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Derivation of the Molière simultaneous distribution between the deflection angle and the lateral displacement

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Abstract: Molière simultaneous distribution between the deflection angle and the lateral displacement is derived by applying numerical Fourier transforms on solution in the frequency space acquired through Kamata-Nishimura formulation of Molière theory. The differences of our result from that under the gaussian approximation and the basic properties of our distribution are investigated closely.

Introduction

Many accurate and significant results have been proposed for transport problems of fast charged particles from the Molière theory of multiple Coulomb scattering [1, 2, 3], although no simultaneous distributions between the deflection angle and the lateral displacement have been obtained yet [4] under the Molière theory. The simultaneous distribution will give more exact analyses for experiments concerning charged particles than the individual distributions. We obtain the simultaneous distribution between the two components in the projected plane under the fixed energy process. We solve the equation for the simultaneous distribution in the frequency space of Fourier transforms by the Molière theory of Kamata-Nishimura formulation [5, 6], and derive the distribution by applying the numerical method for the inverse Fourier transforms [7].

Diffusion equation for the Molière simultaneous distribution

Let $f(\theta, y)d\theta dy$ be the differential probability of fast charged particles to have deflection angle θ and lateral displacement y in the projected plane of y-θ_y. According to the Kamata-Nishimura formulation of the Molière theory, the diffusion equation

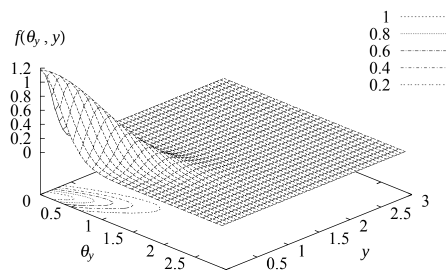

Figure 1: The simultaneous distribution between the deflection angle and the lateral displacement, derived through FFT integrations at $t = 1$ and for $\Omega = 15$, with θ_y and y in units of θ_{G} and $\theta_{\mathrm{G}} t$, respectively.

becomes

$$\frac{\partial}{\partial t}\tilde{f}(\zeta,\eta) = \eta\frac{\partial}{\partial \zeta}\tilde{f} - \frac{K^2\zeta^2}{4E^2}\tilde{f}\left\{1 - \frac{1}{\Omega}\ln\frac{K^2\zeta^2}{4E^2}\right\}, \tag{1}$$

where $\tilde{f}(\zeta,\eta)$ denotes the double Fourier transforms of the simultaneous distribution. As Nishimura did in Eq. (21.9) in his text [6], we can solve the equation as

$$\ln 2\pi\tilde{f} = \frac{1}{\Omega}\frac{\theta_{\mathrm{G}}^2}{12\eta t}\left\{(\zeta+\eta t)^3 \ln\frac{\theta_{\mathrm{G}}^2(\zeta+\eta t)^2}{4te^{2/3+\Omega}} - \zeta^3\ln\frac{\theta_{\mathrm{G}}^2\zeta^2}{4te^{2/3+\Omega}}\right\}, \tag{2}$$

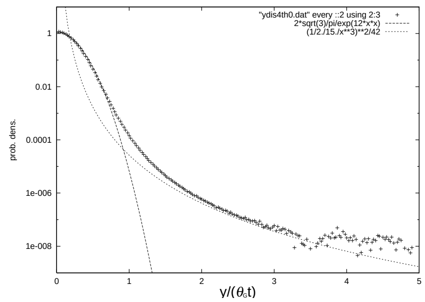

Figure 2: y distribution with $\theta_y = 0$ at $t = 1$ and $\Omega = 15$. In the central region it agrees with that of Fermi distribution, and in the large y region it is explained by the double large angle scattering.

where $\theta_{\rm G}^2$ denotes the spatial mean square angle under the gaussian approximation:

$$\theta_{\rm G}^2 = K^2 t/E^2. \tag{3}$$

At the limiting case of $\Omega \to \infty$, we get

$$\tilde{f}(\zeta,\eta) = \frac{1}{2\pi}\exp\left[-\frac{\theta_{\rm G}^2}{4}\left(\zeta^2 + \zeta\eta t + \frac{\eta^2 t^2}{3}\right)\right], \tag{4}$$

so that we have the simultaneous distribution under the gaussian approximation, well known as the Fermi simultaneous distribution [8]:

$$\begin{aligned} & f_{\rm G}(\theta, y; t) d\theta dy \\ = & \frac{2\sqrt{3}}{\pi\theta_{\rm G}^2 t}\exp\left[-\frac{4}{\theta_{\rm G}^2}\left(\theta^2 - \frac{3y\theta}{t} + \frac{3y^2}{t^2}\right)\right] d\theta dy \end{aligned} \tag{5}$$

It should be noted that our solution (2) is equivalent with Molière's result, Eq. (3.3') in 1955 [9], taking account his $\chi_c'^2 l_0$ agrees with χ_c^2 under the fixed energy condition, so that agrees with our $\chi_c^2 = \theta_{\rm G}^2/\Omega$.

Derivation of the Molière simultaneous distribution

Applying inverse double Fourier transforms on Eq. (2), we get the Molière simultaneous distribution $f(\theta, y) d\theta dy$:

$$f(\theta, y) d\theta dy = \frac{d\theta dy}{2\pi}\iint e^{-i\theta\zeta - iy\eta}\tilde{f}(\zeta,\eta) d\zeta d\eta. \tag{6}$$

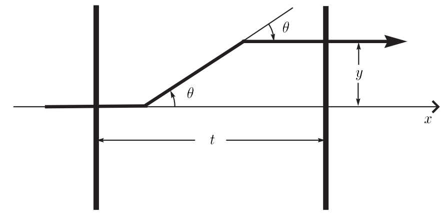

Figure 3: Successive double scattering with the same magnitude.

The probability density $f(\theta, y)$ is symmetric against the origin as easily confirmed from

$$\tilde{f}(-\zeta,-\eta) = \tilde{f}(\zeta,\eta). \tag{7}$$

The Molière simultaneous distribution derived by FFT algorithm

To derive the simultaneous distribution with θ and y on the Cartesian coordinate of constant step sizes, Fourier inverse double transforms by FFT integration is useful. The result so obtained from 2048×2048 values of frequency density is indicated in Fig. 1, where we put $\theta_{\rm G} = 1$ which means we take θ and y in unit of $\theta_{\rm G}$ and $\theta_{\rm G} t$, respectively, and derived the distribution against materials of $\Omega = 15$ at $t = 1$.

Molière lateral distribution at the deflection angle of 0

The Molière simultaneous distribution with the lateral displacement of y at the deflection angle of 0 can be derived by

$$f(0,y) d\theta dy = \frac{d\theta dy}{\pi}\int_0^\infty d\eta \cos(y\eta)\int_{-\infty}^{\infty}\tilde{f}(\zeta,\eta) d\zeta \tag{8}$$

It should be noted that the infinite integral of $\tilde{f}(\zeta,\eta)$ with ζ diverges for $\eta = 0$ as δ-function of angular distribution at $\theta = 0$ due to the survival probability, or more practically in our case due to the ill property of Molière expansion up to ζ^2 for the exponent component [10]. So we use the rectangular formula for the integration with η. The result is indicated in Fig. 2, where we put $\theta_{\rm G} = 1$ which means we take θ and y in unit of $\theta_{\rm G}$ and $\theta_{\rm G} t$,

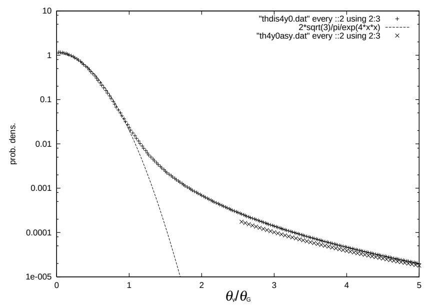

Figure 4: θ_y distribution with $y = 0$ at $t = 1$ and $\Omega = 15$. In the central region it agrees with that of Fermi distribution, and in the large θ region it is explained by the single scattering following after the Fermi distribution.

respectively, and derived the distribution against materials of $\Omega = 15$ at $t = 1$.

At the large displacement region, the distribution is well explained by the successive twice single-scattering with the same magnitude of opposite sign of

$$\frac{N}{A}\sigma_{\mathrm{P}}(\theta_y)d\theta_y dx = \frac{1}{2\Omega}\frac{K^2}{E^2}\frac{1}{(\theta_y^2+\chi_a^2)^{3/2}}d\theta_y dt. \tag{9}$$

Thus

$$\begin{aligned} & f(0,y)d\theta dy \\ \simeq \quad & d\theta dy \int_0^\infty t'\left\{ \tfrac{1}{2\Omega}\tfrac{K^2}{E^2}\tfrac{(t-t')^3}{y^3}\right\}^2 \tfrac{1}{t-t'}dt' \\ = \quad & \tfrac{t^7}{42}\left(\tfrac{1}{2\Omega}\tfrac{K^2}{E^2}\tfrac{1}{y^3}\right)^2 \end{aligned} \tag{10}$$

In the central region it agrees with the Fermi simultaneous distribution (5) at $\theta = 0$,

$$f(0,y)d\theta dy \simeq \frac{\sqrt{3}}{2\pi}e^{-3y^2}d\theta dy. \tag{11}$$

Molière angular distribution at the lateral displacement of 0

The Molière simultaneous distribution with the deflection angle of θ_y at the lateral displacement of 0 can be derived by

$$f(\theta_y,0)d\theta dy = \tfrac{d\theta dy}{\pi}\int_0^\infty d\zeta \cos(\theta_y\zeta)\int_{-\infty}^\infty \tilde{f}(\zeta,\eta)d\eta, \tag{12}$$

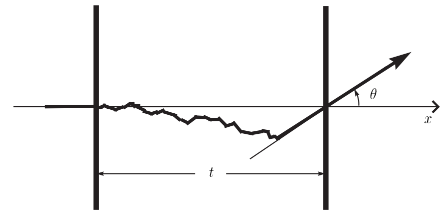

Figure 5: The single scattering passing through $y = 0$ following after the multiple scattering.

same as the preceding section. The results are indicated in Fig. 4, where we also took θ and y in unit of θ_{G} and $\theta_{\mathrm{G}}t$, respectively, and derived the distribution against materials of $\Omega = 15$ at $t = 1$.

At the large angle region, the distribution is well explained by the screened single scattering following after the Fermi simultaneous distribution (5), as indicated in Fig. 5, namely

$$\begin{aligned} & f(\theta_y,0)d\theta dy \\ \simeq \quad & \tfrac{d\theta_y dy}{2\Omega}\tfrac{K^2}{E^2}\int_0^t dt'\int_{-\infty}^\infty \tfrac{f_{\mathrm{G}}(\theta',(t-t')\theta;t')}{|\theta_y-\theta'|^3}d\theta' \end{aligned} \tag{13}$$

In the central region it agrees with the Fermi distribution (5) at $y = 0$,

$$f(\theta_y,0)d\theta dy \simeq \frac{\sqrt{3}}{2\pi}e^{-\theta_y^2}d\theta dy. \tag{14}$$

Asymmetry of the Molière simultaneous distribution between the deflection angle and the lateral displacement

If we measure the deflection angle θ and the lateral displacement y in new units as

$$u \equiv \theta/\theta_{\mathrm{G}} \quad \text{and} \quad v \equiv y/\frac{\theta_{\mathrm{G}}t}{\sqrt{3}} \tag{15}$$

as Eq. (2.13) in our previous investigation [11], Fermi simultaneous distribution can be expressed symmetrically with the deflection angle and the lateral displacement, u and v:

$$f(\theta,y)d\theta dy = \frac{2}{\pi}e^{-4\{u^2-\sqrt{3}uv+v^2\}}dudv. \tag{16}$$

We investigate symmetric properties of the Molière simultaneous distribution in this section.

New Fourier variables corresponding to u and v become

$$\mu \equiv \theta_{\rm G}\zeta \quad \text{and} \quad \nu \equiv \frac{\theta_{\rm G} t}{\sqrt{3}}\eta. \tag{17}$$

It satisfies

$$\tilde{f}(-\mu,-\nu) = \tilde{f}(\mu,\nu), \tag{18}$$

so Molière simultaneous distribution is symmetric against the origin, as mentioned at Eq. (7), but not symmetric with μ and ν, so that with u and v, in the exact sense. We investigate asymmetric features of the Molière simultaneous distribution.

As mentioned in Eq. (12), the simultaneous distribution at u with $v=0$ is derived by

$$f(u,0)dudv = \frac{dudv}{\pi}\int_0^\infty d\mu \cos(u\mu)\int_{-\infty}^\infty \tilde{f}(\mu,\nu)d\nu \tag{19}$$

We can derive the simultaneous distribution at arbitrary combination of u and v by the same calculus, rotating the u-v coordinate with an adequate angle and applying the numerical double Fourier inverse transforms (19). Results of simultaneous distribution thus derived with the rotating angle of 0, $\pi/4$, $\pi/2$, and $3\pi/4$, are indicated in Fig. 6. The rotating angle of $\pi/4$ corresponds to the major axis of the simultaneous distribution, and $3\pi/4$ corresponds to the minor axis. Although Fermi simultaneous distribution shows gaussian decrease for any combination of large u and v, Molière simultaneous distribution shows power decrease for every examined combination of large u and v. We got the index of power decrease, -3.7, -3.7, -6.0, and -5.7, for azimuthal direction of 0, $\pi/4$, $\pi/2$, and $3\pi/4$, in the u-v coordinate.

Conclusions

Kamata-Nishimura equation of the Molière theory for the simultaneous distribution has been solved analytically in the Fourier frequency space. Applying numerical Fourier inverse transforms on the solution, we have derived the Molière simultaneous distribution between the deflection angle and the lateral displacement. The simultaneous distribution will have less redundancy in probability than the individual distributions, themselves. So

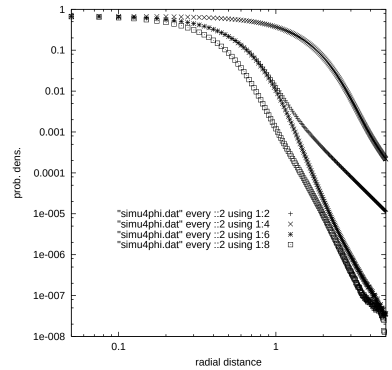

Figure 6: The simultaneous distribution with $u \equiv \theta_y/\theta_{\rm G}$ and $v \equiv y/(\theta_{\rm G}t/\sqrt{3})$ on the radial line rotated from the u-axis with azimuthal angles of $\pi/4$, 0, $\pi/2$, and $3\pi/4$, from top to bottom.

the results will useful and valuable to improve the accuracy and the reliability in designing and analyses of experiments concerning fast charged particles.

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