The flavour permutational symmetry S_3

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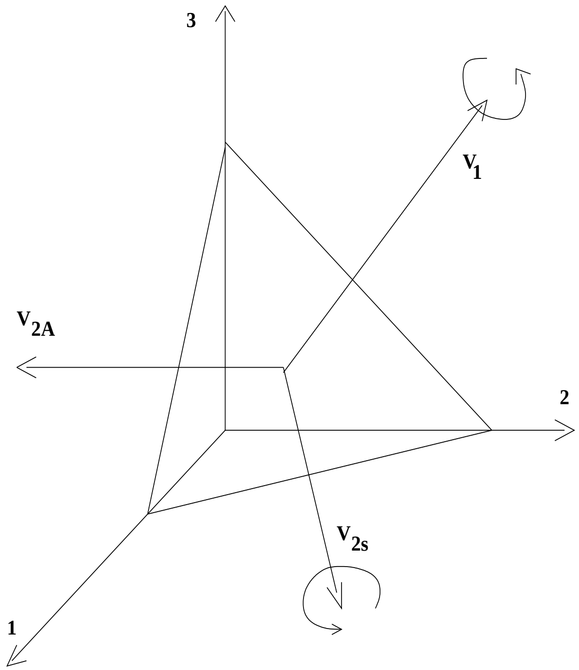
Contents

The Group S_3

The group S_3 of permutations of three objects



Permutations		Rotations
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$\Longleftrightarrow$	a $120^\circ-$ rotation around the invariant vector $\mathbf{V}_1$
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	$\Longleftrightarrow$	a 180°rotation around the invariant vector $\mathbf{V}_{2s}$

Symmetry adapted basis

$$|v_{2A}>= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} , \ |v_{2s}>= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} , \ |v_1>= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Irreducible representations of S_3

The group S_3 has two one-dimensional irreps (singlets) and one two-dimensional irrep (doublet)

- one dimensional: $\mathbf{1}_A$ antisymmetric singlet, $\mathbf{1}_s$ symmetric singlet
- Two - dimensional: 2 doublet

Direct product of irreps of S_3

$$\mathbf{1}_s \otimes \mathbf{1}_s = \mathbf{1}_s, \quad \mathbf{1}_s \otimes \mathbf{1}_A = \mathbf{1}_A, \quad \mathbf{1}_A \otimes \mathbf{1}_A = \mathbf{1}_s, \quad \mathbf{1}_s \otimes \mathbf{2} = \mathbf{2}, \quad \mathbf{1}_A \otimes \mathbf{2} = \mathbf{2}$$

$$\boxed{\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_s \oplus \mathbf{1}_A \oplus \mathbf{2}}$$

the direct (tensor) product of two doublets

$$\mathbf{p_D} = \begin{pmatrix} p_{D1} \\ p_{D2} \end{pmatrix} \qquad \text{and} \qquad \mathbf{q}_D = \begin{pmatrix} q_{D1} \\ q_{D2} \end{pmatrix}$$

has two singlets, r_s and r_A, and one doublet r_D^T

$$r_s = p_{D1}q_{D1} + p_{D2}q_{D2} \quad \text{is invariant}, \qquad r_A = p_{D1}q_{D2} - p_{D2}q_{D1} \quad \text{is not invariant}$$

$$r_D^T = \begin{pmatrix} p_{D1}q_{D2} + p_{D2}q_{D1} \\ p_{D1}q_{D1} - p_{D2}q_{D2} \end{pmatrix}$$

A Minimal S_3 invariant extension of the SM

The Higgs sector is modified,

$$\Phi \to H \quad = \quad (\Phi_1, \Phi_2, \Phi_3)^T$$

H is a reducible $\mathbf{1_s} \oplus 2$ rep. of S_3

$$H_s \quad = \quad \frac{1}{\sqrt{3}}\Big(\Phi_1 + \Phi_2 + \Phi_3\Big)$$

$$H_D \quad = \quad \begin{pmatrix} \frac{1}{\sqrt{2}}(\Phi_1 - \Phi_2) \\ \\ \frac{1}{\sqrt{6}}(\Phi_1 + \Phi_2 - 2\Phi_3) \end{pmatrix}$$

Quark, lepton and Higgs fields are

$$Q^T = (u_L, d_L), u_R, d_R, \qquad L^\dagger = (\nu_L, e_L), e_R, \nu_R, \qquad H$$

All these fields have three species (flavours) and belong to a reducible $\mathbf{1} \oplus 2$ rep. of S_3

Leptons' Yukawa interactions

Leptons

$$\begin{aligned}\mathcal{L}_{Y_E} &= -Y_1^e \overline{L}_I H_S e_{IR} - Y_3^e \overline{L}_3 H_S e_{3R} - Y_2^e [\, \overline{L}_I \kappa_{IJ} H_1 e_{JR} + \overline{L}_I \eta_{IJ} H_2 e_{JR} \,] \\ &- Y_4^e \overline{L}_3 H_I e_{IR} - Y_5^e \overline{L}_I H_I e_{3R} + h.c.,\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{Y_\nu} &= -Y_1^\nu \overline{L}_I (i\sigma_2) H_S^* \nu_{IR} - Y_3^\nu \overline{L}_3 (i\sigma_2) H_S^* \nu_{3R} \\ &- Y_2^\nu [\, \overline{L}_I \kappa_{IJ} (i\sigma_2) H_1^* \nu_{JR} + \overline{L}_I \eta_{IJ} (i\sigma_2) H_2^* \nu_{JR} \,] \\ &- Y_4^\nu \overline{L}_3 (i\sigma_2) H_I^* \nu_{IR} - Y_5^\nu \overline{L}_I (i\sigma_2) H_I^* \nu_{3R} + h.c.\end{aligned}$$

$$\kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad I, J = 1, 2$$

Furthermore, the Majorana mass terms for the right handed neutrinos are

$$\mathcal{L}_M = -\nu_{IR}^T C \mathbf{M}_I \nu_{IR} - M_3 \nu_{3R}^T C \nu_{3R},$$

C is the charge conjugation matrix.

Mass matrices

We will assume that

$$< H_{D1} > \;=\; < H_{D2} > \neq 0 \qquad \text{and} \qquad < H_3 > \neq \; 0$$

and

$$< H_3 >^2 + < H_{D1} >^2 + < H_{D2} >^2 \;\approx\; \Big(\frac{246}{2} GeV\Big)^2$$

Then, the Yukawa interactions yield mass matrices of the general form

$$\mathbf{M} = \begin{pmatrix} \mu_1 + \mu_2 & \mu_2 & \mu_5 \\ \mu_2 & \mu_1 - \mu_2 & \mu_5 \\ \mu_4 & \mu_4 & \mu_3 \end{pmatrix}$$

The Majorana masses for ν_L are obtained from the see-saw mechanism

$$M_\nu = M_{\nu D} \tilde{\mathsf{M}}^{-1} (M_{\nu D})^T \qquad \text{with} \qquad \tilde{\mathsf{M}} = \mathrm{diag}(M_1, M_2, M_3)$$

Mixing matrices

The mass matrices are diagonalized by unitary matrices

$$U^{\dagger}_{d(u,e)L}\mathbf{M}_{d(u,e)}\mathbf{U}_{d(u,e)R} \quad = \quad \mathsf{diag}\Big(m_{d(u,e)}m_{s(c,\mu)}m_{b(t,\tau)}\Big)$$

and

$$U_{\nu}^{T}M_{\nu}U_{\nu} \quad = \quad \mathsf{diag}\Big(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}\Big)$$

The masses can be complex, and so, U_{eL} is such that

$$U^{\dagger}_{eL}M_eM_e^{\dagger}U_{eL} \quad = \quad \mathsf{diag}\Big(|m_e|^2, |m_\mu|^2, |m_\tau|^2\Big), \qquad etc.$$

The quark mixing matrix is

$$V_{CKM} \quad = \quad U^{\dagger}_{uL}U_{dL}$$

and, the neutrino mixing matrix is

$$\mathbf{V}_{MNS} \quad = \quad U^{\dagger}_{eL}U_{\nu}$$

Masses and mixings in the quark sector

The mass matrices for the quark sector take the general form

$$\mathbf{M}_{u(d)} = \begin{pmatrix} \mu_1^{u(d)} + \mu_2^{u(d)} & \mu_2^{u(d)} & \mu_5^{u(d)} \\ \mu_2^{u(d)} & \mu_1^{u(d)} - \mu_2^{u(d)} & \mu_5^{u(d)} \\ \mu_4^{u(d)} & \mu_4^{u(d)} & \mu_3^{u(d)} \end{pmatrix}$$

$$U^{\dagger}_{u(d)L} M_{u(d)} M^{\dagger}_{u(d)} U_{u(d)L} = \mathrm{diag}\Big(|m_{u(d)}|^2, |m_{c(s)}|^2, |m_{t(b)}|^2 \Big),$$

$$V_{CKM} = U^{\dagger}_{uL} U_{dL}$$

The set of dimensionless parameters

$$\mu_1^u/\mu_0^u = -0.000293, \quad \mu_2^u/\mu_0^u = -0.00028, \quad \mu_3^u/\mu_0^u = 1,$$
$$\mu_4^u/\mu_0^u = 0.031, \quad \mu_5^u/\mu_0^u = 0.0386,$$
$$\mu_1^d/\mu_0^d = 0.0004, \quad \mu_2^d/\mu_0^d = 0.00275, \quad \mu_3^d/\mu_0^d = 1 + 1.2I,$$
$$\mu_4^d/\mu_0^d = 0.283, \quad \mu_5^d/\mu_0^d = 0.058,$$

The quark mixing matrix

Yields the mass hierarchy and the mixing matrix

$$m_u/m_t = 2.5469 \times 10^{-5}\,, \qquad m_c/m_t = 3.9918 \times 10^{-3},$$
$$m_d/m_b = 1.5261 \times 10^{-3}\,, \qquad m_s/m_b = 3.2319 \times 10^{-2},$$

The computed mixing matrix is

$$\mathbf{V}_{CKM} = \begin{pmatrix} 0.968 + 0.117I & 0.198 + 0.0974I & -0.00253 - 0.00354I \\ -0.198 + 0.0969I & 0.968 - 0.115I & -0.0222 - 0.0376I \\ 0.00211 + 0.00648I & 0.0179 - 0.0395I & 0.999 - 0.00206I \end{pmatrix}$$

$$|\mathbf{V}^{th}_{CKM}| = \begin{pmatrix} 0.975 & 0.221 & 0.00435 \\ 0.221 & 0.974 & 0.0437 \\ 0.00682 & 0.0434 & 0.999 \end{pmatrix}$$

which should be compared with

$$|V^{\exp}_{\rm CKM}| \;=\; \begin{pmatrix} 0.97383 \pm 0.00024 & 0.2272 \pm 0.0010 & (3.96 \pm 0.09) \times 10^{-3} \\ 0.2271 \pm 0.010 & 0.97296 \pm 0.00024 & (42.21 \pm 0.45 \pm 0.09) \times 10^{-3} \\ (8.14 \pm 0.5) \times 10^{-3} & (41.61 \pm 0.12) \times 10^{-3} & 0.9991 \pm 0.000034 \end{pmatrix}$$

The Jarlskog invariant is

$$J = \mathrm{Im}\,[(V_{\rm CKM})_{11}(V_{\rm CKM})_{22}(V^*_{\rm CKM})_{12}(V^*_{\rm CKM})_{21}] = 2.9 \times 10^{-5}, \quad J^{\bf exp} = (3.0 \pm 0.3) \times 10^{-5}$$

The leptonic sector

To achieve a further reduction of the number of parameters, in the leptonic sector, we introduce an additional discrete Z_2 symmetry

$-$	$+$
$H_I,\ \nu_{3R}$	$H_S,\ L_3,\ L_I,\ e_{3R},\ e_{IR},\ \nu_{IR}$

then,

$$Y_1^e = Y_3^e = Y_1^\nu = Y_5^\nu = 0$$

Hence, the leptonic mass matrices are

$$M_e = \begin{pmatrix} \mu_2^e & \mu_2^e & \mu_5^e \\ \mu_2^e & -\mu_2^e & \mu_5^e \\ \mu_4^e & \mu_4^e & 0 \end{pmatrix} \qquad \text{and} \qquad M_{\nu D} = \begin{pmatrix} \mu_2^\nu & \mu_2^\nu & 0 \\ \mu_2^\nu & -\mu_2^\nu & 0 \\ \mu_4^\nu & \mu_4^\nu & \mu_3^\nu \end{pmatrix}$$

The Mass Matrix of the charged leptons as function of its eigenvalues

The mass matrix of the charged leptons is

$$M_e \approx m_\tau \begin{pmatrix} \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}}\sqrt{\frac{1+x^2-\tilde{m}_\mu^2}{1+x^2}} \\ \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & -\frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}}\sqrt{\frac{1+x^2-\tilde{m}_\mu^2}{1+x^2}} \\ \frac{\tilde{m}_e(1+x^2)}{\sqrt{1+x^2-\tilde{m}_\mu^2}}e^{i\delta_e} & \frac{\tilde{m}_e(1+x^2)}{\sqrt{1+x^2-\tilde{m}_\mu^2}}e^{i\delta_e} & 0 \end{pmatrix}.$$

$x = m_e/m_\mu$, $\tilde{m}_\mu = m_\mu/m_\tau$ and $\tilde{m}_e = m_e/m_\tau$

This expression is accurate to order 10^{-9} in units of the τ mass

There are no free parameters in $\mathbf{M}_e$ other than the Dirac Phase δ!!

The Unitary Matrix U_{eL}

The unitary matrix U_{eL} is calculated from

$$U_{eL}^{\dagger} M_e M_{eL}^{\dagger} U_{eL} = diag\left(m_e^2, m_\mu^2, m_\tau^2\right)$$

We find

$$U_{eL} = \Phi_{eL} O_{eL}, \qquad \Phi_{eL} = diag\left[1, 1, e^{i\delta_D}\right]$$

and

$$\mathbf{O}_{eL} \approx \begin{pmatrix} \frac{1}{\sqrt{2}} x \frac{(1+2\tilde{m}_\mu^2+4x^2+\tilde{m}_\mu^4+2\tilde{m}_e^2)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^4-\tilde{m}_\mu^6+\tilde{m}_e^2+12x^4}} & -\frac{1}{\sqrt{2}} \frac{(1-2\tilde{m}_\mu^2+\tilde{m}_\mu^4-2\tilde{m}_e^2)}{\sqrt{1-4\tilde{m}_\mu^2+x^2+6\tilde{m}_\mu^4-4\tilde{m}_\mu^6-5\tilde{m}_e^2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} x \frac{(1+4x^2-\tilde{m}_\mu^4-2\tilde{m}_e^2)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^4-\tilde{m}_\mu^6+\tilde{m}_e^2+12x^4}} & \frac{1}{\sqrt{2}} \frac{(1-2\tilde{m}_\mu^2+\tilde{m}_\mu^4)}{\sqrt{1-4\tilde{m}_\mu^2+x^2+6\tilde{m}_\mu^4-4\tilde{m}_\mu^6-5\tilde{m}_e^2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{1+2x^2-\tilde{m}_\mu^2-\tilde{m}_e^2}(1+\tilde{m}_\mu^2+x^2-2\tilde{m}_e^2)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^4-\tilde{m}_\mu^6+\tilde{m}_e^2+12x^4}} & -x \frac{(1+x^2-\tilde{m}_\mu^2-2\tilde{m}_e^2)\sqrt{1+2x^2-\tilde{m}_\mu^2-\tilde{m}_e^2}}{\sqrt{1-4\tilde{m}_\mu^2+x^2+6\tilde{m}_\mu^4-4\tilde{m}_\mu^6-5\tilde{m}_e^2}} & \frac{\sqrt{1+x^2}\tilde{m}_e\tilde{m}_\mu}{\sqrt{1+x^2-\tilde{m}_\mu^2}} \end{pmatrix},$$

$x = m_e/m_\mu$, $\tilde{m}_\mu = m_\mu/m_\tau$ and $\tilde{m}_e = m_e/m_\tau$

The neutrino mass matrix I

The Majorana masses for ν_L are obtained from the see-saw mechanism

$$\mathbf{M}_\nu = \mathbf{M}_{\nu D}\tilde{\mathbf{M}}_R^{-1}\mathbf{M}_{\nu D}^T$$

with

$$\tilde{\mathbf{M}}_R = diag\left[M_1, M_2, M_3\right] \qquad M_1 \neq M_2 \neq M_3$$

and

$$\mathbf{M}_\nu \quad = \quad \begin{pmatrix} \mu_2 & \mu_2 & 0 \\ \mu_2 & -\mu_2 & 0 \\ \mu_4 & \mu_4 & \mu_3 \end{pmatrix}$$

Then

$$M_\nu = \left(\begin{array}{ccc} \frac{2\mu_2^2}{\bar{M}} & 2\lambda\mu_2^2 & \frac{2\mu_2\mu_4}{\bar{M}} \\ 2\lambda\mu_2^2 & \frac{2\mu_2^2}{\bar{M}} & 2\lambda_2\mu_2\mu_4 \\ \frac{2\mu_2\mu_4}{\bar{M}} & 2\lambda\mu_2\mu_4 & \frac{2\mu_4^2}{\bar{M}} + \frac{\mu_3^2}{\bar{M}} \end{array} \right).$$

$$\frac{1}{\bar{M}} = \frac{1}{2}\left(\frac{1}{M_1} + \frac{1}{M_2}\right) \quad \text{and} \quad \lambda = \frac{1}{2}\left(\frac{1}{M_1} - \frac{1}{M_2}\right)$$

The neutrino mass matrix II

$\mathbf{M}_\nu^{(o)}$ is reparametrized in term of the neutrinomasses

$$M_\nu = M_\nu^{(0)} + \delta M_\mu$$

$$M_\nu^{(0)} = \begin{pmatrix} m_{\nu_3} & 0 & \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_2} - m_{\nu_3})} e^{-i\delta_\nu} \\ 0 & m_{\nu_3} & 0 \\ \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_2} - m_{\nu_3})} e^{-i\delta_\nu} & 0 & (m_{\nu_1} + m_{\nu_2} - m_{\nu_3}) e^{-2i\delta_\nu} \end{pmatrix}$$

$$\delta M_\nu = 2\lambda\mu_2^2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{(1 - \frac{m_{\nu 1}}{m_{\nu 2}})(\frac{m_{\nu 2}}{m_{\nu 3}} - \frac{m_{\nu 1}}{m_{\nu 3}})} \\ 0 & \sqrt{(1 - \frac{m_{\nu 1}}{m_{\nu 2}})(\frac{m_{\nu 2}}{m_{\nu 3}} - \frac{m_{\nu 1}}{m_{\nu 3}})} & 0 \end{pmatrix}$$

$$2\lambda\mu_2 = \frac{m_{\nu 3}}{\mu_2}$$

The Unitary Matrix U_ν

The complex symmetric matrix M_ν is diagonalized as

$$\mathbf{U}_\nu^T \mathbf{M}_\nu \mathbf{U}_\nu \quad = \quad \begin{pmatrix} |m_{\nu_1}| e^{i(\phi_1 - \phi_\nu)} & 0 & 0 \\ 0 & |m_{\nu_2}| e^{i(\phi_2 - \phi_\nu)} & 0 \\ 0 & 0 & |m_{\nu_3}| \end{pmatrix}$$

where

$$U_\nu = K \begin{pmatrix} \cos\eta & \sin\eta & (1 - \sqrt{\frac{(m_{\nu_3} - m_{\nu_2})}{(m_{\nu_1} - m_{\nu_3})}}) f(n) \\ f(n) & -\sqrt{\frac{(m_3 - m_{\nu_2})}{(m_1 - m_{\nu_3})}} f(n) & 1 - O((\lambda\mu)^2) \\ -\sin\eta & \cos\eta & (1 + \sqrt{\frac{(m_{\nu_3} - m_{\nu_2})}{(m_{\nu_1} - m_{\nu_3})}}) f(n) \end{pmatrix}$$

and

$$f(n) = \frac{2\lambda\mu^2}{m_{\nu_1} - m_{\nu_2}} (\cos\eta - \sqrt{(1 - \frac{m_{\nu_1}}{m_{\nu_3}} (\frac{m_{\nu_2}}{m_{\nu_3}} - \frac{m_{\nu_1}}{m_{\nu_3}})} \sin\eta)$$

$$\sin^2\eta = \frac{(m_{\nu_3} - m_{\nu_1})}{(m_{\nu_2} - m_{\nu_1})}, \qquad \cos^2\eta = \frac{(m_{\nu_2} - m_{\nu_3})}{(m_{\nu_2} - m_{\nu_1})}, \qquad K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta_\nu} \end{pmatrix}$$

the mass eigenvalues, m_{ν_1}, m_{ν_2} and m_{ν_3} are, in general, complex numbers

Unitarity condition on U_ν

All the phases in M_ν except for one, ϕ_ν, can be absorbed in a rephasing of the fields

The phases ϕ_1 and ϕ_2 are fixed by the unitarity condition on U_ν

$$|m_{\nu_3}| \sin \phi_\nu = |m_{\nu_2}| \sin \phi_2 = |m_{\nu_1}| \sin \phi_1$$

therefore

$$\frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_3} - m_{\nu_1}} = \frac{(\Delta m_{12}^2 + \Delta m_{13}^2 + |m_{\nu_3}|^2 \cos^2 \phi_\nu)^{1/2} - |m_{\nu_3}||\cos \phi_\nu|}{(\Delta m_{13}^2 + |m_{\nu_3}|^2 \cos^2 \phi_\nu)^{1/2} + |m_{\nu_3}||\cos \phi_\nu|}.$$

The neutrino mixing matrix

$$V_{PMNS}^{th} = U_{eL}^{\dagger} U_{\nu}$$

The theoretical mixing matrix V_{PMNS}^{th} is

$$V_{PMNS}^{th} = \begin{pmatrix} O_{11}\cos\eta + O_{31}\sin\eta e^{i\delta} & O_{11}\sin\eta - O_{31}\cos\eta e^{i\delta} & -O_{31}\left(\sqrt{\frac{(m_{\nu_3}-m_{\nu_2})}{(m_{\nu_1}-m_{\nu_3})}} - 1\right)f(\eta) \\ -O_{12}\cos\eta + O_{32}\sin\eta e^{i\delta} & -O_{12}\sin\eta - O_{32}\cos\eta e^{i\delta} & O_{22} + \mathcal{O}\left((\lambda mu)^2\right) \\ O_{13}\cos\eta - O_{33}\sin\eta e^{i\delta} & O_{13}\sin\eta + O_{33}\cos\eta e^{i\delta} & O_{23}f(\eta) \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix}$$

where O_{ij} are the absolute values of the elements of $\mathbf{O}_e$

$$\mathbf{V}_{PMNS}^{PDG} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix}$$

From a comparison of $\mathbf{V}_{PMNS}^{th}$ with $\mathbf{V}_{PMNS}^{exp}$, we obtain the neutrino mixing angles as function of the lepton masses.

Neutrino Mixing Angles

The solar angle θ_{12} is strongly dependent on the neutrino masses but depends only very weakly on the charged lepton masses

$$\tan\theta_{12}^2 = \frac{(\Delta m_{12}^2 + \Delta m_{13}^2 + |m_{\nu_3}|^2\cos^2\phi_\nu)^{1/2} - |m_{\nu_3}||\cos\phi_\nu|}{(\Delta m_{13}^2 + |m_{\nu_3}|^2\cos^2\phi_\nu)^{1/2} + |m_{\nu_3}||\cos\phi_\nu|}.$$

the numerical value of $\tan^2\theta_{12}$ fixes the scale of the neutrino masses

The mixing angle θ_{23} depends mostly on the charged lepton masses

$$\sin\theta_{23} \approx \frac{1}{\sqrt{2}}\frac{1-2\tilde{m}_\mu^2+\tilde{m}_\mu^4}{\sqrt{1-4\tilde{m}_\mu^2+x^2+6\tilde{m}_\mu^4}} = 0.7071;$$

$$x = m_e/m_\mu = 4.84\times^{-3}, \quad \tilde{m}_\mu = m_\mu/m_\tau = 5.95\times^{-2}$$

The reactor mixing angle θ_{13} is mostly determined by the interplay of the S_3 symmetry and the mass spliting of the right-handed neutrinos in the see saw mechanism plus a very small contribution from the charged leptons,

$$\sin\theta_{13} \approx \frac{2(\lambda\mu)m_{\nu_3}}{m_{\nu_1}-m_{\nu_2}}\left(1-\sqrt{\frac{(m_{\nu_2}-m_{\nu_3})}{m_{\nu_3}-m_{\nu_1}}}\right)\left(\cos\eta - \sqrt{\left(1-\frac{m_{\nu_1}}{m_{\nu_3}}\right)\left(\frac{m_{\nu_2}}{m_{\nu_3}}-\frac{m_{\nu_1}}{m_{\nu_3}}\right)}\sin\eta\right)$$

we get $\quad \sin\theta_{13}^{th} \approx 0.137 \quad$ with $\quad (\lambda\mu) \approx 0.02$

Neutrino Mixing Angles: Theory vs Experiment

The most recent experimental values of the neutrino mixing angles θ_{13} and θ_{23} (T. Schwetz, M. Tortola and J.W.F. Valle, New J. Phys. **13**, 063004 (2011) and arXiv: 1108.137 v1[hep-ph] 4 Aug 2011; G.L. Fogli, E. Lisi, A. Marrone, A. Palazzo and A. M. Rotunno, arXiv: 1106.6028 v2 [hep-ph] 26 Aug 2011)

$$
\begin{aligned}
\bar{\theta}^{o\ exp}_{13} &= 19.8^{+1.2}_{-1.2} \rightarrow \quad \sin^2 \bar{\theta}^{o\ exp}_{13} = 0.120^{+0.007}_{-0.007}; \ \text{(T2K and MINOS)} \\
\bar{\theta}^{o\ exp}_{23} &= 46^{+3}_{-3} \qquad\qquad \sin^2 \bar{\theta}^{o\ exp}_{23} = 0.52^{+0.06}_{-0.06}
\end{aligned}
$$

Our theoretical values are (Phys. Rev. D **76**, 076003 (2007) and this work)

$$
\begin{aligned}
\theta^{o\ th}_{13} &= 20.0^{+2.0}_{-2.0} \qquad \sin^2 \theta^{o\ th}_{13} = 0.117 \ \text{(not a prediction)} \\
\theta^{o\ th}_{23} &= 44.97^{+1.2}_{-1.2} \qquad \sin^2 \theta^{o\ th}_{23} = 0.50
\end{aligned}
$$

in very good agreement with the experimental values !!!

Majorana Phases

The Majorana phases are

$$
\begin{aligned}
\sin 2\alpha &= \sin(\phi_1 - \phi_2) \\
&= \frac{|m_{\nu_3}|\sin\phi_\nu}{|m_{\nu_1}||m_{\nu_2}|}\left(\sqrt{|m_{\nu_2}|^2 - |m_{\nu_3}|^2\sin^2\phi_\nu} + \sqrt{|m_{\nu_1}|^2 - |m_{\nu_3}|^2\sin^2\phi_\nu}\right)
\end{aligned}
$$

$$
\begin{aligned}
\sin 2\beta &= \sin(\phi_1 - \phi_\nu) \\
&= \frac{\sin\phi_\nu}{|m_{\nu_1}|}\left(|m_{\nu_3}|\sqrt{1 - \sin^2\phi_\nu} + \sqrt{|m_{\nu_1}|^2 - |m_{\nu_3}|^2\sin^2\phi_\nu}\right).
\end{aligned}
$$

The neutrino mass spectrum I

In the present model, the experimental restriction

$$|\Delta m_{21}^2| < |\Delta m_{23}^2|$$

implies an inverted neutrino mass spectrum $m_{\nu_3} < m_{\nu_1}, m_{\nu_2}$

From our previous expressions for $\tan\theta_{12}$

$$|m_{\nu_3}| = \frac{\sqrt{\Delta m_{13}^2}}{2\cos\phi_\nu \tan\theta_{12}} \frac{1 - \tan^4\theta_{12} + r^2}{\sqrt{1+\tan^2\theta_{12}}\sqrt{1+\tan^2\theta_{12}+r^2}},$$

where $r = \Delta m_{21}^2/\Delta m_{23}^2$.

The mass $|m_{\nu_3}|$ assumes its minimal value when $\sin\phi_\nu = 0$,

$$|m_{\nu_3}| \approx \frac{1}{2}\frac{\sqrt{\Delta m_{13}^2}}{\tan\theta_{12}}(1 - \tan^2\theta_{12})$$

Neutrino mass spectrum II

- We wrote the neutrino mass differences, $m_{\nu_i} - m_{\nu_j}$, in terms of the differences of the squared masses $\Delta^2_{ij} = m^2_{\nu_i} - m^2_{\nu_j}$ and one of the neutrino masses, say m_{ν_3}.
- The mass m_{ν_2} was taken as a free parameter in the fitting of our formula for $\tan\theta_{12}$ to the experimental value
- with

$$\Delta m^2_{21} = 7.6 \times 10^{-5} eV^2 \qquad \Delta m^2_{13} = 2.4 \times 10^{-3} eV^2$$

and

$$\tan\theta_{12} = 0.696$$

we get

$$|m_{\nu_3}| \approx 0.019\ eV \implies |m_{\nu_2}| \approx 0.053\ eV \quad \text{and} \quad |m_{\nu_1}| \approx 0.052\ eV$$

- The neutrino mass spectrum has an inverted hierarchy of masses

FCNC I

In the Standard Model the FCNC at tree level are suppressed by the GIM mechanism.

Models with more than one Higgs $SU(2)$ doublet have tree level FCNC due to the exchange of scalar fields. The mass matrix written in terms of the Yukawa couplings is

$$\mathcal{M}_Y^e = Y_w^{E1} H_1^0 + Y_w^{E2} H_2^0,$$

FCNC processes:

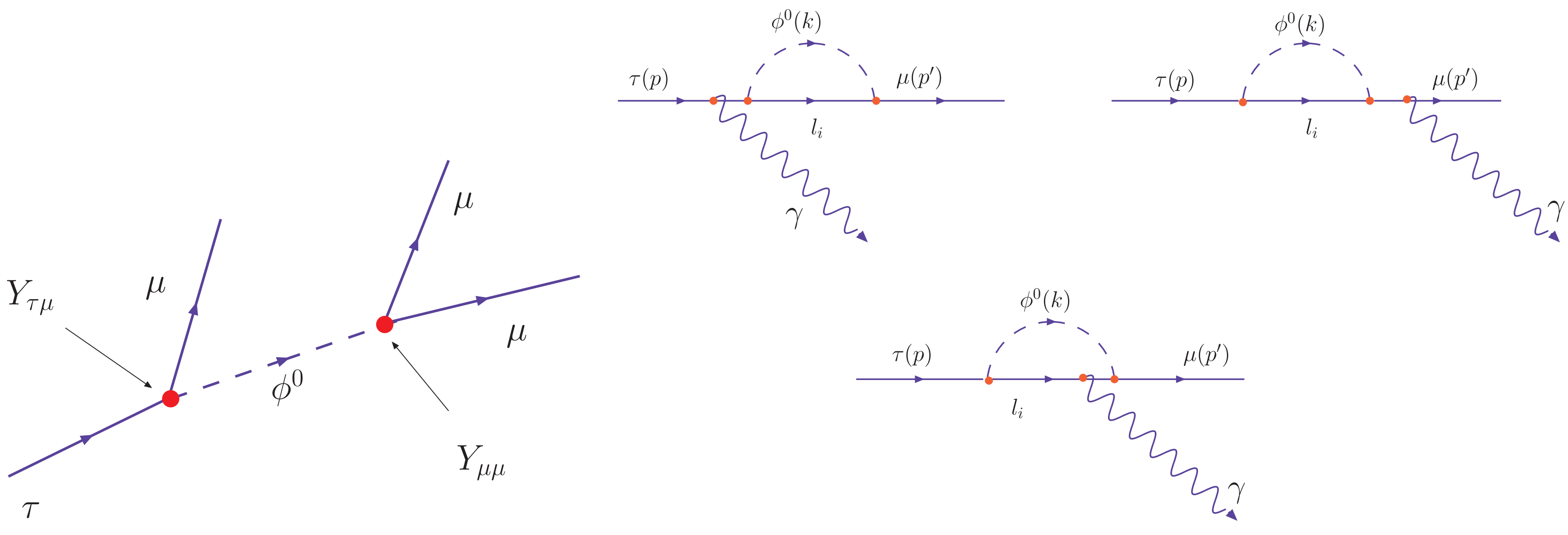


Figure 1: The diagram in the left contributes to the process $\tau^- \to 3\mu$. The three diagrams in the right contribute to the process $\tau \to \mu\gamma$.

The Yukawa matrices

The Yukawa matrices in the weak basis are

$$Y_w^{E1} = \frac{m_\tau}{v_1} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}}\sqrt{\frac{1+x^2-\tilde{m}_\mu^2}{1+x^2}} \\ \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & 0 & 0 \\ \frac{\tilde{m}_e(1+x^2)}{\sqrt{1+x^2-\tilde{m}_\mu^2}}e^{i\delta_e} & 0 & 0 \end{pmatrix}$$

and

$$Y_w^{E2} = \frac{m_\tau}{v_2} \begin{pmatrix} \frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}}\frac{\tilde{m}_\mu}{\sqrt{1+x^2}} & \frac{1}{\sqrt{2}}\sqrt{\frac{1+x^2-\tilde{m}_\mu^2}{1+x^2}} \\ 0 & \frac{\tilde{m}_e(1+x^2)}{\sqrt{1+x^2-\tilde{m}_\mu^2}}e^{i\delta_e} & 0 \end{pmatrix}.$$

Yukawa matrices in the mass representation

The Yukawa matrices in the mass basis defined by

$$\tilde{Y}_m^{EI} = U_{eL}^{\dagger} Y_w^{EI} U_{eR}$$

$$\tilde{Y}_m^{E1} \approx \frac{m_\tau}{v_1} \begin{pmatrix} 2\tilde{m}_e & -\frac{1}{2}\tilde{m}_e & \frac{1}{2}x \\ -\tilde{m}_\mu & \frac{1}{2}\tilde{m}_\mu & -\frac{1}{2} \\ \frac{1}{2}\tilde{m}_\mu x^2 & -\frac{1}{2}\tilde{m}_\mu & \frac{1}{2} \end{pmatrix}_m ,$$

and

$$\tilde{Y}_m^{E2} \approx \frac{m_\tau}{v_2} \begin{pmatrix} -\tilde{m}_e & \frac{1}{2}\tilde{m}_e & -\frac{1}{2}x \\ \tilde{m}_\mu & \frac{1}{2}\tilde{m}_\mu & \frac{1}{2} \\ -\frac{1}{2}\tilde{m}_\mu x^2 & \frac{1}{2}\tilde{m}_\mu & \frac{1}{2} \end{pmatrix}_m ,$$

all off diagonal terms give rise to FCNC processes!!

Branching ratios

We define the partial branching ratio (only leptonic decays)

$$Br(\tau \to \mu e^+ e^-) = \frac{\Gamma(\tau \to \mu e^+ e^-)}{\Gamma(\tau \to e\nu\bar{\nu}) + \Gamma(\tau \to \mu\nu\bar{\nu})}, \quad \Gamma(\tau \to \mu e^+ e^-) \approx \frac{m_\tau^5}{3 \times 2^{10}\pi^3} \frac{(Y_{\tau\mu}^{1,2} Y_{ee'}^{1,2})^2}{M_{H_{1,2}}^4}$$

thus

$$Br(\tau \to \mu e^+ e^-) \approx \frac{9}{4} \left(\frac{m_e m_\mu}{m_\tau^2}\right)^2 \left(\frac{m_\tau}{M_{H_{1,2}}}\right)^4,$$

Similar computations lead to

$$Br(\tau \to e\gamma) \approx \frac{3\alpha}{8\pi} \left(\frac{m_\mu}{M_H}\right)^4,$$

$$Br(\tau \to \mu\gamma) \approx \frac{3\alpha}{128\pi} \left(\frac{m_\mu}{m_\tau}\right)^2 \left(\frac{m_\tau}{M_H}\right)^4,$$

$$Br(\tau \to 3\mu) \approx \frac{9}{64} \left(\frac{m_\mu}{M_H}\right)^4,$$

$$Br(\mu \to 3e) \approx 18 \left(\frac{m_e m_\mu}{m_\tau^2}\right)^2 \left(\frac{m_\tau}{M_H}\right)^4,$$

$$Br(\mu \to e\gamma) \approx \frac{27\alpha}{64\pi} \left(\frac{m_e}{m_\mu}\right)^4 \left(\frac{m_\tau}{M_H}\right)^4.$$

Numerical results

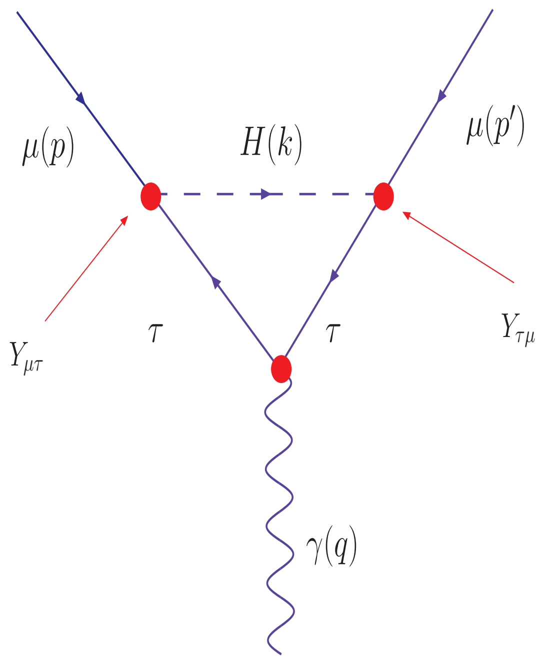
Table 1: Leptonic processes via FCNC

FCNC processes	Theoretical BR	Experimental upper bound BR	References
$\tau \to 3\mu$	8.43×10^{-14}	5.3×10^{-8}	B. Aubert *et. al.* (2007)
$\tau \to \mu e^+ e^-$	3.15×10^{-17}	8×10^{-8}	B. Aubert *et. al.* (2007)
$\tau \to \mu\gamma$	9.24×10^{-15}	6.8×10^{-8}	B. Aubert *et. al.*(2005)
$\tau \to e\gamma$	5.22×10^{-16}	1.1×10^{-11}	B. Aubert *et. al.* (2006)
$\mu \to 3e$	2.53×10^{-16}	1×10^{-12}	U. Bellgardt *et al.* (1998)
$\mu \to e\gamma$	2.42×10^{-20}	1.2×10^{-11}	M. L. Brooks *et al.* (1999)

Small FCNC processes mediating non-standard quark-neutrino interactions could be important in the theoretical description of the **gravitational core collapse and shock generation** in the explosion stage of a supernova

Muon Anomalous Magnetic Moment

The anomalous magnetic moment of the muon is related to the gyroscopic ratio by



$$a_\mu = \frac{\mu_\mu}{\mu_B} - 1 = \frac{1}{2}(g_\mu - 2)$$

In models with more than one Higgs $SU(2)$ doublet, the exchange of flavour changing neutral scalars also contribute to the anomalous magnetic moment of the muon

$$\delta a_\mu^{(H)} = \frac{Y_{\mu\tau} Y_{\tau\mu}}{16\pi^2} \frac{m_\mu m_\tau}{M_H^2} \left(log \left(\frac{M_H^2}{m_\tau^2} \right) - \frac{3}{2} \right)$$

From our results: $Y_{\mu\tau} Y_{\tau\mu} = \frac{m_\mu m_\tau}{4 v_1 v_2}$

$$\delta a_\mu^{(H)} = \frac{m_\tau^2}{(246\ GeV)^2} \frac{(2 + \tan^2\beta)}{32\pi^2} \frac{m_\mu^2}{M_H^2} \left(log \left(\frac{M_H^2}{m_\tau^2} \right) - \frac{3}{2} \right), \quad \tan\beta = \frac{v_s}{v_1}$$

From the experimental upper bound on $(\mu \to 3e)$, we get $\tan\beta \leq 14$, Hence

$$\delta a_\mu = 1.7 \times 10^{-10}$$

Contribution to the anomaly of the muon's magnetic moment

The difference between the experimental value and the Standard Model prediction for the anomaly is

$$\Delta a_\mu = a_\mu^{exp} - a_\mu^{SM} = (28.7 \pm 9.1) \times 10^{-10}$$

$\Delta a_\mu \sim 3\sigma$ (three standard deviations) !!

But, the uncertainty in the computation of higher order hadronic effects is large

$$\delta a_\mu^{LBL}(3, had) \approx 1.59 \times 10^{-9}; \quad \delta a_\mu^{VP}(3, had) \approx -1.82 \times 10^{-9}$$

$$\frac{\delta a_\mu^{(H)}}{\Delta a_\mu} \approx \frac{1.7}{28} \approx 6\% \quad \text{and} \quad \delta a_\mu^{(H)} < \delta a_\mu(3, had)$$

The contribution of the exchange of flavour changing scalars to the anomaly of the muon's magnetic moment, $\delta a_\mu^{(H)}$, is small but not negligible, and it is compatible with the best, state of the art, measurements and theoretical predictions.

Summary

- By introducing three $SU(2)_L$ Higgs doublet fields, in the theory, we extended the concept of flavour and generations to the Higgs sector and formulated a minimal S_3-invariant Extension of the Standard Model
- The neutrino mixing angles θ_{12}, θ_{23} and θ_{13}, are determined by an interplay of the $S_3 \times Z_2$ symmetry, the see-saw mechanism and the lepton mass hierarchy
- The fit of, $\sin^2\theta_{13}^{th}$ to $\sin^2\theta_{13}^{exp}$ breaks the mass degeneracy of the right handed neutrinos.
- The solar mixing angle, θ_{12}, fixes the scale and origin of the neutrino mass spectrum which has an inverted mass hierarchy with values
$$|m_{\nu_2}| \approx 0.056 eV, \qquad |m_{\nu_1}| \approx 0.055 eV, \qquad |m_{\nu_3}| \approx 0.022 eV$$
- The branching ratios of all flavour changing neutral processes in the leptonic sector are strongly suppressed by the $S_3 \times Z_2$ symmetry and powers of the small mass ratios m_e/m_τ, m_μ/m_τ, and $\left(m_\tau/M_{H_{1,2}}\right)^4$, but could be important in astrophysical processes
- The anomalous magnetic moment of the muon gets a small but non-negligible contribution from the exchange of flavour changing scalar fields