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Studies On Curvature Tensor and Geodesic Deviation Equation

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Abstract: Modified Henon-Heiles system describing geodesic in gravitational waves has already been investigated. There chaotic sense is inferred by the presence of fractal structure of the boundaries separating the basins of possible escapes. In this paper, observed chaotic behaviour is characterized by the formalism appropriating the signature of curvature tensor associated with the space-time geometry concerned. A simple but appealing analysis of geodesic deviation equation is also followed.

Introduction

Modified Henon-Heiles Hamiltonian system can describe the geodesic motion of the test particle in plane-fronted gravitational waves [1]. The reasons for regular motion with parallel rays (pp-waves) and chaotic nature in non-homogeneous case can be attributed to the complex function $f(u,\xi)$ found in the line element for the well-known class of gravitational wave solutions in vacuum

$$ds^2 = 2d\xi d\bar{\xi} - 2dudv - (f+\bar{f})du^2 \qquad (1)$$

where $f(u,\xi)$ is an arbitrary complex function of the retarded time u and the complex spatial coordinate ξ.

Though originally in complex form, after some simple transformation, the equation of motion can be reduced in the real coordinates q_1 and q_2 to

$$\ddot{q}_1 + q_1 + q_1^2 - q_2^2 = 0 \qquad (2)$$

and

$$\ddot{q}_2 - q_2 - 2q_1q_2 = 0 \qquad (3)$$

These equations can be obtained from standard Hamiltonian with the potential

$$V(q_1,q_2) = \frac{1}{2}\left(q_1^2 - q_2^2\right) + \frac{1}{3}\left(q_1^3 - 3q_1q_2^2\right) \qquad (4)$$

This potential can be termed as "modified Henon-Heiles potential" comparing with the standard Henon-Heiles potential [2].

The combined effect of both homogeneous and non-homogeneous gravitational wave contributions to the geodesic of the test particle can be attributed to the function f. The space - time represented by the expression (1) with the function f linear in ξ represents the trivial flat Minkowski universe. The quadratic function $f \sim \xi^2$ describes homogeneous pp - waves, in which the particle motions are integrable. $f \sim \xi^n$ describes geodesic in non-homogeneous pp - waves whose investigations have only recently been started [3], [4].

Change of space, - from flat Minkowski to other motivates us to follow a different path for the discussion of behaviour of the motion. In our paper, we observe the effect in terms of newly generated Jacobi metric out of the metric of Minkowski space -time due the presence of Henon - Heiles type potential. Curvature associated with the Jacobi metric gives the information of the motions of the test particle since the Ricci Scalar (Gaussian curvature) is either positive or negative accordingly the system is integrable or non - integrable [5]. Moreover, in very simplified form, the role of curvature in the analytical solution of the equation of motion can be obtained.

Stability Analysis by Gaussian Curvature

The presence of potential makes the existing space-time metric g_{ij} be changed to $\hat{g}_{ij}$ by

$$\hat{g}_{ij} = 2\,(E - V)\,g_{ij}$$

where E represents the total energy of the system.

Thus the problem of local instability (integrability) appears as the problem of studying the geodesic deviation equation. The investigation of the geodesic deviation equation shows that the Gauss curvature contains information about the divergence of nearby geodesic. The expression of the Gauss curvature K is given by

$$K = \frac{R_{hijk}}{g_{hj}g_{ik} - g_{kk}g_{ij}} \tag{5}$$

where R_{hijk} is the Riemannian tensor of the first kind.

Now assuming "Hamiltonian Constrain" i.e. Hamiltonian $H \equiv E = 0$, we can have the metric to represent pseudo-Riemannian space. In the present case, as we have the space - time represented by the expression (1) with the function f linear in ξ represents the trivial flat Minkowski universe, the potential generated new Jacobi metric can be written as

$$\hat{g}_{\alpha\beta} = 2 \mid \frac{1}{2}q_1^2 - \frac{1}{2}q_2^2 + \frac{1}{3}\left(q^3 - 3q_1q_2^2\right) \mid \eta_{\alpha\beta} \tag{6}$$

where

$$V(q_1, q_2) = \frac{1}{2}q_1^2 - \frac{1}{2}q_2^2 + \frac{1}{3}\left(q^3 - 3q_1q_2^2\right)$$

and

$$\eta_{\alpha\beta} = diag\,(-1, 1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ultimately $\hat{K}$ comes out as

$$\hat{K} = -\frac{\hat{R}_{1212}}{4V^2}$$

Curvature in Equation of Motion

The differential equation, known as Jacobi's equation, or the geodesic deviation equation can be written as ([5], [6])

$$\frac{d^2n}{ds^2} + Kn = 0 \tag{7}$$

where for our case n, the deviation vector is considered in the space associated with Jacobi metric and K, the Gauss curvature is a function of arc length $\alpha : K = K\,(\alpha(s))$.

Now to solve the differential equation (7.1), let us put

$$n = ue^{-\frac{s^2}{2}}$$

And after proper replacement we get

$$\frac{d^2u}{ds^2} - 2s\frac{du}{ds} + \left[s^2 + K(s) - 1\right]u = 0 \tag{8}$$

which is the Hermite differential equation. Therefore, its solution is known in series form.

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