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Nonlinear Field Line Random Walk and Generalized Compound Diffusion of Charged Particles

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Abstract: A new nonlinear theory for the perpendicular transport of charged particles is presented. This approach is based on an improved nonlinear treatment of field line random walk in combination with a generalized compound diffusion model. The generalized compound diffusion model is much more systematic and reliable, in comparison to previous theories. Furthermore, the new theory shows remarkably good agreement with test-particle simulations and heliospheric observations.

Introduction

An early treatment of cosmic ray transport in a turbulent electromagnetic field has relied on a quasilinear description of cosmic ray propagation [1]. In the quasilinear theory (QLT) it is assumed that particles follow the magnetic field lines while they move unperturbed in the direction parallel to the background field. For the slab turbulence model, the quasilinear perpendicular mean-square deviation (MSD) of the particle increases linearly with time, viz. $\left\langle (\Delta x)^2 \right\rangle = 2\kappa_{xx} t$. This linear time dependence is usually referred to as a classical Markovian diffusion process. Thirty-four years later, Kóta & Jokipii [2] formulated a compound diffusion model that assumes that the particle moves along the magnetic field lines while it is scattered diffusively in the parallel direction. Relying on the Taylor-Green-Kubo-formulation, in combination with the assumption of diffusive field line random walk (FLRW), Kóta & Jokipii [2] have found a subdiffusive behavior of particle transport of the form $\left\langle (\Delta x)^2 \right\rangle \sim \sqrt{t}$. In the same years, particle propagation in magnetized plasmas was explored by making use of test-particle simulations [3, 4], where it was clearly confirmed that $\left\langle (\Delta x)^2 \right\rangle \sim \sqrt{t}$, so long as a slab model is considered. If the slab model is replaced by a slab/2D composite model, however, diffusion is recovered (though only partially, as demonstrated in this article). This recovery of diffusion cannot been explained by the method of Kóta & Jokipii [2].

A promising theory, namely the nonlinear guiding center theory (NLGC-theory), has been derived by Matthaeus et al. [5]. Although this theory shows agreement with some test-particle simulations in slab/2D geometry, the theory cannot reproduce subdiffusion for the slab model. An extended nonlinear guiding center (ENLGC) theory was therefore formulated by Shalchi [6], which agrees with simulations for slab and non-slab models. However, this theory is very close to the original NLGC-theory and uses nearly the same crude approximation (exponential form of the velocity correlation function, magnetic fields and particle velocities are uncorrelated). In this paper we propose a more reliable theoretical approach that uses less ad-hoc assumptions and *ansätze* than previous theories.

Nonlinear description of FLRW

The key input into our new formulation is the MSD of the magnetic field lines $< (\Delta x(z))^2 >_{FL}$. In a recent article [7], an improved analytical formulation for nonlinear FLRW in magnetostatic turbulence has been developed. This approach is a direct generalization of the diffusion theory proposed by

Matthaeus et al. [8]. However, the new theory can also be applied in non-diffusive transport cases.

In view of modeling FLRW, the turbulence model has to be specified in terms of the magnetic correlation tensor $P_{ij}(\vec{k}) =< \delta B_i(\vec{k})\delta B_j^*(\vec{k}) >$. According to Bieber et al. [9] the slab/2D composite model is a realistic model for solar wind turbulence. In this model the correlation tensor has the form: $P_{xx}(\vec{k}) = P_{xx}^{slab}(\vec{k}) + P_{xx}^{2D}(\vec{k})$ with $P_{xx}^{slab}(\vec{k}) = g^{slab}(k_\parallel)\delta(k_\perp)/k_\perp$ and $P_{xx}^{2D}(\vec{k}) = g^{2D}(k_\perp)\delta(k_\parallel)k_y^2/k_\perp^3$ and with the two wave spectra

$$g^{slab}(k_\parallel) = \frac{C(\nu)}{2\pi} l_{slab}\delta B_{slab}^2 \left(1 + k_\parallel^2 l_{slab}^2\right)^{-\nu}$$

$$g^{2D}(k_\perp) = \frac{2C(\nu)}{\pi} l_{2D}\delta B_{2D}^2 \left(1 + k_\perp^2 l_{2D}^2\right)^{-\nu}. \quad (1)$$

Here we used the normalization constant $C(\nu) = \Gamma(\nu)/(2\sqrt{\pi}\Gamma(\nu - 1/2))$, the slab- and 2D bendover scales l_{slab} and l_{2D}, the strength of the turbulent fields δB_{slab} and δB_{2D}, and the inertial-range spectral index 2ν.

It can easily be demonstrated that, for pure slab geometry, the field lines behaves diffusively $\left\langle (\Delta x(z))^2 \right\rangle_{|z|\to\infty} \approx 2\kappa_{FL} \mid z \mid$. In several previous papers [8] it has been explicitly assumed that FLRW is also diffusive for two-component turbulence. However, by applying the improved formulation of FLRW, Shalchi & Kourakis [7] have shown that

$$\left\langle (\Delta x(z))^2 \right\rangle_{|z|\to\infty} = \left(9\sqrt{\frac{\pi}{2}}C(\nu)\right)^{2/3} \times \left(\frac{\delta B_{2D}}{B_0}\right)^{4/3} l_{2D}^2 \left(\frac{\mid z \mid}{l_{2D}}\right)^{4/3}. \quad (2)$$

The only assumptions that have been applied to derive this result are the validity of the random phase approximation and the assumption of a Gaussian distribution of field lines.

Compound transport of particles

FLRW is described as a function of z. However, charged particles experience parallel scattering while moving through the turbulence. Thus, the parameter z becomes a random variable in particle transport studies. If we assume that the particles (or, more precisely, their guiding centers) follow the magnetic field lines (GC approximation), we have

$$\left\langle (\Delta x(t))^2 \right\rangle_P = \int_{-\infty}^{+\infty} dz \left\langle (\Delta x(z))^2 \right\rangle_{FL} f_P(z,t). \quad (3)$$

Here the index P denotes the perpendicular MSD of the charged particle, and $f_P(z,t)$ is the particle distribution in the parallel direction. Furthermore, we assume a Gaussian particle distribution.

$$f_P(z,t) = \left(2\pi\left\langle (\Delta z(t))^2 \right\rangle_P\right)^{-1/2} e^{-\frac{z^2}{2\left\langle (\Delta z(t))^2 \right\rangle_P}}. \quad (4)$$

By using Eq. (2) for the field line MSD in combination with Eq. (4), we can evaluate Eq. (3) to find

$$\left\langle (\Delta x)^2 \right\rangle_P = \alpha(\nu)\left(\frac{\delta B_{2D}}{B_0}\right)^{4/3} \times \left[l_{2D}\left\langle (\Delta z(t))^2 \right\rangle_P\right]^{2/3}. \quad (5)$$

with

$$\alpha(\nu) = \frac{\Gamma(7/6)}{\sqrt{\pi}}\left(18\sqrt{\frac{\pi}{2}}C(\nu)\right)^{2/3}. \quad (6)$$

In observed spectra, it was clearly found that $\nu = 5/6$ and thus $\alpha(5/6) \approx 0.5$. A (time-dependent) diffusion coefficient as obtained from test-particle simulations can be defined as $\kappa_{xx}(t) =< (\Delta x)^2 > /(2t)$. In general, one may adopt the assumption $< (\Delta z(t))^2 >_P \sim t^{b_\parallel + 1}$, implying a parallel diffusion coefficient $\kappa_{zz} \sim t^{b_\parallel}$. By assuming $\kappa_{xx} \sim t^{b_\perp}$, it is straightforward to find from Eq. (5) the relation

$$b_\perp = \frac{2b_\parallel - 1}{3}. \quad (7)$$

Therefore, knowledge of $b_\parallel$ (e.g., from simulation data) leads to an evaluation of $b_\perp$, within this model. For instance, if parallel transport behaves diffusively ($b_\parallel = 0$), we find $b_\perp = -1/3$ (subdiffusion). We refer to this new approach, which allows a systematic and reliable discription of perpendicular transport, as the *Generalized Compound Diffusion (GCD)*-model.

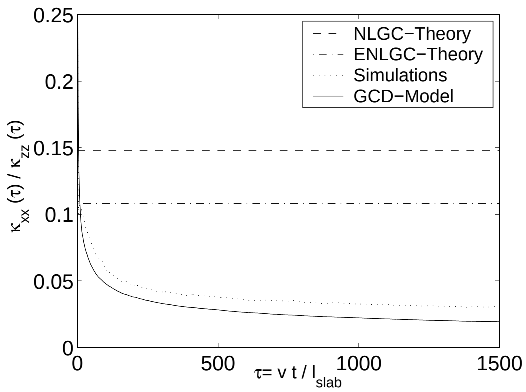

Figure 1: The ratio of perpendicular and parallel diffusion coefficients ($\kappa_{xx}(t)/\kappa_{zz}(t)$) for $R = R_L/l_{slab} = 0.001$. The results from test-particle simulations (dotted line) are compared to various theoretical results: NLGC-theory (dashed line), ENLGC-theory (dash-dotted line), and our GCD-model (solid line).

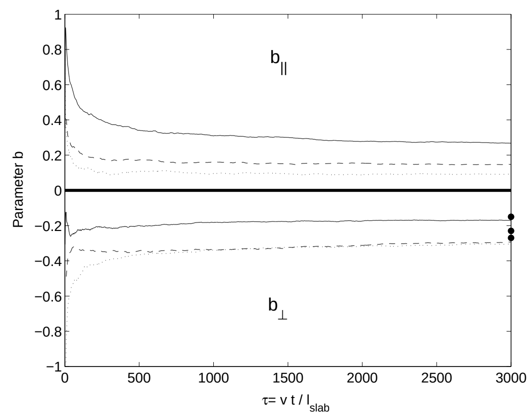

Figure 2: The parameters $b_\parallel$ and $b_\perp$ as a function of time for different values of the dimensionless rigidity: $R = 10^{-3}$ (dotted line), $R = 10^{-2}$ (dashed line), and $R = 10^{-1}$ (solid line). The dots denote the values predicted by the GCD-model.

Test particle simulations

For slab/2D composite geometry test-particle simulations can be performed easily by using procedures described previously [3, 4]. We performed simulations for the following set of parameters: $l_{2D} = 0.1\, l_{slab}$, $\nu = 5/6$, and 20%/80% slab/2D composite geometry. In Fig. 1, we depict the ratio κ_{xx}/κ_{zz} as a function of the dimensionless time $\tau = vt/l_{slab}$ for the dimensionless rigidity value $R = R_L/l_{slab} = 0.001$. We have chosen a low value of R to ensure that the guiding center approximation is valid. The simulations are compared with NLGC-theory, ENLGC-theory, and the GCD-model. For the NLGC-results we have assumed a parameter value of $a^2 = 1$, which corresponds to the assumption that guiding centers follow magnetic field lines. Obviously the GCD-model provides a result much closer to the simulations than the other theories.

By assuming the form $\tilde{\kappa}(\tau) = a\tau^b$, we can deduce the time dependence from numerical data by using $b = (\ln \tilde{\kappa}(\tau) - \ln a)/\ln \tau \approx (\ln \tilde{\kappa}(\tau))/\ln \tau$ in the high time limit ($\tilde{\kappa}$ denotes the dimensionless diffusion coefficients obtained by the simulations). The exponents for the parallel $b_\parallel$ and perpendicular $b_\perp$ diffusion coefficients are depicted in Fig. 2 for different values of the parameter R. Clearly we find a weakly superdiffusive behavior of parallel transport ($b_\parallel > 0$) and a weakly subdiffusive behavior of perpendicular transport ($b_\perp < 0$). In all cases considered, the GCD-model agrees well with the simulations.

Comparison with observations

It is difficult to directly compare our non-diffusive result with solar wind observations. In this section, we attempt a rough comparison by averaging our non-diffusive result over the characteristic scattering time $t_c = \lambda_\parallel/v$, where we have defined the parallel mean free path $\lambda_\parallel$ and the velocity v of the charged particle. First, we replace the parallel mean-square deviation in Eq. (5) by a diffusive behavior ($< (\Delta z(t))^2 >_P \approx 2t\kappa_\parallel$) and thus one obtains for the perpendicular diffusion coefficient

$$\kappa_\perp(t) = \frac{\alpha(\nu)}{2^{1/3}} \left(\frac{\delta B_{2D}}{B_0}\right)^{4/3} \frac{\left(l_{2D}\kappa_\parallel\right)^{2/3}}{t^{1/3}}. \quad (8)$$

To proceed, we average over the scattering time and we use $\lambda_\parallel = 3\kappa_\parallel/v$ and $\lambda_\perp = 3\kappa_\perp/v$ to find for the perpendicular mean free path

$$\overline{\lambda}_\perp = \left(\frac{3}{2}\right)^{4/3} \alpha(\nu) \left(\frac{\delta B_{2D}}{B_0}\right)^{4/3} l_{2D}^{2/3} \lambda_\parallel^{1/3}. \quad (9)$$

Assumption	NLGC	GCD
GC approximation	YES	YES
Gaussian statistics	YES	YES
Random phase approx.	YES	YES
Uncorrelated velocities and fields	YES	NO
Exponential velocity correlation function	YES	NO
Diffusion approximation	YES	NO

Table 1: Comparison between the assumptions used in our GCD-model and the assumptions used in the NLGC-theory.

For $\nu = 5/6$ and $\delta B_{2D}^2/B_0^2 = 0.8$, as proposed by Bieber et al. [9], we obtain

$$\overline{\lambda}_\perp = 0.75\, l_{2D}^{2/3}\, \lambda_\parallel^{1/3}. \tag{10}$$

Palmer [10] suggested that the parallel mean free path in the solar wind is $0.08AU \leq \lambda_{\parallel,Palmer} \leq 0.3AU$ and the perpendicular mean free path is $\lambda_{\perp,Palmer} \approx 0.007AU$. By taking the average value for the parallel mean free path $\lambda_{\parallel,Palmer} \approx 0.2$ and by applying Eq. (10) we find $\lambda_{\perp,GCD} \approx 0.009AU$ (for $l_{2D} = 0.1 l_{slab} \approx 0.003AU$, as suggested by e.g. Matthaeus et al. [5]), which is very close to the measurements.

Summary and conclusion

By combining a compound diffusion model (Eq. (3)) with a nonlinear treatment of FLRW (Eq. (2)), a new theoretical treatment for the perpendicular transport of cosmic rays is presented in this article. In Table 1, the assumptions of this new theory are compared to the NLGC-theory, as representative of existing transport theories. Obviously the new approach relies on less approximations and model assumptions. Furthermore, the theory is very tractable due to its simple analytical form (see Eqs. (5) and (6)). Through comparison with test particle simulations, we have demonstrated that the GCD-model behaves very well and provides a noticeably improved description of perpendicular transport compared to several other theories. Furthermore, by averaging over the scattering time, we have derived a simple formula (Eq. (9)) for the perpendicular mean free path which agrees with previous measurements in the solar wind.

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