Effective Action in arbitrary background fields

Adolfo Huet
IFM-UMSNH

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Outline

General aspects, computation

The method

New results

The effective action in QED

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But the vacuum is never empty! Virtual (e^+, e^-) pairs turn the vacuum effectively into a medium.

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{Maxwell}} + \frac{2(\hbar/mc)^3}{45mc^2}\left(\frac{e^2}{4\pi\hbar c}\right)^2 [(\vec{E}^2 - \vec{B}^2)^2 + 7(\vec{E}\cdot\vec{B})^2] + \cdots$$

Quantum fluctuations in the vacuum introduce *non-linear* corrections to the Lagrangian.

Physical consequences

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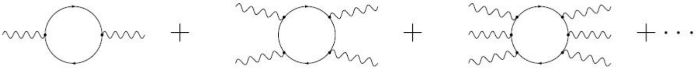

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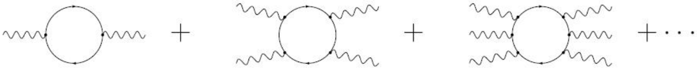
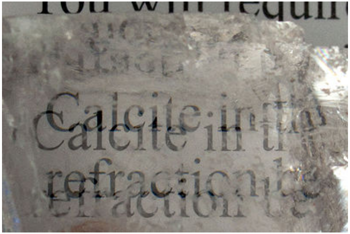


- Vacuum-birefringence



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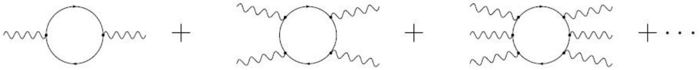

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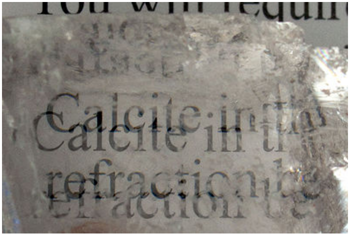

These effects are encoded in the Effective Action.

Effective Action and Determinants

The generating functional for QED is

$$Z = \int \mathcal{D}A \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{-i\int F_{\mu\nu}^2} e^{-i\int \bar{\psi}(i\not{D}-m)\psi}$$

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- Spinor Effective Action :

$$\Gamma = -i \ln\det(i\not{D} - m) \, ,$$

- Scalar Effective Action :

$$\Gamma = \frac{i}{2} \ln\det(D_\mu^2 + m^2) \, ,$$

where $\not{D} \equiv \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - ieA_\mu(x))$.

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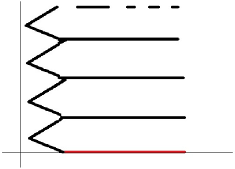

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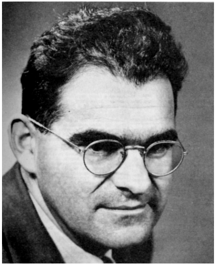

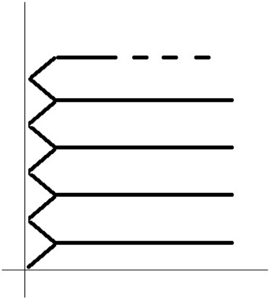

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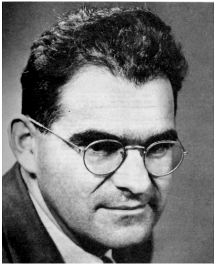

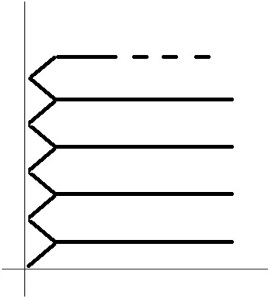

- The corresponding Landau levels are $E_n = (2n + 1)B$.
- Note that zero-modes are possible in the Spinor case but not in the Scalar case.

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For a more general $F_{\mu\nu}$, the $m \to \infty$ limit can be systematically studied by means of the *heat-kernel* expansion :

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However, for $m \to 0$, there is no general approach.

The Derivative Expansion

We may expand about the soluble constant field cases

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- In fact, the DE has been shown to be very accurate in Scalar theories. What happens in Spinor theories?

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In this talk I present a recent method for calculating the Effective Action that improves these three aspects.

Outline

The Gel'Fand-Yaglom theorem

A typical way to calculate a determinant

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$$\Rightarrow \quad \det\left[-\frac{d^2}{dx^2}+V(x)\right]=\phi(L)\,.$$

Example: The Helmholtz operator

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$$\Rightarrow \quad \frac{\det\left[-\frac{d^2}{dx^2} + m^2 \right]}{\det\left[-\frac{d^2}{dx^2} \right]} = \frac{\phi(L)}{\phi_0(L)} = \frac{\sinh(mL)}{mL}$$

The G-Y Theorem

The Gel'fand-Yaglom Theorem states that for a 1-dim operator we can compute the determinant without calculating the eigenvalues

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- In fact, very interesting systems like instantons, sphalerons, monopoles and vortices are separable.
- The G-Y theorem has been successfully applied to the Scalar theory and renormalization carried out.
- New results : Spinor theories.

Radially Symmetric Backgrounds

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For this system it is possible to set up a *partial-wave decomposition* :

$$\ln\left[\frac{\det(\not{D}^2 - m^2)}{\det(\not{\partial}^2 - m^2)}\right] = \sum_{l=0}^{\infty} \Omega(l) \ln\left[\frac{\det(\mathcal{H}_l + m^2)}{\det(\mathcal{H}_l^0 + m^2)}\right]$$

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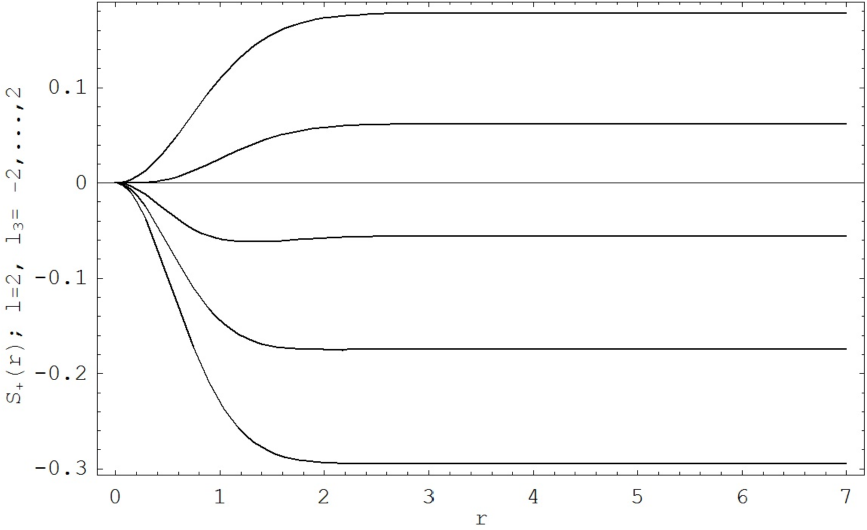
GY Theorem (initial value problem):

$$\frac{d^2 S_l(r)}{dr^2} + \left(\frac{dS_l(r)}{dr}\right)^2 + \left(\frac{1}{r} + 2m\frac{I'_{2l+1}(mr)}{I_{2l+1}(mr)}\right)\frac{dS_l(r)}{dr} = V(r)$$

$$\{\, S_l(0) = 0\,,\ S'_l(0) = 0\,\}$$

where $S_l(r) \equiv \ln \frac{\psi(r)}{\psi^0(r)}$ and $V(r)$ depends on $g(r)$.

We can find $S_l(r)$ numerically



- Example: $S_l(r)$, $\{l = 2, l_3 = -2, \cdots, 2\}$

Is it that simple?

However,

$$\sum_{l=0}^{\infty} \Omega(l) \ln \left[\frac{\det(\mathcal{H}_l + m^2)}{\det(\mathcal{H}_l^0 + m^2)} \right] = \sum_{l=0}^{\infty} \Omega(l) S_l(\infty) \sim \infty\,!$$

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In fact

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- Not really a surprise, in more than one dimension, we need renormalization.

The strategy

$$\sum_{l=0}^{\infty}\Omega(l)S_l(\infty) = \sum_{l=0}^{L}\Omega(l)S_l(\infty)+\sum_{l=L+1/2}^{\infty}\Omega(l)S_l(\infty) = \Gamma_{\mathrm{Low}}+\Gamma_{\mathrm{High}}$$

Low-modes : $\Rightarrow$ GY Theorem (numerical solution)
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$$\frac{\det(\mathcal{H}_l + m^2)}{\det(\mathcal{H}_l^0 + m^2)} = -\int_0^\infty \frac{ds}{s}\, e^{-m^2 s} \int_0^\infty dr \{\Delta_l(r,r;s) - \Delta_l^0(r,r;s)\}$$

where $\Delta_l(r,r';s) \equiv \langle r | e^{-s\mathcal{H}_l} | r' \rangle$.

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where $\Delta_l(r,r';s)\equiv\langle r|e^{-s\mathcal{H}_l}|r'\rangle$.

$$\Delta_l(r,r;s)=\frac{e^{-s\mathcal{V}_l(r)}}{\sqrt{4\pi s}}\left[1+\left(\frac{s^3}{12}(\mathcal{V}_l'(r))^2-\frac{s^2}{6}\mathcal{V}_l''(r)\right)+\cdots\right]$$

where $\mathcal{V}_l(r)$ includes a centrifugal term that depends on l.

The calculation

- First we perform the infinite sum over the angular momentum l. We use the Euler-Maclaurin formula for this:

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \frac{f(a)+f(b)}{2} + \cdots$$

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- We are left with an integral over dr

$$\Gamma^{\mathrm{ren}}_{\mathrm{High}} = \int_0^\infty dr \left(Q_{log}(r) \ln L + \sum_{n=0}^{2} Q_n(r) L^n + \sum_{n=1}^{N} Q_{-n}(r) \frac{1}{L^n} \right) + \mathcal{O}\left(\frac{1}{L^{N+1}} \right)$$

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$$\Gamma^{\rm ren}_{\rm High} \sim L^2 \qquad ; \qquad L \to \infty$$

But nevertheless, the sum of this two divergent terms, one numerically calculated, and the other one analytically calculated, produces a finite result :

$$\Gamma_{\rm Low} + \Gamma^{\rm ren}_{\rm High} < \infty \qquad ; \qquad L \to \infty$$

Dependence on the cutoff L

- We require $\Gamma^{\mathrm{ren}} = \Gamma_{\mathrm{Low}} + \Gamma_{\mathrm{High}}$ to be finite, but also independent of the arbitrary cutoff L.

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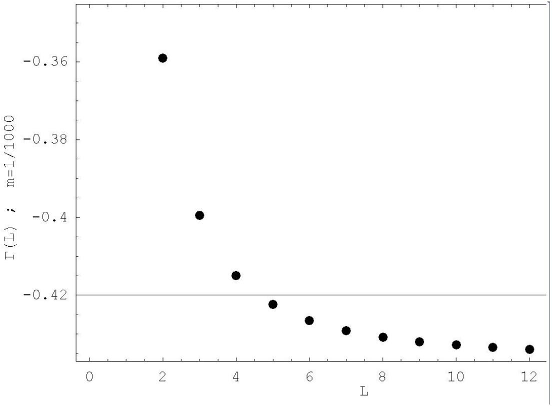
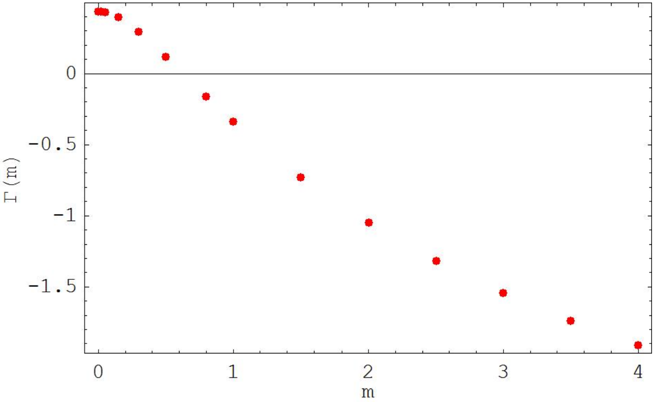


- This shows an example of $\Gamma^{\mathrm{ren}}(L)$ for Spinor QED, with

$$g(r) = B(1 - \mathrm{Tanh}[\beta\sqrt{B}r - \xi]) \quad \{B = 1,\ \beta = 1,\ \xi = 1\}$$

The G-Y method works in a wide mass-range



- This shows an example of $\Gamma^{\rm ren}(m)$ for Spinor QED, with

$$g(r) = B(1 - \mathrm{Tanh}[\beta\sqrt{B}r - \xi]) \qquad \{B = 1,\ \beta = 1,\ \xi = 1\}$$

Outline

Different background fields

We can now study different field configurations, for example :

$$A_\mu(r) = \eta^3_{\mu\nu} x_\nu g(r), \quad g(r) = B(1 - \mathrm{Tanh}[\beta\sqrt{B} r - \xi]) \, .$$

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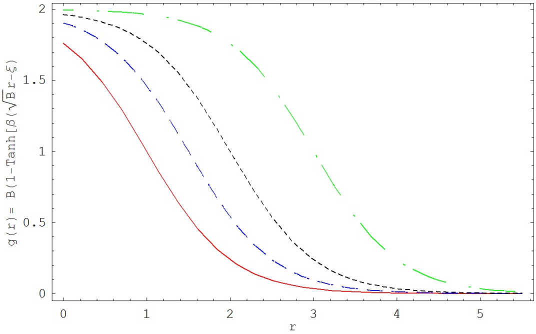
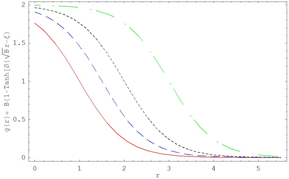
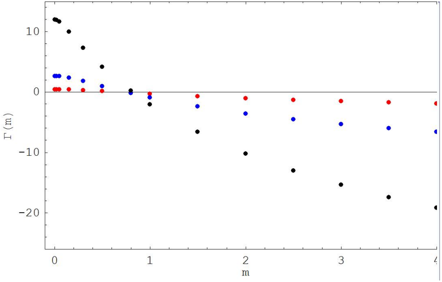


- The graph shows $g(r)$ for different choices of the *range* parameter :

$$\xi = 1,\ 3/2,\ 2,\ 3$$

Γ in different backgrounds fields



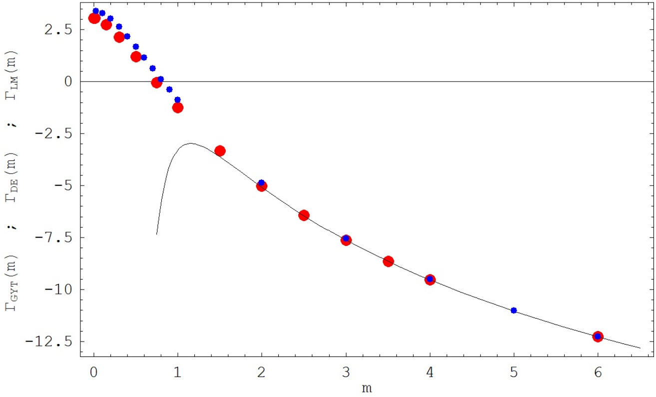


- The graph on the right shows $\Gamma(m)$, as calculated for different values of ξ.
- We can study how some general properties of $\Gamma(m)$ depend on some specific characteristics of the background.

G-Y versus approximations : Scalar case

G-Y versus approximations : Scalar case

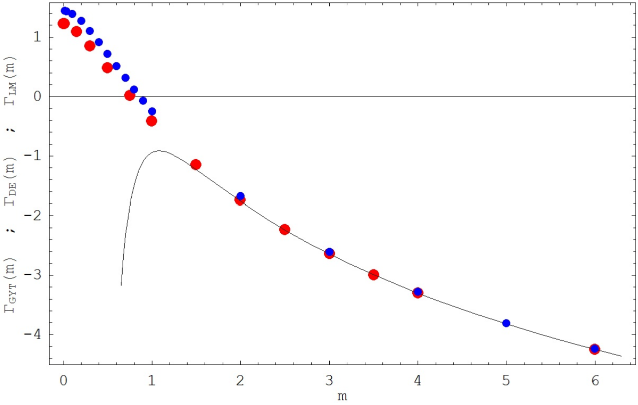
- First the Scalar case :



- The graph shows the Effective Action as calculated with the large-mass expansion(line), the derivative expansion(blue), and the G-Y method(red).

$$g(r) = B(1 - \mathrm{Tanh}[\beta\sqrt{B}r - \xi]); \quad \xi = 2, \quad \beta = 1, B = 1$$

G-Y versus approximations : Scalar case



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$$g(r) = B(1 - \mathrm{Tanh}[\beta\sqrt{B}r - \xi]); \quad \xi = \frac{3}{2}, \quad \beta = 1,\ B = 1$$

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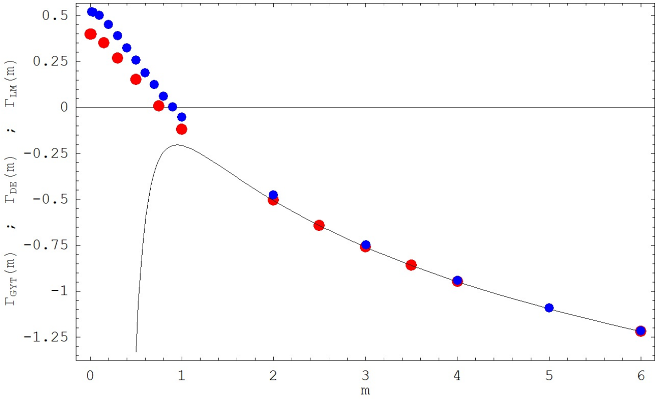

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G-Y versus approximations : Spinor Case

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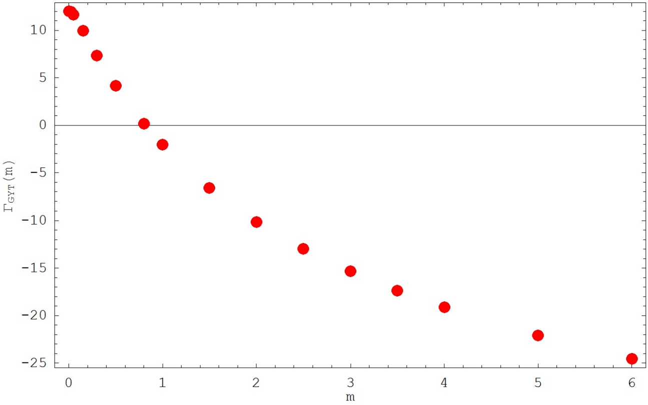
- The G-Y method behaves well in all mass regimes.



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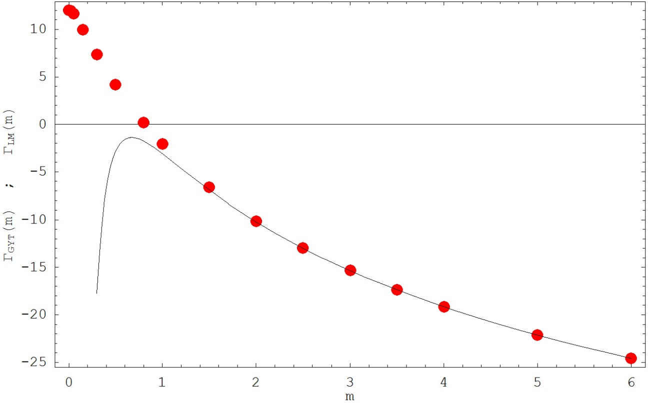
G-Y versus approximations : Spinor Case

- The large-mass expansion behaves as expected.

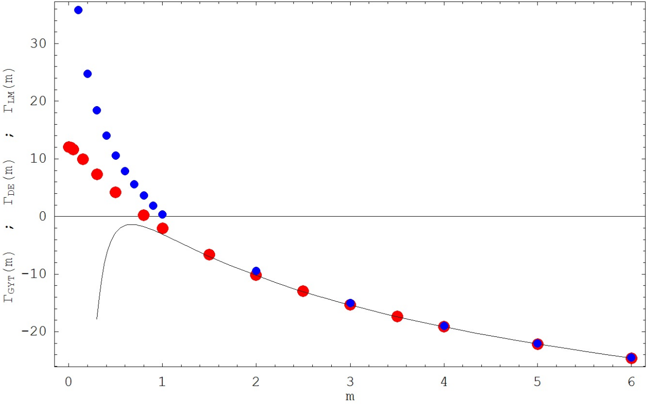


- Large-mass expansion(line) and G-Y method(red).

$$g(r) = B(1 - \mathrm{Tanh}[\beta\sqrt{B}r - \xi]); \quad \xi = 2, \quad \beta = 1, \; B = 1$$

G-Y versus approximations : Spinor Case

- Add the Derivative Expansion and we have a surprise!



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$$g(r) = B(1 - \mathrm{Tanh}[\beta\sqrt{B}r - \xi]); \quad \xi = 2, \quad \beta = 1, B = 1$$

What is wrong in the Derivative Expansion?

$$\mathcal{L}_{\text{spinor}}(a,b) = -\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\, e^{-m^2 s}\Big\{(ab)s^2 \coth(as)\coth(bs) - 1 - \frac{s^2}{3}(a^2+b^2)\Big\},$$

$$a^2+b^2 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = 2g(r) \quad ; \quad ab = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = 2g(r)+rg'(r)$$

- The small-mass limit corresponds to $s \to \infty$, this gives :

$$\mathcal{L}_{\text{spinor}}(a,b) \sim -\frac{1}{8\pi^2}\left[ab - \frac{1}{3}(a^2+b^2)\right]\int_0^\infty \frac{ds}{s}\, e^{-m^2 s}$$

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- Note that $\mathcal{L}_{\rm spinor}(a,b) = \mathcal{L}_{\rm spinor}(-a,-b)$, the Derivative Expansion is calculating $|a(r)b(r)|$ instead of $a(r)b(r)$.

What is different in the Scalar case?

$$\mathcal{L}_{\text{scalar}}(a,b) = \frac{1}{16\pi^2}\int_0^\infty \frac{ds}{s^3}\, e^{-m^2 s}\left\{\frac{(ab)s^2}{\sinh(as)\sinh(bs)} - 1 + \frac{s^2}{6}(a^2+b^2)\right\},$$

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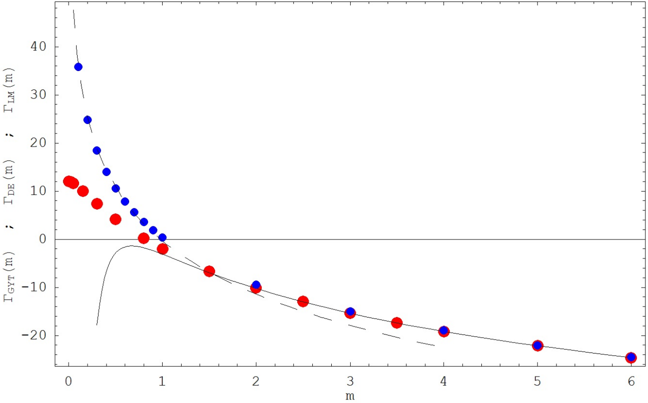
- This means the $|a(r)b(r)|$ contribution vanishes.

G-Y versus approximations

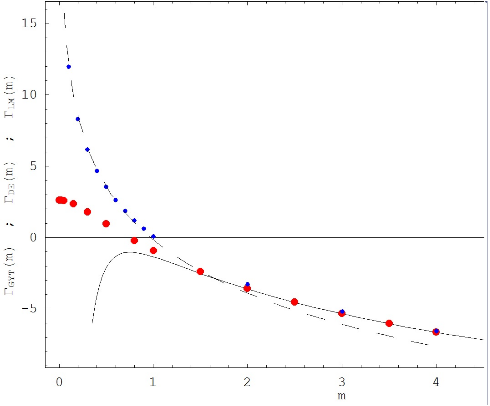
- The Spinor case.



- Large-mass expansion(line), Derivative Expansion(blue), G-Y method(red), and
- Here I also plot $f(m) = (\frac{1}{4\pi^2}\int d^4x|a(r)b(r)|)\ln m$ (dash).

G-Y versus approximations : Spinor

- $\xi = \frac{3}{2}$

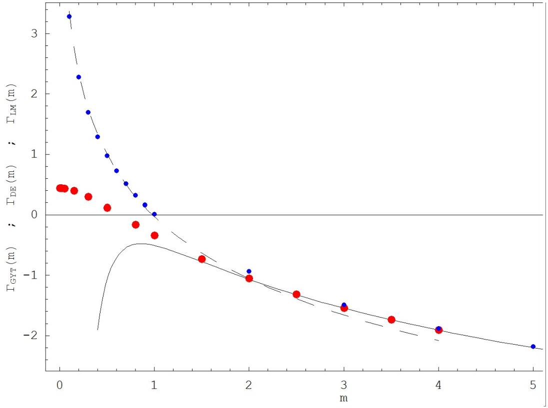


- Large-mass expansion(line), derivative expansion(blue), G-Y method(red), and $f(m)$ (dash).

$$g(r) = B(1 - \mathrm{Tanh}[\beta\sqrt{B}r - \xi]); \quad \xi = 3/2, \quad \beta = 1, B = 1$$

G-Y versus approximations : Spinor

- $\xi = 1$



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  $g(r) = B(1 - \mathrm{Tanh}[\beta\sqrt{B}r - \xi]); \quad \xi = 1, \quad \beta = 1, B = 1$

What about zero-modes?

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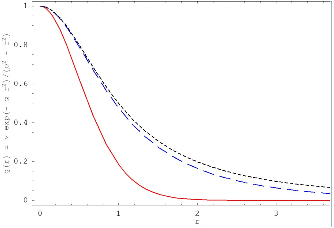

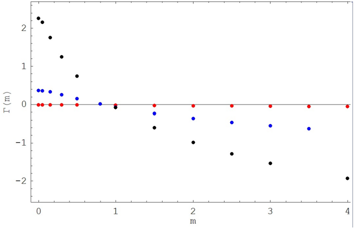

- $g(r) = e^{-\alpha r^2}/(1 + r^2)$ with $\alpha = \{1,\ 1/20,\ 1/400\}$.
- The second graph shows $\Gamma(m)$ for each case, in the limit $\alpha \to 0$ the system has zero-modes.

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- Thanks

END