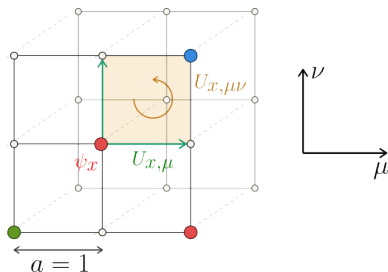


Gauge fields under Gradient Flow

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June 17 2026



Functional integral formalism

We work with Euclidean spacetime $\eta_{\mu\nu} \rightarrow \delta_{\mu\nu}$.

Sum over ALL field configurations $[\phi(x)]$

$$Z = \int \mathcal{D}\phi e^{-S[\phi]},$$

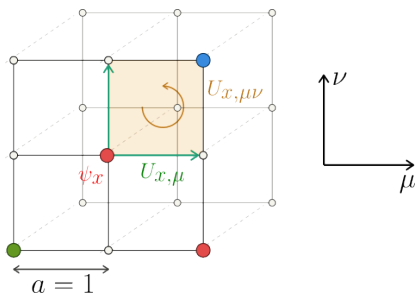
where $S[\phi]$ is the Euclidean action.

n -point function

$$\langle O[\phi(x_1)] \cdots \rangle = \frac{1}{Z} \int \mathcal{D}\phi O[\phi(x_1)] \cdots e^{-S[\phi]}.$$

Lattice regularization

Discretization of spacetime:



- ▶ $\psi(x) \rightarrow \psi_x$
- ▶ $\partial_\mu \psi(x) \rightarrow \psi_{x+\hat{\mu}} - \psi_x$
- ▶ $\int d^4x \rightarrow \sum_x, \int \frac{d^4p}{(2\pi)^4} \rightarrow \int_B \frac{d^4p}{(2\pi)^4}$

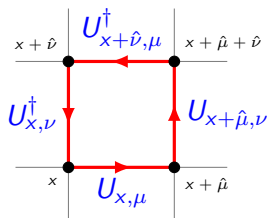
NON-PERTURBATIVE and GAUGE INVARIANT approach.

Lattice gauge field theory

Compact formulation (NO gauge fixing):

$$A_{x,\mu} \in \mathfrak{su}(N) \rightarrow U_{x,\mu} \in \text{SU}(N).$$

$$\text{Link variable: } U_{x,\mu} = e^{iA_{x,\mu}}.$$



plaquette:

$$U_{x,\mu\nu} = U_{x,\nu}^{\dagger} U_{x+\hat{\nu},\mu}^{\dagger} U_{x+\hat{\mu},\nu} U_{x,\mu}$$

(smallest Wilson loop)

Wilson's standard action

$$S[U] = \frac{\beta}{N} \sum_x \sum_{\mu < \nu} \text{Tr Re} [\mathbb{1} - U_{x,\mu\nu}],$$

where $\beta = 2N/g^2$, with g the gauge coupling.

Fermions on the lattice

Fermi statistics implemented by *anticommuting* Grassmann variables

$$\psi_i \psi_j = -\psi_j \psi_i, \quad i, j \in \{1, \dots, N\}.$$

Fermion functional integral (fermion determinant)

$$Z_F = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp(-\bar{\Psi} D_{\text{Dirac}} \Psi) = \det D_{\text{Dirac}},$$

where D_{Dirac} is the Dirac operator.

Dirac operator

In the lattice the components of the Dirac operator or Dirac matrix take the form

$$(D_{\text{Dirac}})_{xy,\alpha\beta}^{ab} = \sum_{\mu} (\gamma_{\mu})_{\alpha\beta} \frac{U_{x,\mu}^{ab} \delta_{x+\hat{\mu},y} - U_{x,-\mu}^{ab} \delta_{x-\hat{\mu},y}}{2} + m_0 \delta_{\alpha\beta} \delta^{ab} \delta_{x,y}.$$

Fermion fields ψ satisfy anti-periodic boundary conditions in the Euclidean time direction and periodic elsewhere.

Computation of fermion determinant very inefficient.

Better to invert the Dirac matrix.

Pseudofermions

On the lattice, instead of working with fermions one works with *pseudo-fermions*.

Pseudofermions are bosonic variables with a similar partition function to the fermion determinant for two degenerate mass flavors

$$\begin{aligned} Z_F &= \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp(-\bar{\Psi}_u D_{\text{Dirac}} \Psi_u + \bar{\Psi}_d D_{\text{Dirac}} \Psi_d) \\ &= (\det D_{\text{Dirac}})^2 \end{aligned}$$

$$\begin{aligned} Z_{\text{pseudo}} &= \int \mathcal{D}(\text{Re } \phi) \mathcal{D}(\text{Im } \phi) \exp(-\pi \phi^\dagger (D_{\text{Dirac}} D_{\text{Dirac}}^\dagger)^{-1} \phi) \\ &= (\det D_{\text{Dirac}})^2 \end{aligned}$$

Markov chains

Markov chain: A chain of stochastic configurations where the current configuration only depends on the previous one.

$$[\phi_1] \rightarrow [\phi_2] \rightarrow \dots$$

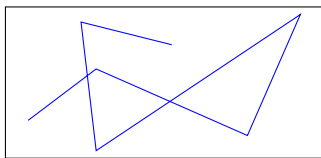
$$\langle O \rangle = \frac{1}{N} \sum_{i=1}^N O[\phi_i]$$

where $[\phi_i]$ is generated with probability $\frac{1}{Z} e^{-S[\phi_i]}$.

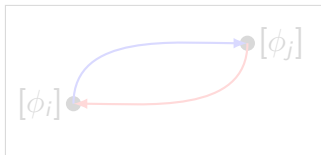
Importance sampling

In order to generate good configurations the algorithm needs to satisfy

- ▶ Ergodicity



- ▶ Detailed balance

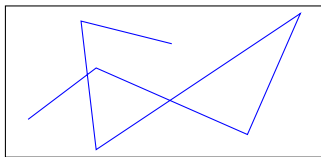


$$p[\phi_i]p[\phi_i \rightarrow \phi_j] = p[\phi_j]p[\phi_j \rightarrow \phi_i]$$

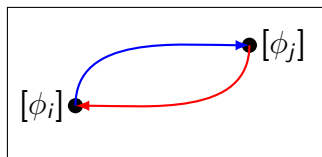
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$$p[\phi_i]p[\phi_i \rightarrow \phi_j] = p[\phi_j]p[\phi_j \rightarrow \phi_i]$$

Update algorithms

Local

- ▶ Metropolis
- ▶ Glauber
- ▶ Heatbath
- ▶ etc. . .

Global

- ▶ Wolff
- ▶ Swendsen-Wang
- ▶ Hybrid Monte Carlo (HMC)

Monte Carlo simulation

- ▶ Initial configuration *hot* or *cold start*.
- ▶ Generation of configurations.
- ▶ Thermalization: attain equilibrium.
- ▶ Taking of well separated measurements.
- ▶ Computation of averages and errors.
- ▶ Extrapolation to infinite volume.
- ▶ Interpretation of results.

Two flavor Schwinger model

Lagrangian of the Schwinger model with gauge group $U(1)$

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \sum_{f=1}^2 \bar{\psi}^f (\gamma_\mu D_\mu + m_0) \psi^f,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu + ieA_\mu$.

Action on the lattice

$$\begin{aligned} S[\psi, \bar{\psi}, U] = & \beta \sum_x \text{Re}(1 - U_{x,12}) + \\ & \sum_{x,f} \bar{\psi}_x^f \sum_{\mu=1}^2 \gamma_\mu \frac{U_{x,\mu} \psi_{x+\hat{\mu}}^f - U_{x,-\mu} \psi_{x-\hat{\mu}}^f}{2} + \\ & \sum_{x,f,\mu} m_0 \bar{\psi}_x^f \psi_x^f + \text{other terms}, \end{aligned}$$

where $\beta = 1/e^2$, and $U_{x,\mu} = e^{iA_{x,\mu}} \in U(1)$.

γ -matrices in 2d

In 2 dimensions we take the Pauli matrices

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

and $\gamma_5 = -i\gamma_1\gamma_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$

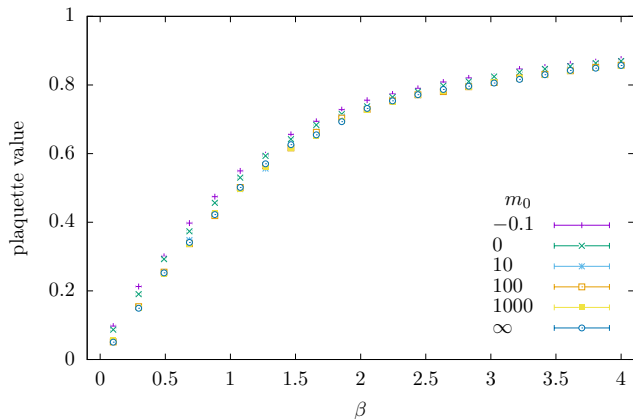
Plaquette value

One of the simplest observables to measure is the sum of plaquette variables

$$\text{plaquette value} = \frac{1}{V} \left\langle \sum_x \text{Re } U_{x,12} \right\rangle.$$

Plaquette value

Schwinger model with two fermion flavors, $V = 8^2$



Pion correlation function

Pion propagator $C_\pi(t) = \langle O_x \bar{O}_y \rangle$, where $O_x = \bar{d}_x \gamma_5 u_y$ and $t = |x - y|$

$$C_\pi(t) = \left\langle \sum_{\alpha, \beta} |(D_{\text{Dirac}}^{-1})_{xy, \alpha\beta}|^2 \right\rangle.$$

Decays exponentially in Euclidean time

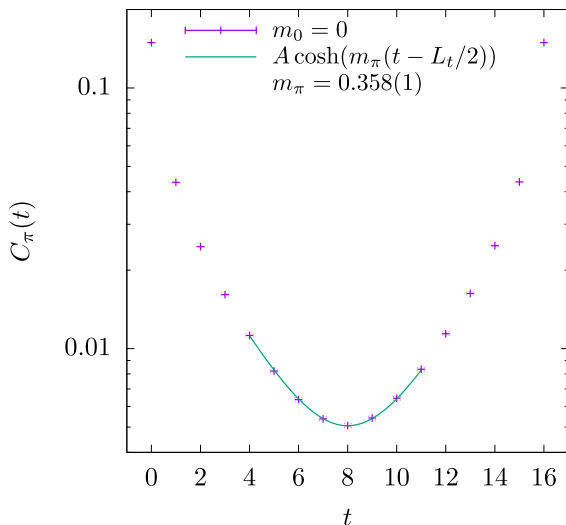
$$C_\pi(t) \propto \exp(-m_\pi t).$$

To take into account periodicity we use

$$C_\pi(t) \propto \cosh(m_\pi(t - L_t/2)).$$

Pion correlation function

$$V = 16^2, \beta = 4$$



Topological charge

In the continuum, gauge fields can be classified in topological sectors characterized by the topological charge $Q \in \mathbb{Z}$.

In 2d the topological charge takes the form

$$Q = \frac{1}{4\pi} \int d^2x \epsilon_{\mu\nu} F_{\mu\nu} \in \mathbb{Z}.$$

On the lattice

$$Q = \frac{1}{2\pi} \sum_x \text{Arg} U_{x,12} \in \mathbb{Z}.$$

Topological susceptibility

An important quantity is the topological susceptibility

$$\chi_t = \frac{\langle Q^2 \rangle}{V}$$

the topological charge squared per spacetime volume V .

Physically, in QCD it is related to the mass of the η' meson via the Witten-Veneziano formula

$$\chi_t = \frac{F_\pi^2 M_{\eta'}^2}{2N_f}.$$

Topological susceptibility with slab method

Divide the periodic volume V into sub-volumes or *slabs* of size xV , where $x \in [0, 1]$.

Extract χ_t from the fluctuations of the topological charge q within these slabs.

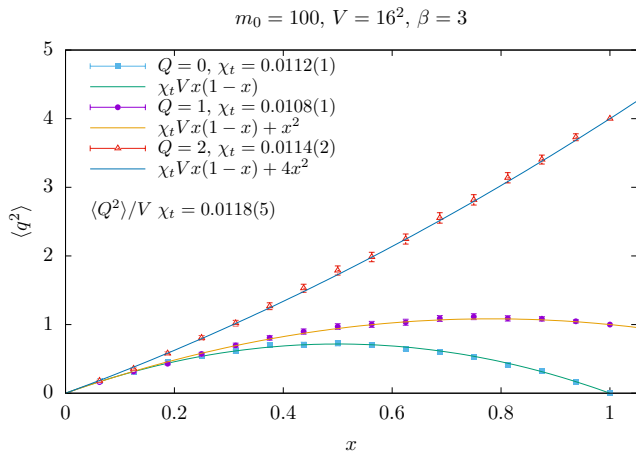
At fixed topological charge Q

$$p(q)p(Q - q) \propto \exp\left(-\frac{1}{2\chi_t V} \frac{(q - xQ)^2}{x(1-x)}\right),$$

from which we infer

$$\langle q^2 \rangle = \chi_t V x(1-x) + Q^2 x^2.$$

Topological susceptibility with slab method



Gradient Flow

On the continuum the Yang Mills flow is defined by

$$\frac{d}{d\tau} B_\mu(x, \tau) = -\frac{\partial S_{\text{gauge}}(B)}{\partial B_\mu(x, \tau)}, \quad B_\mu(x, \tau)|_{\tau=0} = A_\mu(x)$$

where τ is the *flow time*.

On the lattice, the Gradient Flow (or *Wilson flow*) is defined by

$$\frac{d}{d\tau} V_{x,\mu}(\tau) = Z_{x,\mu}(\tau) V_{x,\mu}(\tau), \quad V_{x,\mu}(\tau)|_{\tau=0} = U_{x,\mu}$$

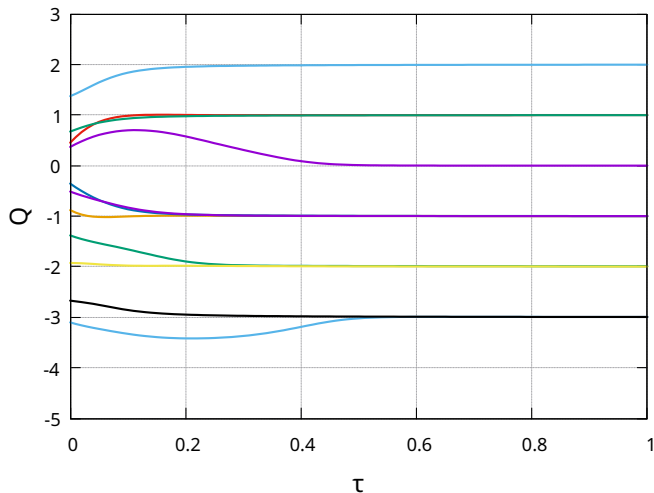
where

$$Z_{x,\mu} = -\{U_{x,\mu} \Sigma_{x,\mu}^\dagger\}_{\text{TA}}$$

$$\Sigma_{x,\mu} = \sum_{\nu \neq \mu} \left[U_{x,\nu} U_{x+\hat{\nu},\mu} U_{x+\hat{\mu},\nu}^\dagger + U_{x-\hat{\nu},\nu}^\dagger U_{x-\hat{\nu},\mu} U_{x+\hat{\mu}-\hat{\nu},\nu} \right]$$

$$W_{\text{TA}} = \frac{W - W^\dagger}{2} - \frac{\text{Tr}(W - W^\dagger)}{2N}$$

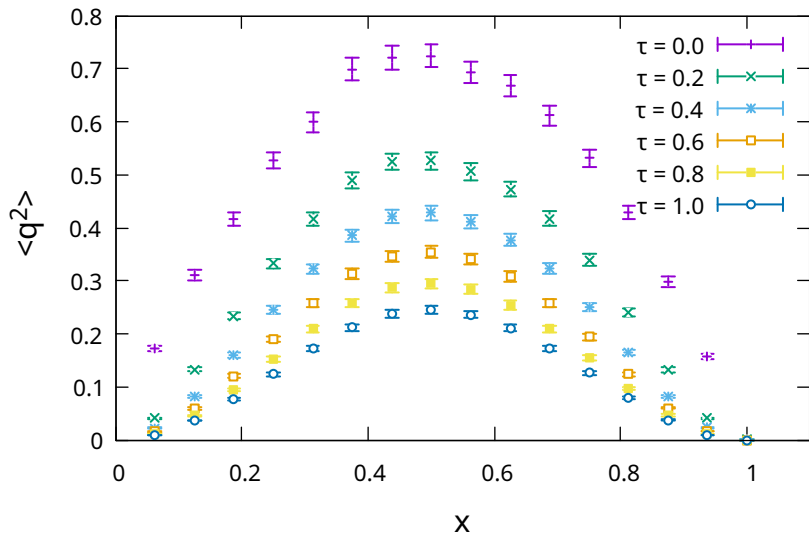
Topological charge under Gradient Flow



$$Q = \frac{1}{2\pi} \sum_x \text{Im} U_{x,12}$$

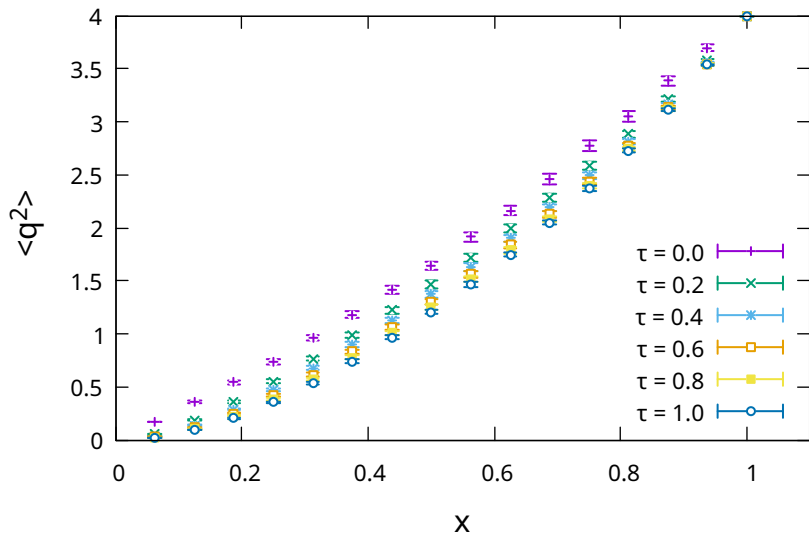
Topological charge under Gradient Flow

$$Q = 0, V = 16^2, \beta = 3$$



Topological charge under Gradient Flow

$$|Q| = 2, V = 16^2, \beta = 3$$



Summary

Lattice regularization is a non-perturbative and gauge invariant approach.

Compact formulation does NOT require gauge fixing.

Lattice formulation very robust.

Schwinger model: toy model for QCD.

LQCD simulations in progress ;).

The slab method efficient when topological freezing occurs.