy to show that $B(p)$ constructed in such a way is rephasing invariant and that it cannot be factored out into the squares of the CKM matrix elements and other rephasing invariant, so it is indeed the PRI.

The one to one mapping between the p matrix and PRI $B(p)$ has the following additional properties

$$\begin{aligned} &(p_1 + p_2) \rightarrow \mathrm{B}(p_1 + p_2) = \mathrm{B}(p_1) \cdot \mathrm{B}(p_2), \quad n \cdot p \rightarrow (\mathrm{B}(p))^n, \quad n \text{ integer}, \\ &\text{if } p \rightarrow \mathrm{B}(p), \text{ then } (-p) \rightarrow (\mathrm{B}(p))^*. \end{aligned} \tag{10}$$

Example 3.- For the no pure rephasing invariant

$$\begin{aligned} \mathrm{P}(m_3,n_3) &= V_{11}V_{22}V_{33}V_{12}^*V_{21}^*V_{33}^* \\ &= |V_{33}|^2 V_{11}V_{22}V_{12}^*V_{21}^* \end{aligned} \tag{11}$$

the matrices m_3, n_3 are:

$$m_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

And

$$\mathrm{B}(p) = V_{11}V_{22}V_{12}^*V_{21}^* \tag{13}$$

represent a pure rephasing invariant,

$$p = m_3 - n_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

Example 4.- If p is equal to

$$p = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ -3 & 2 & 1 \end{pmatrix},$$

The PRI defined by it is:

$$\mathrm{B}(p) = V_{11} V_{13}^2 V_{21}^2 V_{22} V_{32}^2 V_{33} \left(V_{12}^3 V_{23}^3 V_{31}^3 \right)^* .$$

It is easy to show that $\mathrm{B}(p)$ is rephasing invariant and it cannot be factored into the square of the CKM matrix elements and other rephasing invariants so it is PRI.

$$p = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ -3 & 2 & 1 \end{pmatrix},$$

The matrix p can be decomposed into the sum of matrices, all of them with the same properties that p, for example

$$p = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad (15)$$

another possible decomposition for p is the following one [3]

$$p = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \quad (16)$$

The eqs. (20) and (21) indicate us that exist a subgroup within the pure rephasing invariants from which all pure rephasing invariants can be constructed. It takes us to the following definition.

[3] there is no unique way for the decomposition of p.

Fundamental rephasing invariant

Definition 2

The *fundamental rephasing invariant (FRI)* is such a pure rephasing invariant monomial that is the product of 4 or 6 CKM matrix elements and its complex conjugates.

there are 30 fundamental rephasing invariants, 18 of them are the products of 4 CKM matrix elements and their complex conjugates

Set 1: 4-th order FRIs ($J_1, J_2, \ldots, J_{18}$)

$$
\begin{aligned}
J_1 &= V_{11}V_{22}V_{12}^*V_{21}^*, \quad J_5 = V_{11}V_{33}V_{13}^*V_{31}^*, \\
J_2 &= V_{11}V_{23}V_{13}^*V_{21}^*, \quad J_6 = V_{12}V_{33}V_{13}^*V_{32}^*, \\
J_3 &= V_{12}V_{23}V_{13}^*V_{22}^*, \quad J_7 = V_{21}V_{32}V_{22}^*V_{31}^*, \\
J_4 &= V_{11}V_{32}V_{12}^*V_{31}^*, \quad J_8 = V_{21}V_{33}V_{23}^*V_{31}^*, \\
J_9 &= V_{22}V_{33}V_{23}^*V_{32}^* \\
J_{9+i} &= (J_i)^*, \quad i = 1, \ldots, 9.
\end{aligned} \tag{17}
$$

Fundamental rephasing invariant

and 12 are the products of 6 of the CKM matrix elements and their complex conjugates

Set 2: 6-th order FRIs ($I_1, I_2, \ldots, I_{12}$)

$$\begin{aligned} I_1 &= V_{11}V_{22}V_{33}V_{13}^*V_{21}^*V_{32}^*, \quad & I_4 &= V_{11}V_{23}V_{32}V_{13}^*V_{22}^*V_{31}^*, \\ I_2 &= V_{11}V_{22}V_{33}V_{12}^*V_{23}^*V_{31}^*, \quad & I_5 &= V_{12}V_{23}V_{31}V_{13}^*V_{21}^*V_{32}^*, \\ I_3 &= V_{11}V_{23}V_{32}V_{12}^*V_{21}^*V_{33}^*, \quad & I_6 &= V_{12}V_{21}V_{33}V_{13}^*V_{22}^*V_{31}^* \\ I_{6+i} &= (I_i)^*, \quad i = 1, \ldots, 6. \end{aligned} \tag{18}$$

For each FRI in Eqs. (17) and (18) there corresponds a p matrix, e.g.,

$$J_1 \to p_{J_1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad I_1 \to p_{I_1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \text{ etc.} \tag{19}$$

All the matrices p_{J_i} and p_{I_i} corresponding to the invariants in Eqs. (17) and (18) can be obtained by the permutations of the rows and columns the p matrices of J_1 and I_1 that are given in Eq. (19). This means that an arbitrary permutation of the rows and columns of a p_J matrix maps it into another p_J matrix. The same applies to the p_I matrices.

Theorem 2

Any pure rephasing invariant can be expressed in a unique way as the product of positive powers of at most 4 fundamental rephasing invariants. Not more than one of these invariants can be from the 6-th order FRIs $(I_1, I_2, \ldots, I_{12})$ and the remaining are from the 4-th order FRIs $(J_1, J_2, \ldots, J_{18})$.

There are two important conditions in theorem 2:

- The powers of the invariants are *positive*.
- In the decomposition there may be no more than 1 fundamental rephasing invariant of the 6-th order.

Without these conditions the decomposition of a pure rephasing invariant into the fundamental invariants is not unique. The proof of this theorem is given in the Appendix[4].

[4] The inverse theorem is not true, the product of two or more FRIs has to be rephasing invariant but it does not have to be PRI.

$$p = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ -3 & 2 & 1 \end{pmatrix},$$

The matrix p can be decomposed into the sum of FRIs

$$p = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad (20)$$

another possible decomposition for p is the following one.

$$p = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \quad (21)$$

The Main Theorem for the RI

The Main Theorem

Any rephasing invariant monomial of the CKM matrix for 3 generations is the product of no more than 5 factors: 4 fundamental rephasing invariants with positive powers and the product of the squares of the absolute values of the CKM matrix elements also with positive powers. Only one fundamental invariant is from 6-th order FRIs $(I_1, I_2, \ldots, I_{12})$.

The unitary of the CKM matrix

From the unitarity of the CKM matrix it follows that the 6-th order FRIs in Eq. (18) can be expressed by the 4-th order FRIs from Eq. (17) and the squares of the CKM matrix elements[5]. We have for example,

$$I_1 = V_{11}V_{22}V_{33}V_{13}^*V_{21}^*V_{32}^* = |V_{22}|^2 V_{12}V_{33}V_{13}^*V_{32}^* - |V_{13}|^2 V_{22}V_{33}V_{23}^*V_{32}^* = |V_{22}|^2 J_6 - |V_{13}|^2 J_9 \tag{22}$$

and there are analogous formulas for the remaining I_i's.

From this and The Main Theorem follows immediately that the imaginary part of any rephasing invariant monomial is proportional to the Jarlskog invariant or equal to 0.

[5] It should be emphasized that without the unitarity of the CKM matrix there are no simple relations between the invariants of the 4-th and 6-th order. Thus relation (22) is also a test of the unitarity of the CKM matrix.

The unitary of the CKM matrix

From the unitary of the CKM matrix it follows that the 6 FRIs can be expressed by the 9 FRIs and the squares of the CKM matrix elements. We have for example that

$$\begin{aligned} V_{11}V_{23}V_{32}V_{12}^*V_{21}^*V_{33}^* &= V_{11}V_{32}V_{12}^*V_{21}^*\left(V_{23}V_{33}^*\right) \\ &= -V_{11}V_{32}V_{12}^*V_{21}^*\left(V_{21}V_{31}^*+V_{22}V_{32}^*\right) \\ &= -|V_{21}|^2V_{11}V_{32}V_{12}^*V_{31}^*-|V_{32}|^2V_{11}V_{22}V_{12}^*V_{21}^*. \end{aligned} \tag{23}$$

$$\begin{aligned} V_{13}V_{21}V_{32}V_{12}^*V_{23}^*V_{31}^* &= V_{21}V_{32}V_{12}^*V_{31}^*\left(V_{13}V_{23}^*\right) \\ &= -V_{21}V_{32}V_{12}^*V_{31}^*\left(V_{11}V_{21}^*+V_{12}V_{22}^*\right) \\ &= -|V_{21}|^2V_{11}V_{32}V_{12}^*V_{31}^*-|V_{12}|^2V_{21}V_{32}V_{22}^*V_{31}^*. \end{aligned} \tag{24}$$

$$\begin{aligned} V_{13}V_{21}V_{32}V_{12}^{*}V_{23}^{*}V_{31}^{*} &= V_{13}V_{21}V_{23}^{*}V_{31}^{*}\left(V_{32}V_{12}^{*}\right) \\ &= -V_{13}V_{21}V_{23}^{*}V_{31}^{*}\left(V_{31}V_{11}^{*}+V_{33}V_{13}^{*}\right) \\ &= -|V_{31}|^{2}V_{13}V_{21}V_{11}^{*}V_{23}^{*}-|V_{13}|^{2}V_{21}V_{33}V_{23}^{*}V_{31}^{*}. \end{aligned} \tag{25}$$

$$\begin{aligned} V_{13}V_{21}V_{32}V_{12}^{*}V_{23}^{*}V_{31}^{*} &= V_{13}V_{32}V_{12}^{*}V_{23}^{*}\left(V_{21}V_{31}^{*}\right) \\ &= -V_{13}V_{32}V_{12}^{*}V_{23}^{*}\left(V_{22}V_{32}^{*}+V_{23}V_{33}^{*}\right) \\ &= -|V_{32}|^{2}V_{22}V_{13}V_{23}^{*}V_{12}^{*}-|V_{23}|^{2}V_{13}V_{32}V_{12}^{*}V_{33}^{*}. \end{aligned} \tag{26}$$

From the Eqs. 24, 25 and 26, we conclude that:

$$\begin{aligned} \mathrm{Im}\left(V_{13}V_{21}V_{32}V_{12}^{*}V_{23}^{*}V_{31}^{*}\right) &= \left(|V_{21}|^{2}-|V_{12}|^{2}\right)J, \\ &= \left(|V_{32}|^{2}-|V_{23}|^{2}\right)J, \\ &= \left(|V_{13}|^{2}-|V_{31}|^{2}\right)J, \end{aligned} \tag{27}$$

therefore

$$|V_{21}|^{2}-|V_{12}|^{2}=|V_{32}|^{2}-|V_{23}|^{2}=|V_{13}|^{2}-|V_{31}|^{2}. \tag{28}$$

The unitarity of the CKM matrix gives us the next two theorems:

Theorem 3

The imaginary part of any monomial $\mathrm{P}(m,n)$ rephasing invariant is proportional to J or zero.

Theorem 4

The imaginary part of any polynomial constructed from the rephasing invariants $\mathrm{P}(m,n)$ with real coefficients, is proportional to J or zero

Conclusions

1.- In this presentation, we presented a recursive construction technique that extends any RI in terms of the FRIs and the absolute values of the CKM matrix elements (the main theorem).

2.- The most important result is stated in The Main Theorem and it is mathematically a strong result. It tells that any rephasing invariant monomial of the CKM matrix can be expressed as the product of 5 factors which are functions of a finite, small number of the FRI monomials.

3.- The unitarity of the CKM matrix allows to express the 6-th order rephasing invariant monomials by the 4-th monomials.
From this fact and the main theorem follows that:

4.- The imaginary part of any monomial $\mathrm{P}(m,n)$ rephasing invariant is proportional to J or zero.

5.- The imaginary part of any polynomial constructed from the rephasing invariants $\mathrm{P}(m,n)$ with real coefficients, is proportional to J or zero

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Parametrizations of the CKM Matrix

Introductory remarks

The CKM matrix is unitary but the rephasing freedom of the quarks fields reduces the number of physically significant parameters. The $n \times n$ unitary matrix is described by n^2 parameters.The up and down quarks rephasing freedom reduces the numbers of the parameters by $2n-1$, so the CKM matrix for n quark families is described by $(n-1)^2$ parameters. These $(n-1)^2$ parameters are divided into two classes: angles and phases. Angles are the parameters of the $n \times n$ real unitary matrix (orthogonal matrix) and there are $\frac{n(n-1)}{2}$ angles. The remaining $\frac{(n-1)(n-2)}{2}$ parameters are phases. One can observe that if the number of generations is increased from $(n-1)$ to n-then the number of angles increases by $(n-1)$ and the number of the phases by $(n-2)$. We want to present the recursive construction of the parametrization of the $n \times n$ CKM matrix $V^{(n)}$ from the $(n-1) \times (n-1)$ CKM matrix $V^{(n-1)}$. We introduce such a notation of the CKM matrix where the parameters (angles an phases) are labeled according to the generation to which they belong.

Wolfenstein Parametrization of the CKM Matrix

The CKM matrix has 9 complex matrix elements, which are parameterized by 4 real parameters. The choice of the parameters is not unique. There exist various equivalent parameterizations, which were chosen to fulfill various needs. Let us start with the standard parameterization of the PDG [6] This parameterization is exact and uses 3 angles and 1 phase and can be represented as the product of 3 real rotation matrices and the diagonal matrices with phase terms.

$$\hat{V}_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix},$$

For phenomenological applications, it would be useful to have a parametrization of the CKM matrix that makes the hierarchy arising. In order to derive such a parametrization, one introduces a set of new parameters, λ, A, ρ and η, by imposing the following relations:

$$s_{12} \equiv \lambda = 0{,}22, \quad s_{23} \equiv A\lambda^2, \quad s_{13}e^{-i\delta_{13}} \equiv A\lambda^3(\rho - i\eta). \tag{29}$$

[6] K. Nakamura and P.D. Group, Journal of Physics G: Nuclear and Particle Physics 37, 075021 (2010)

Another widely used parameterization is the one proposed by Wolfenstein [7]. Initially it was considered to be an approximate representation of the CKM matrix, because it was chosen in such a way as to reproduce the suppression for the weak transitions of quarks between the generations. Later, it was made exact [8].

The famous "Wolfenstein parametrization":

$$\hat{V}_{\text{CKM}} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathscr{O}(\lambda^4),$$

Using the exact standard parametrization one can calculate straightforwardly each CKM element to the desired accuracy in λ

[7] Phys. Rev. Lett.51, 1945 (1983)

[8] A.J. Buras, Phys. Rev. D50 34336 (1994)

The Jarlskog Invariant

$$J = c_{12}c_{23}c_{13}^2 s_{12}s_{23}s_{13}\sin\delta\,,$$

The Jarlskog Invariant

$$J = A^2\lambda^6\eta + \mathscr{O}(\lambda^8).$$

Recursive construction of the CKM matrix

Angles and Phases

We will present here the recursive construction of the $(n \times n)$ CKM matrix $V^{(n)}$, assuming that the $(n-1) \times (n-1)$ CKM matrix $V^{(n-1)}$ is known.

Recursive construction of the CKM matrix

Let us introduce the notation, where the parameters of the CKM matrix (angle-like and phases) are labeled according to the generation number to which they belong:

Angles and Phases

$$\begin{array}{ll} \theta_{1,k}, \theta_{2,k}, \ldots, \theta_{k-1,k} & k = 2, 3, \ldots \\ \delta_{1,k}, \delta_{2,k}, \ldots, \delta_{k-2,k} & k = 3, 4, \ldots \end{array}$$

where $\theta_{k-1,k}$ and $\delta_{k-1,k}$ are angles and phase, respectively

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix},$$

$$c_{ij} = \cos\theta_{ij}, \qquad s_{ij} = \sin\theta_{ij},$$

Recursive construction of the CKM matrix

In such a way we form the hierarchy of the parameters

Angles and Phases

n-Generations	Parameters
2	$\theta_{1,2}$
3	$\theta_{1,3},\ \theta_{2,3},\ \delta_{1,3}$
4	$\theta_{1,4},\ \theta_{2,4},\ \delta_{3,4}, \delta_{1,4}, \delta_{2,4}$
$\vdots$	$\vdots$

in the table above the parameters labelling the n-th family are those listed up to the line with n generations.

Recursive construction of the CKM matrix

In this section we will show how to construct the $n \times n$ CKM matrix $V^{(n)}$ if we know the matrix $V^{(n-1)}$. Let us first introduce notation of the columns of the

CKM matrix

The CKM matrix

$$V^{(n)} = \left(\mathbf{v}_1^{(n)}, \mathbf{v}_2^{(n)}, \ldots, \mathbf{v}_n^{(n)}\right),$$

here $V^{(n)}$ is the CKM matrix and $\mathbf{v}_1^{(n)}, \ldots, \mathbf{v}_n^{(n)}$ are the columns, e.g.

$$\mathbf{v}_k^{(n)} = \begin{pmatrix} v_{1k}^{(n)} \\ v_{2k}^{(n)} \\ \vdots \\ v_{nk}^{(n)} \end{pmatrix},$$

Recursive construction of the CKM matrix

The recursive construction of the CKM matrix is done in two steps:

a) Construction of n real column vectors

$$\mathbf{e}_1, \ldots, \mathbf{e}_n,$$

that depend on $(n-1)$ parameters

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij},$$

Recursive construction of the CKM matrix

b) The columns of the matrix $V^{(n)}$ are then constructed from the vectors $\mathbf{e}_k$ and the elements of the matrix $V_{ij}^{(n-1)}$

$$
\begin{aligned}
\mathbf{v}_k^{(n)} &= V_{1k}^{(n-1)}\mathbf{e}_1 + \sum_{l=2}^{n-1} V_{lk}^{(n-1)} e^{-i\delta_{l-1,n}}\mathbf{e}_l, \qquad k = 1,\ldots,n-1 \\
\mathbf{v}_n^{(n)} &= \mathbf{e}_n
\end{aligned}
$$

The matrix $V^{(n)}$ is thus equal

$$V^{(n)} = (\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n).$$

The matrix $V^{(n)}$ constructed in such a way has the following properties:

Recursive construction of the CKM matrix

- It is unitary. This follows from the unitarity of the matrix $V^{(n-1)}$ and the orthogonality of the vectors $\{\mathbf{e}_1, \ldots \mathbf{e}_n\}$.
- The resulting parametrization of the matrix $V^{(n)}$ depends on the parametrization of $V^{(n-1)}$ and of the vectors $(\mathbf{e}_1, \ldots \mathbf{e}_n)$.
- It depends on $\frac{n(n-1)}{2}$ angles and $\frac{(n-1)(n-2)}{2}$ phases.

As an example we will show how one can obtain the standard parameterization of the CKM matrix for 3 generations and we will obtain the analogue of the standard parametrization for 4 generations.

Example 1. Standard Parametrization for 3-generations

The matrix $V^{(2)}$ depends on one angle θ_{12} as it is 2×2 rotation matrix

2×2 rotation matrix

$$V^{(2)} = \begin{pmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{pmatrix},$$

The vector $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are choose in the following way.

$$\mathbf{e}_1 = \begin{pmatrix} c_{13} \\ -s_{13}s_{23} \\ -s_{13}c_{23} \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ c_{23} \\ -s_{23} \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} s_{13} \\ c_{13}s_{23} \\ c_{13}c_{23} \end{pmatrix},$$

$$c_{ij} = \cos\theta_{ij}, \qquad s_{ij} = \sin\theta_{ij},$$

The vectors in the equation above are orthonormal.

Example 1. Standard Parametrization

The columns of the matrix $V^{(3)}$ are equal

$$\begin{aligned}
\mathbf{v}_1^{(3)} &= V_{11}^{(2)}\mathbf{e}_1 + V_{21}^{(2)} e^{-i\delta_{13}}\mathbf{e}_2, \\
\mathbf{v}_2^{(3)} &= V_{12}^{(2)}\mathbf{e}_1 + V_{22}^{(2)} e^{-i\delta_{13}}\mathbf{e}_2, \\
\mathbf{v}_3^{(3)} &= \mathbf{e}_3,
\end{aligned}$$

$$\mathbf{v}_1^{(3)} = \left(\begin{array}{c} c_{12}c_{13} \\ -c_{12}s_{13}s_{23} - s_{12}c_{23}e^{i\delta_{13}} \\ -c_{12}s_{13}c_{23} + s_{12}s_{23}e^{i\delta_{13}} \end{array}\right), \mathbf{v}_2^{(3)} = \left(\begin{array}{c} s_{12}c_{13} \\ -s_{12}s_{13}s_{23} + c_{12}c_{23}e^{-i\delta_{13}} \\ -s_{12}s_{13}c_{23} - c_{12}s_{23}e^{-i\delta_{13}} \end{array}\right),$$

$$\mathbf{v}_3^{(3)} = \left(\begin{array}{c} s_{13} \\ c_{13}s_{23} \\ c_{13}c_{23} \end{array}\right),$$

La matriz V_{CKM}

$$V^{(3)} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -c_{12}s_{13}s_{23} - s_{12}c_{23}e^{i\delta_{13}} & -s_{12}s_{13}s_{23} + c_{12}c_{23}e^{-i\delta_{13}} & c_{13}s_{23} \\ -c_{12}s_{13}c_{23} + s_{12}s_{23}e^{i\delta_{13}} & -s_{12}s_{13}c_{23} - c_{12}s_{23}e^{-i\delta_{13}} & c_{13}c_{23} \end{pmatrix},$$

The form of the matrix in not exactly the same as that of the standard parametrization. However due to the rephasing freedom they are equivalent. Multiplying the first and second column by $e^{i\delta_{13}}$ on the first row by $e^{-i\delta_{13}}$ one obtains exactly the standard parametrization PDG.

Standard parametrization of the CKM matrix

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix},$$

Example 2. Standard Parametrization for 4-generations

For 4 generations one obtains the $V^{(4)}$ from the standard $V^{(3)}$ in on analogous way. First we construct the vectors

$$\mathbf{e}_1 = \begin{pmatrix} c_{14} \\ -s_{14}s_{24} \\ -s_{14}c_{24}s_{34} \\ -s_{14}c_{24}c_{34} \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ c_{24} \\ -s_{24}s_{34} \\ -s_{24}c_{24} \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ c_{34} \\ s_{34} \end{pmatrix}, \mathbf{e}_4 = \begin{pmatrix} s_{14} \\ c_{14}s_{24} \\ c_{14}c_{24}s_{34} \\ c_{14}c_{24}c_{34} \end{pmatrix},$$

$$c_{ij} = \cos\theta_{ij}, \qquad s_{ij} = \sin\theta_{ij},$$

The vectors in equation above are orthonormal. Then we construct the vectors $\mathbf{v}_i^{(4)}$ and the matrix $V^{(4)}$, which may be called the standard parametrization of the CKM matrix for 4 generations. The explicit form of this matrix is the following:

$$\begin{aligned} \mathbf{v}_1^{(4)} &= V_{11}^{(3)}\mathbf{e}_1 + V_{21}^{(3)}e^{-i\delta_{14}}\mathbf{e}_2 + V_{31}^{(3)}e^{-i\delta_{24}}\mathbf{e}_3, \\ \mathbf{v}_2^{(4)} &= V_{12}^{(3)}\mathbf{e}_1 + V_{22}^{(3)}e^{-i\delta_{14}}\mathbf{e}_2 + V_{32}^{(3)}e^{-i\delta_{24}}\mathbf{e}_3, \\ \mathbf{v}_3^{(4)} &= V_{13}^{(3)}\mathbf{e}_1 + V_{23}^{(3)}e^{-i\delta_{14}}\mathbf{e}_2 + V_{33}^{(3)}e^{-i\delta_{24}}\mathbf{e}_3, \\ \mathbf{v}_4^{(4)} &= \mathbf{e}_4, \end{aligned}$$

Example 2. Standard Parametrization for 4-generations

$$
\begin{aligned}
\mathbf{v}_1^{(4)} &= V_{11}^{(3)}\mathbf{e}_1 + V_{21}^{(3)}e^{-i\delta_{14}}\mathbf{e}_2 + V_{31}^{(3)}e^{-i\delta_{24}}\mathbf{e}_3,\\
\mathbf{v}_2^{(4)} &= V_{12}^{(3)}\mathbf{e}_1 + V_{22}^{(3)}e^{-i\delta_{14}}\mathbf{e}_2 + V_{32}^{(3)}e^{-i\delta_{24}}\mathbf{e}_3,\\
\mathbf{v}_3^{(4)} &= V_{13}^{(3)}\mathbf{e}_1 + V_{23}^{(3)}e^{-i\delta_{14}}\mathbf{e}_2 + V_{33}^{(3)}e^{-i\delta_{24}}\mathbf{e}_3,\\
\mathbf{v}_4^{(4)} &= \mathbf{e}_4,
\end{aligned}
$$

If the vectors $\mathbf{e}_i$ fulfill the conditions

$$
(\mathbf{e}_i)_j\Big|_{\substack{\theta_{k,n}=0\\ \delta_{l,n}=0}} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} \qquad \begin{array}{l} k=1,\ldots,n-1 \\ l=1,\ldots,n-2. \end{array} \tag{30}
$$

then one obtains

$$
V^{(n)}\Big|_{\substack{\theta_{k,n}=0\\ \delta_{l,n}=0}} = \begin{pmatrix} V^{(n-1)} & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{array}{l} k=1,\ldots,n-1 \\ l=1,\ldots,n-2. \end{array} \tag{31}
$$

$$
\mathbf{e}_1 = \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \mathbf{e}_4 = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix},
$$

Example 3. Wolfenstein Parametrization for 4 generations

The Wolfenstein parametrization has the form

$$V_{\mathsf{CKM}} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathscr{O}(\lambda^4),$$

Now we will generalise this parametrization to 4 generations using the method outlined earlier.

First we have to construct the vectors $\mathbf{e}_i$. We express them in the spirit of the Wolfenstein parametrization in terms of powers of λ.

Example 3. Wolfenstein Parametrization

The vectors $\mathbf{e}_i$ are real and are chosen in the following way.

$$\mathbf{e}_1 = N_1 \begin{pmatrix} 1+x_1^2+x_3^2 \\ -x_1x_2 \\ -x_1x_3 \\ -x_1 \end{pmatrix}, \mathbf{e}_2 = N_2 \begin{pmatrix} 0 \\ 1+x_3^2 \\ -x_2x_3 \\ -x_2 \end{pmatrix}, \mathbf{e}_3 = N_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -x_3 \end{pmatrix}, \mathbf{e}_4 = N_4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix},$$

Here $x_i = A_i^{(4)}\lambda^{k_i}$, $A_i^{(4)} \sim 1$ $k_i \geq 1$ are integers and N_i are the normalizations factors. It is easy to show that the vectors $\mathbf{e}_i$ satisfy

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

The columns of the CKM matrix for 4 generations are then equal

$$\begin{aligned}
\mathbf{v}_1^{(4)} &= V_{11}^{(3)}\mathbf{e}_1 + V_{21}^{(2)}e^{i\delta_{14}}\mathbf{e}_2 + V_{31}^{(2)}e^{i\delta_{24}}\mathbf{e}_3 \\
\mathbf{v}_2^{(4)} &= V_{12}^{(3)}\mathbf{e}_1 + V_{22}^{(2)}e^{i\delta_{14}}\mathbf{e}_2 + V_{32}^{(2)}e^{i\delta_{24}}\mathbf{e}_3 \\
\mathbf{v}_2^{(4)} &= V_{13}^{(3)}\mathbf{e}_1 + V_{23}^{(2)}e^{i\delta_{14}}\mathbf{e}_2 + V_{32}^{(2)}e^{i\delta_{24}}\mathbf{e}_3 \\
\mathbf{v}_4^{(4)} &= \mathbf{e}_4
\end{aligned}$$

Example 3. Wolfenstein Parametrization

and

$$V^{(4)} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4,) \tag{32}$$

$$\mathbf{v}_1^{(4)} = N_1 \begin{pmatrix} v_{11}^{(3)}(1+x_2^2+x_3^2) \\ -v_{11}^{(3)}x_1x_2 + e^{i\delta_{14}}v_{21}^{(3)}(1+x_3^2) \\ -v_{11}^{(3)}x_1x_3 - e^{i\delta_{14}}v_{21}^{(3)}x_2x_3 + e^{i\delta_{24}}v_{31}^{(3)} \\ -v_{11}^{(3)}x_1 - e^{i\delta_{14}}v_{21}^{(3)}x_2 - e^{i\delta_{24}}v_{31}^{(3)}x_3 \end{pmatrix}, \mathbf{v}_2^{(4)} = N_2 \begin{pmatrix} v_{12}^{(3)}(1+x_2^2+x_3^2) \\ -v_{12}^{(3)}x_1x_2 + e^{i\delta_{14}}v_{22}^{(3)}(1+x_3^2) \\ -v_{12}^{(3)}x_1x_3 - e^{i\delta_{14}}v_{22}^{(3)}x_2x_3 + e^{i\delta_{24}}v_{32}^{(3)} \\ -v_{12}^{(3)}x_1 - e^{i\delta_{14}}v_{22}^{(3)}x_2 - e^{i\delta_{24}}v_{32}^{(3)}x_3 \end{pmatrix}$$

$$\mathbf{v}_3^{(4)} = N_3 \begin{pmatrix} v_{13}^{(3)}(1+x_2^2+x_3^2) \\ -v_{13}^{(3)}x_1x_2 + e^{i\delta_{14}}v_{23}^{(3)}(1+x_3^2) \\ -v_{13}^{(3)}x_1x_3 - e^{i\delta_{14}}v_{23}^{(3)}x_2x_3 + e^{i\delta_{24}}v_{33}^{(3)} \\ -v_{13}^{(3)}x_1 - e^{i\delta_{14}}v_{23}^{(3)}x_2 - e^{i\delta_{24}}v_{33}^{(3)}x_3 \end{pmatrix}, \mathbf{v}_4^{(4)} = N_4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix},$$

The matrix $V^{(4)}$ is described by 9 parameters: λ, A, ρ η of the matrix $V^{(3)}$ and $A_1^{(4)}$, $A_2^{(4)}$, $A_3^{(4)}$, δ_{14}, δ_{24}, of the 4-th generation, Not all these parameters can be determined from the experimental data.

On the other hand we can derive some restriction on the powers k_i that determine the suppression in $V^{(4)}$. We have the following information

$$
\begin{aligned}
|V_{12}| \sim \lambda, \qquad & |V_{21}| \sim \lambda, \\
|V_{23}| \sim \lambda^2, \qquad & |V_{32}| \sim \lambda^2, \\
||V_{13}| \sim \lambda^3, \qquad & |V_{31}| \sim \lambda^3, \\
& (|V_{12}| - |V_{21}|) \sim \lambda^3, \\
& (|V_{23}| - |V_{32}|) \sim \lambda^4,
\end{aligned}
$$

If one uses the unitarity of the 3×3 CKM matrix then one has $(|V_{12}| - |V_{21}|) \sim \lambda^5$, but the element $|V_{21}|$ is not measured whith such a precision and experimentally $(|V_{12}| - |V_{21}|) \sim \lambda^3$ holds as in eq. above. Now using the information in the explicit form of the 4×4 CKM matrix we obtain the following restrictions on the powers k_i

$$k_i \geq 1, \qquad k_1 + k_2 \geq 3, \qquad k_2 + k_3 \geq 4 \qquad , k_1 + k_3 \geq 3,$$

which can be resolved and give

$$k_1 \geq 1, \qquad k_2 \geq 2, \qquad k_3 \geq 2.$$

The vector $\mathbf{v}_4$ for minimal values of k_i in Eq. (21) has the following form

$$\mathbf{v}_1^{(4)} = \begin{pmatrix} A_1^{(4)}\lambda \\ A_2^{(4)}\lambda^2 \\ A_3^{(4)}\lambda^2 \\ 1 \end{pmatrix},$$

This result is rather surprising because in the case of the 3 generations this suppression has totally different structure [9]. We would like to note that the real suppression may be different, because we have only obtained the lower limits of the suppression powers. To conclude this section we will compare the values of two Jarlskog invariants of the 3×3 sub-matrix

$$J_A = \mathrm{Im}\,(V_{12}V_{23}V_{13}^*V_{22}^*)\,,$$
$$J_B = \mathrm{Im}\,(V_{21}V_{33}V_{23}^*V_{31}^*)\,,$$

that describe the CP violations effects in the strange and bottom sectors.
In 3 dimensions we have from the unitarity of the CKM matrix

$$J_A + J_B = 0$$

In 4 dimensions, if we keep the only the terms of the highest order we obtain

$$J_A + J_B = \mathrm{Im}\left(\mathrm{e}^{-i\delta_{24}}V_{21}^{(3)}V_{33}^{(3)}V_{23}^{(3)}V_{11}^{(3)}\right)x_1x_2 \sim \lambda^6$$

[9] The full analysis of the 4×4 CKM matrix based on Eq. (32) will be published elsewhere.

which is of the same order λ^6 as J_A and J_B themselves. If means that for supperession described by Eq.(21) the CP violation in the K and B sector have diferent phases. This result follows from the condition

$$k_1 + k_3 = 3,$$

If $k_1 + k_3 = 4$ then $J_A + J_B \sim \lambda^7$ and for $k_1 + k_3 = 5$ we would have $J_A + J_B \sim \lambda^8$ and the CP violation in K and B sector for four vectors would differ at the level of 5 %.

Conclusions

To conclude this section let us note that the presence of the 4-th generation in the CKM matrix $V^{(3)}$ can be observed through the violation of the unitarity of $V^{(3)}$. This can be done by experimental verification of the asymmetry of the CKM matrix

$$|V_{12}|^2 - |V_{21}|^2 = |V_{23}|^2 - |V_{32}|^2 \tag{33}$$

or by observation of different CP violating phases in K and B decays.

La matriz V_{CKM}

La matriz V_{CKM}

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix},$$

El tringulo unitario

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0,$$

There are two important bases. One where the masses are diagonal, called the **mass basis**, and the other where the $W^{\pm}$ interactions are diagonal, called the **interaction basis**. The CKM matrix is the matrix that rotates between these two bases.

Since most measurements are done in the mass basis, we write the interactions in that basis. Upon the replacement $\mathrm{Re}(\phi^0) \to (v+H^0)/\sqrt{2}$, we decompose the $SU(2)_{\mathrm{L}}$ quark doublets into their components:

$$Q^I_{Li} = \begin{pmatrix} U^I_{Li} \\ D^I_{Li} \end{pmatrix},$$

and then the Yukawa interactions, , give rise to mass terms:

$$-\mathscr{L}^q_M = (M_d)_{ij}\overline{D^I_{Li}}D^I_{Rj} + (M_u)_{ij}\overline{U^I_{Li}}U^I_{Rj} + \mathrm{h.c.}, \qquad M_q = \frac{v}{\sqrt{2}}Y^q, \tag{34}$$

The mass basis corresponds, by definition, to diagonal mass matrices. We can always find unitary matrices U_L and U_R such that

$$U_L M_u U_R^{\dagger} = M_u^{\mathrm{diag}}, \qquad U_L M_d D_R^{\dagger} = M_d^{\mathrm{diag}}, \tag{35}$$

with M_q^{diag} diagonal and real. The quark mass eigenstates are then identified as

$$u_{Li} = (U_L)_{ij}u^I_{Lj}, \qquad u_{Ri} = (U_R)_{ij}u^I_{Rj}, \tag{36}$$

$$d_{Li} = (D_L)_{ij}d^I_{Lj}, \qquad d_{Ri} = (D_R)_{ij}d^I_{Rj}, \tag{37}$$

The charged current interactions for quarks are the interactions of the $W^{\pm}_{\mu}$, which in the interaction basis are described by

$$-\mathscr{L}^{q}_{W^{\pm}} = \frac{g}{\sqrt{2}}\overline{u_{Li}}\gamma^{\mu}(V_{uL}V^{\dagger}_{dL})_{ij}d_{Lj}W^{+}_{\mu} + \text{h.c.}. \qquad (38)$$

It has a more complicated form in the mass basis:
The unitary 3×3 matrix,

$$V = V_{uL}V^{\dagger}_{dL}, \qquad (VV^{\dagger} = \mathbf{1}), \qquad (39)$$

is the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix for quarks. As a result of the fact that V is not diagonal, the $W^{\pm}$ gauge bosons couple to mass eigenstates quarks of different generations. Within the SM, this is the only source of flavor changing quark interactions.
The form of the CKM matrix is not unique. We already counted and concluded that only one of the phases is physical. This implies that we can find bases where V has a single phase. This physical phase is the Kobayashi-Maskawa phase that is usually denoted by δ_{KM}.