# Nonlinear Field Line Random Walk and Generalized Compound Diffusion of Charged Particles

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- P A R T - I: wandering of magnetic field lines (FLRW=Field Line Random Walk)
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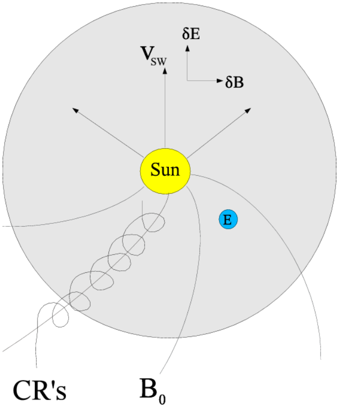
# Introduction




Fields:

- $\vec{B}_0 = B_0 \vec{e}_z =$ mean magnetic field
- $\delta\vec{E}$, $\delta\vec{B} =$ turbulent fields

Model for the turbulence correlation tensor:

$$P_{xx}(\vec{k},t) =< \delta B_x(\vec{k},t)\delta B_x^*(\vec{k},0) >$$

- Magnetostatic turbulence: $P_{xx}(\vec{k},t) = P_{xx}(\vec{k})$
- Slab/2D composite geometry:

$$P_{xx}(\vec{k}) = P_{xx}^{slab}(\vec{k}) + P_{xx}^{2D}(\vec{k})$$

with

$$P_{xx}^{slab}(\vec{k}) = g^{slab}(k_{\|})\frac{\delta(k_\perp)}{k_\perp}$$

and

$$P_{xx}^{2D}(\vec{k}) = g^{2D}(k_\perp)\frac{\delta(k_{\|})}{k_\perp}\frac{k_y^2}{k_\perp^2}$$

- Standard form of the wave spectrum:

$$g^{slab}(k_{\parallel}) = \frac{C(\nu)}{2\pi} l_{slab} \delta B_{slab}^2 \left(1 + k_{\parallel}^2 l_{slab}^2\right)^{-\nu}$$

and

$$g^{2D}(k_{\perp}) = \frac{2C(\nu)}{\pi} l_{2D} \delta B_{2D}^2 \left(1 + k_{\perp}^2 l_{2D}^2\right)^{-\nu}$$

with:

the normalization constant $C(\nu)$,

the bendover scales $l_{slab}$ and $l_{2D}$,

the strength of the turbulent fields $\delta B_{slab}$ and $\delta B_{2D}$,

and the inertial range spectral index $2\nu$;

# Field Line Random Walk (FLRW)



Field line equation (for $\delta B_z \ll B_0$):

$$dx = \frac{\delta B_x(\vec{x}(z))}{B_0} dz$$

Field line MSD (Mean Square Deviation)

$$\left\langle (\Delta x(z))^2 \right\rangle = \frac{1}{B_0^2} Re \int_0^z dz' \int_0^z dz'' \, R_{xx}(z', z'')$$

with

$$R_{xx}(z', z'') = \left\langle \delta B_x(\vec{x}(z')) \delta B_x^*(\vec{x}(z'')) \right\rangle$$

## To proceed:

- we replace the turbulent fields by a Fourier transformation
- we assume homogeneous and axisymmetric turbulence
- we assume a Gaussian statistics of the field lines
- we employ Corrsin's independence hypothesis:

$$\begin{aligned} &\left\langle \delta B_x(\vec{k}) \delta B_x^*(\vec{k}) e^{i\vec{k}\cdot\Delta\vec{x}(z)} \right\rangle \\ = \; &\left\langle \delta B_x(\vec{k}) \delta B_x^*(\vec{k}) \right\rangle \left\langle e^{i\vec{k}\cdot\Delta\vec{x}(z)} \right\rangle \end{aligned}$$

to get

$$\left\langle (\Delta x(z))^2 \right\rangle = \frac{2}{B_0^2} \int d^3k \; P_{xx}(\vec{k})$$
$$\times \quad \int_0^z dz' \; (z - z') \cos(k_\parallel z') e^{-\frac{1}{2}\left\langle (\Delta x(z'))^2 \right\rangle k_\perp^2}.$$

By applying the operator $d^2/dz^2$ we get an ODE

$$\frac{d^2}{dz^2}\left\langle (\Delta x(z))^2 \right\rangle = \frac{2}{B_0^2} \int d^3k \; P_{xx}(\vec{k}) \times \cos(k_\parallel z) e^{-\frac{1}{2}\left\langle (\Delta x(z))^2 \right\rangle k_\perp^2}.$$

# Analytical results for the slab/2D composite model

- For pure slab geometry we have:

$$\left\langle (\Delta x(z))^2 \right\rangle = 2\kappa_{FL}|z|$$

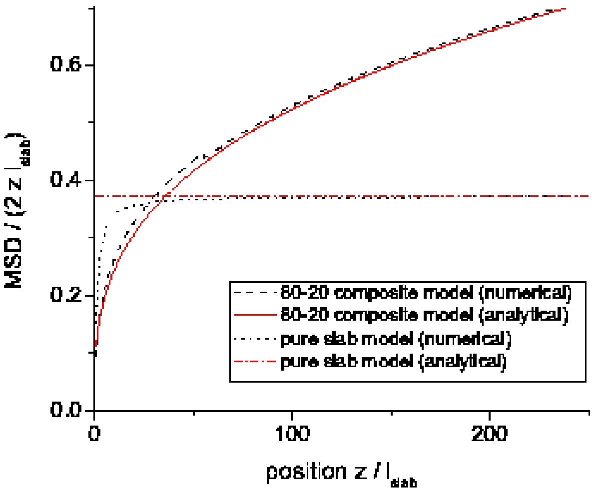
$\Rightarrow$ (Markovian-)Diffusion of field lines

- For slab/2D composite geometry we find:

$$\left\langle (\Delta x)^2 \right\rangle = \left[ 9C(\nu)\sqrt{\frac{\pi}{2}} l_{2D} \frac{\delta B_{2D}^2}{B_0^2} \right]^{2/3} |z|^{4/3}$$

$\Rightarrow$ Superdiffusion of field lines

# Numerical investigation:



# Perpendicular scattering of charged particles

Previous approaches:

- Quasilinear theory of particle transport (Jokipii 1966)
- Nonlinear closure approximation (Owens 1974)
- The Bieber and Matthaeus model (BAM, Bieber & Matthaeus 1997)
- The compound transport model (Kota & Jokipii 2000)
- The nonlinear guiding center theory (NLGC-theory, Matthaeus et al. 2003)
- The weakly nonlinear theory (WNLT, Shalchi et al. 2004)
- The extended NLGC-theory (ENLGC-theory, Shalchi 2006)

Test-particle simulations (e.g. Qin et al. 2002a,b):

- Slab geometry:

$$\left\langle (\Delta x)^2 \right\rangle_P \sim \sqrt{t}$$

$\Rightarrow$ subdiffusion

- Slab/2D composite geometry:

$$\left\langle (\Delta x)^2 \right\rangle_P \sim t$$

$\Rightarrow$ recovery of diffusion?

# Generalized Compound Diffusion (GCD)

## Guiding center approximation

$$\left\langle (\Delta x)^2 \right\rangle_P (t) \approx \left\langle (\Delta x)^2 \right\rangle_{FL} (z(t))$$

and thus

$$\left\langle (\Delta x)^2 \right\rangle_P (t) = \int_{-\infty}^{+\infty} dz \, \left\langle (\Delta x)^2 \right\rangle_{FL} (z) f_P(z,t)$$

For $f_P(z,t)$ we assume a Gaussian particle distribution:

$$f_P(z,t) = \frac{1}{\sqrt{2\pi \left\langle (\Delta z)^2 \right\rangle_P}} \, e^{-\frac{z^2}{2\left\langle (\Delta z(t))^2 \right\rangle_P}}$$

For slab geometry and the standard spectrum we have

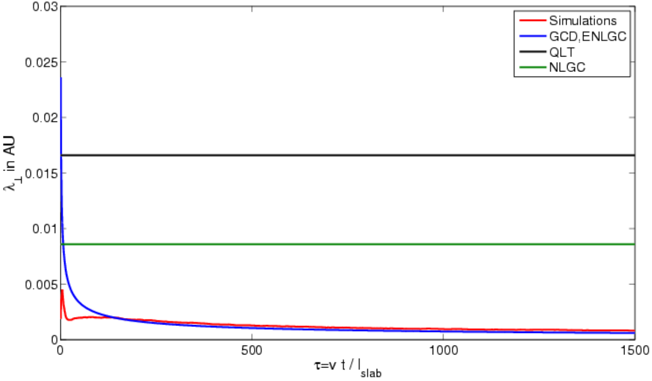
$$\begin{aligned} \left\langle (\Delta x)^2 \right\rangle_{FL}(z) &= 2\kappa_{FL}|z| \\ \left\langle (\Delta z)^2 \right\rangle_{P}(t) &= 2\kappa_{\parallel} t \end{aligned}$$

The GCD-model provides:

$$\left\langle (\Delta x)^2 \right\rangle_P (t) = 4\kappa_{FL}\sqrt{\frac{\kappa_{\parallel} t}{\pi}} \sim \sqrt{t}$$

$\Rightarrow$ Perpendicular particle transport behaves subdiffusively!

The time-dependent perpendicular mean free path $\lambda_\perp \sim \left\langle (\Delta x)^2 \right\rangle_P / (2t)$ for pure slab geometry:



For slab/2D composite turbulence we have

$$\left\langle (\Delta x)^2 \right\rangle_{FL} \sim |z|^{4/3}$$

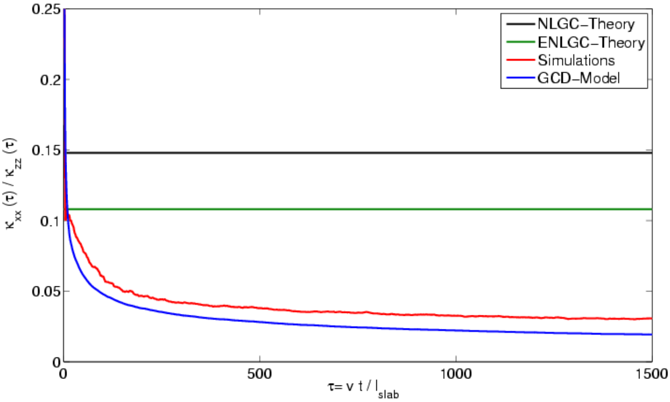
and thus

$$\left\langle (\Delta x)^2 \right\rangle_{P} = \alpha(\nu) \left( \frac{\delta B_{2D}}{B_0} \right)^{4/3} \left[ l_{2D} \left\langle (\Delta z)^2 \right\rangle_{P} \right]^{2/3}$$

with

$$\alpha(\nu) = \frac{\Gamma(7/6)}{\sqrt{\pi}} \left( 18 \sqrt{\frac{\pi}{2}} C(\nu) \right)^{2/3} .$$

The ratio of perpendicular and parallel diffusion coefficients for $R = R_L/l_{slab} = 0.001$:



$\Rightarrow$ Good agreement between the GCD-model and simulations!

However, a detailed evaluation of the simulations has shown that:

- Parallel transport is weakly superdiffusive

$$\left\langle (\Delta z)^2 \right\rangle_P \sim t^{1.2}$$

- Perpendicular transport is weakly subdiffusive

$$\left\langle (\Delta x)^2 \right\rangle_P \sim t^{0.8}$$

# Comparison with observations

Assume diffusion of parallel transport ($< (\Delta z(t))^2 >_P \approx 2t\kappa_\parallel$) and thus

$$\kappa_\perp(t) = \frac{\alpha(\nu)}{2^{1/3}} \left(\frac{\delta B_{2D}}{B_0}\right)^{4/3} \frac{\left(l_{2D}\kappa_\parallel\right)^{2/3}}{t^{1/3}}.$$

To proceed, we average over the scattering time

$$t_c = \lambda_\parallel / v$$

and we use $\lambda_\parallel = 3\kappa_\parallel / v$ and $\lambda_\perp = 3\kappa_\perp / v$ to find

$$\overline{\lambda}_\perp = \left(\frac{3}{2}\right)^{4/3} \alpha(\nu) \left(\frac{\delta B_{2D}}{B_0}\right)^{4/3} l_{2D}^{2/3} \lambda_\parallel^{1/3}.$$

For

$$\begin{aligned}\nu &= 5/6\\ \delta B_{2D}^2/B_0^2 &= 0.8\\ l_{2D} &= 0.1 l_{slab} \approx 0.003 AU\\ \lambda_{\parallel,Palmer} &\approx 0.2 AU\end{aligned}$$

we find

$$\lambda_{\perp,GCD} \approx 0.009 AU$$

in agreement with observations (see e.g. Palmer (1982)) where we have

$$\lambda_{\perp,Palmer} \approx 0.007 AU$$

# Summary and Conclusion



- In most cases: FLRW behaves superdiffusively
- The (generalized) compound transport model is a useful tool for describing perpendicular cosmic ray scattering analytically
- The GCD-model agrees with test-particle simulations for slab and slab/2D composite geometry
- By averaging the result for slab/2D turbulence we can explain observed perpendicular mean free path
- However, there is a weak subdiffusive behavior of perpendicular scattering for slab/2D composite geometry

Comparison between the assumptions used in the GCD-model and the assumptions used in previous theories:

| Assumption | NLGC, WNLT | GCD |
|---|---|---|
| GC approximation | YES | YES |
| Gaussian statistics | YES | YES |
| Corrsin's hypothesis | YES | YES |
| Uncorrelated velocities and fields | YES | NO |
| Exponential velocity correlation function | YES | NO |
| Diffusion approximation | YES | NO |

Assume $\kappa_\perp(t) \sim t^{b_\perp}$:

$$\begin{aligned} b_\perp < 0: &\quad \text{subdiffusion} \\ b_\perp = 0: &\quad \text{(Markovian-)diffusion} \\ b_\perp > 0: &\quad \text{superdiffusion} \end{aligned}$$

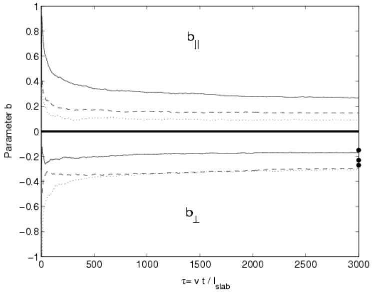
Comparison between results for the parameter $b_\perp$ from various theories:

| Theory | slab turbulence | slab/2D composite |
|---|---|---|
| Simulations | $-0.5$ | $\approx -0.2$ |
| QLT | $0$ | $1.0$ |
| BAM | $0$ | $0$ |
| NLGC | $0$ | $0$ |
| WNLT | $0$ | $0$ |
| ENLGC | $-0.5$ | $0$ |
| GCD-model | $-0.5$ | $\approx -0.2$ |

Assume $\kappa_i(t) \sim t^{b_i}$, $i =_{\parallel,\perp}$:

$$\begin{array}{ll} b_i < 0: & \text{subdiffusion} \\ b_i = 0: & \text{(Markovian-)diffusion} \\ b_i > 0: & \text{superdiffusion} \end{array}$$

The parameters $b_\parallel$ and $b_\perp$ as a function of time for different values of the dimensionless rigidity: $R = 10^{-3}$ (dotted line), $R = 10^{-2}$ (dashed line), and $R = 10^{-1}$ (solid line). The dots denote the values predicted by the GCD-model:



- By assuming homogeneous turbulence

$$R_{xx}(z', z'') = R_{xx}(\mid z' - z'' \mid)$$

one can easily derive

$$\left\langle (\Delta x(z))^2 \right\rangle = \frac{2}{B_0^2} Re \int_0^z dz' \, (z - z') R_{xx}(z')$$

- By applying a fourier transformation we have

$$R_{xx}(z) = \int d^3k \int d^3k' \, \left\langle \delta B_x(\vec{k}) \delta B_x^*(\vec{k}') e^{i\vec{k}\cdot\vec{x}(z) - i\vec{k}'\cdot\vec{x}(0)} \right\rangle$$

- To proceed we apply Corrsin's independence hypothesis

$$\begin{aligned} R_{xx}(z) &= \int d^3k \int d^3k' \, \left\langle \delta B_x(\vec{k}) \delta B_x^*(\vec{k}') \right\rangle \\ &\times \left\langle e^{i\vec{k}\cdot\vec{x}(z) - i\vec{k}'\cdot\vec{x}(0)} \right\rangle \end{aligned}$$

- and for homogeneous turbulence

$$\left\langle \delta B_x(\vec{k}) \delta B_x^*(\vec{k}') \right\rangle = P_{xx}(\vec{k}) \delta(\vec{k} - \vec{k}')$$

we get

$$R_{xx}(z) = \int d^3k \; P_{xx}(\vec{k}) \left\langle e^{i\vec{k}\cdot\Delta\vec{x}(z)} \right\rangle .$$

- By assuming a Gaussian statistics of the field lines

$$\left\langle e^{i\vec{k}\cdot\Delta\vec{x}(z)} \right\rangle = e^{-\frac{1}{2}\left\langle (\Delta x(z))^2 \right\rangle k_x^2 - \frac{1}{2}\left\langle (\Delta y(z))^2 \right\rangle k_y^2 + i k_\parallel z}$$

- and for axisymmetric turbulence

$$\left\langle (\Delta x)^2 \right\rangle = \left\langle (\Delta y)^2 \right\rangle$$

- QLT for FLRW:

QLT is exact for the slab model:

$$\left\langle (\Delta x(z))^2 \right\rangle = 2\kappa_{slab}|z|$$

For the composite model, we find superdiffusion

$$\left\langle (\Delta x(z))^2 \right\rangle \sim z^2$$

$\Rightarrow$ QLT is not valid for FLRW in the slab/2D model!

- The diffusion theory of Matthaeus et al. 1995:

$$\left\langle (\Delta x(z))^2 \right\rangle = 2\kappa \mid z \mid$$

$$\Rightarrow \kappa = \frac{\kappa_{slab} + \sqrt{\kappa_{slab}^2 + 4\kappa_{2D}^2}}{2}$$

with

$$\begin{aligned} \kappa_{2D}^2 &= \frac{1}{B_0^2} \int d^3k \; k_{\perp}^{-2} P_{xx}^{2D}(\vec{k}) \\ &\equiv \frac{\pi}{B_0^2} \int_0^{\infty} dk_{\perp} \; k_{\perp}^{-2} \; g^{2D}(k_{\perp}) \end{aligned}$$

For the standard spectrum, we find

$$\kappa_{2D}^2 \longrightarrow \infty$$