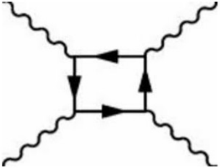
Discontinuity method for evaluating scattering amplitudes within the worldline formalism



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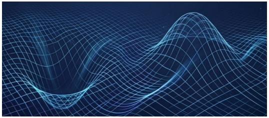
The worldline formalism is an alternative framework to the standard diagrammatic approach in quantum field theory

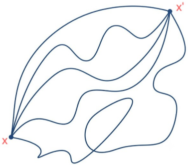
Standard approach

Based on second quantization



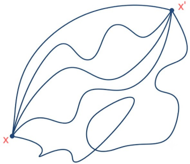
Worldline formalism

Based on first-quantized relativistic path integrals



Example

Propagator of a relativistic scalar particle in an electromagnetic field A^μ in Euclidean space-time



$$D^{xx'} = \int_0^\infty dT\, \mathrm{e}^{-m^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x\, \mathrm{e}^{-S}$$

$$S = \int_0^T d\tau\, \left[\frac{1}{4}\dot{x}^2(\tau) + ie\dot{x}(\tau)\cdot A(x)\right]$$

Example

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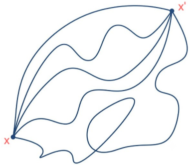


$$D^{xx'} = \int_0^\infty dT\, \mathsf{e}^{-m^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x\, \mathsf{e}^{-S}$$

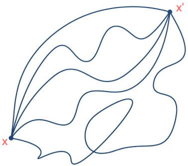
$$S = \int_0^T d\tau\, \left[\frac{1}{4}\dot{x}^2(\tau) + ie\dot{x}(\tau)\cdot A(x)\right]$$

$\int \mathcal{D}x\, \mathsf{e}^{-S}$

Add contributions from every possible worldline connecting x with x'

Example

Propagator of a relativistic scalar particle in an electromagnetic field A^{μ} in Euclidean space-time



$$D^{xx'} = \int_0^{\infty} dT \, \mathrm{e}^{-m^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x \, \mathrm{e}^{-S}$$

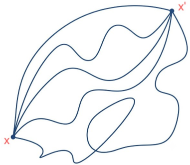
$$S = \int_0^{T} d\tau \, \left[\frac{1}{4}\dot{x}^2(\tau) + ie\dot{x}(\tau)\cdot A(x)\right]$$

$\int_0^{\infty} dT$

Sum over all possible values of the particle's proper time.

Example

Propagator of a relativistic scalar particle in an electromagnetic field A^μ in Euclidean space-time



$$D^{xx'} = \int_0^\infty dT\, \mathsf{e}^{-m^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x\, \mathsf{e}^{-S}$$

$$S = \int_0^T d\tau\, \left[\frac{1}{4}\dot{x}^2(\tau) + ie\dot{x}(\tau)\cdot A(x)\right]$$

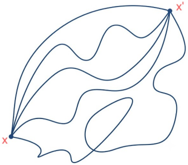
$\int_0^\infty dT\, \mathsf{e}^{-m^2 T}$

The exponential factor suppresses long worldlines.

Large mass m makes long worldlines unlikely.

Example

Propagator of a relativistic scalar particle in an electromagnetic field A^μ in Euclidean space-time



$$D^{xx'} = \int_0^\infty dT\, \mathsf{e}^{-m^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x\, \mathsf{e}^{-S}$$

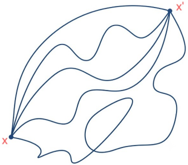
$$S = \int_0^T d\tau \left[\frac{1}{4}\dot{x}^2(\tau) + ie\dot{x}(\tau)\cdot A(x)\right]$$

$\frac{1}{4}\dot{x}^2(\tau)$

The path integral favors smoother trajectories; extremely jagged paths are exponentially suppressed.

Example

Propagator of a relativistic scalar particle in an electromagnetic field A^μ in Euclidean space-time



$$D^{xx'} = \int_0^\infty dT\, \mathsf{e}^{-m^2T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x\, \mathsf{e}^{-S}$$

$$S = \int_0^T d\tau\, \left[\frac{1}{4}\dot{x}^2(\tau) + ie\dot{x}(\tau)\cdot A(x)\right]$$

$ie\int_0^T d\tau\, \dot{x}\cdot A = ie\int_{\text{path}} A_\mu dx^\mu$

Every path accumulates an electromagnetic phase while moving through the gauge field.

Specializing the background field to a superposition of plane waves yields a master formula for the scalar propagator interacting with N photons

$$A^\mu(x) = \sum_{i=1}^{N} \epsilon_i^\mu \mathsf{e}^{ik_i\cdot x}$$

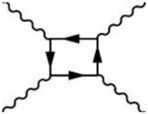
$$\begin{aligned} D^{pp'}(k_1,\epsilon_1;\dots;k_N,\epsilon_N) \quad &= \quad (-ie)^N (2\pi)^D \delta^D(p+p'+\sum_{i=1}^{N} k_i) \int_0^\infty dT\, \mathsf{e}^{-m^2T} \\ &\times \prod_{i=1}^{N} \int_0^T d\tau_i \quad \mathsf{e}^{F(\tau_i,T,p',k_i,\epsilon_i)}\Big|_{\epsilon_1\dots\epsilon_N} \end{aligned}$$

Similar worldline master formulas exist for the one-loop N-photon amplitude in scalar and spinor QED

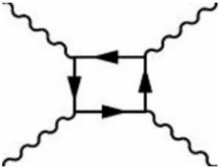
$$\Gamma_{\mathsf{scal}}(p_1,\epsilon_1;\dots;p_N,\epsilon_N) = (-ie)^N(2\pi)^D\delta^D\left(\sum p_i\right)\int_0^\infty \frac{dT}{T}(4\pi T)^{\frac{-D}{2}}\,\mathsf{e}^{-m^2T}$$

$$\times \int_0^T \prod_{k=1}^N d\tau_k \,\mathsf{exp}\left\{\sum_{i,j=1}^N\left[\frac{1}{2}G_{ij}p_i\cdot p_j - i\dot{G}_{ij}\epsilon_i\cdot p_j + \frac{1}{2}\ddot{G}_{ij}\epsilon_i\cdot\epsilon_j\right]\right\}\Bigg|_{\epsilon_1\dots\epsilon_N}$$

$$G(\tau_i,\tau_j) = |\tau_i-\tau_j| - \frac{(\tau_i-\tau_j)^2}{T}$$



Discontinuity method for evaluating scattering amplitudes within the worldline formalism



1. Discontinuity method
2. Application to photon-photon scattering

After integration over T and rescaling $\tau_i = Tu_i$,

the one-loop worldline integrals take the form

$$\int_0^1 \prod_{k=1}^{N} du_k \, \frac{Q(\dot{G})}{\Big[m^2 - \frac{1}{2}\sum_{i,j=1}^{N} p_i \cdot p_j \, G_{ij}\Big]^M} \times \text{Pol. vect.}$$

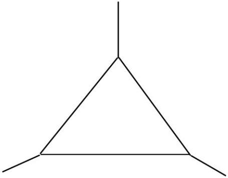
$$\begin{aligned} G(u,u') &= |u-u'| - (u-u')^2, \\ \dot{G}(u,u') &= \mathrm{sign}(u-u') - 2(u-u') \end{aligned}$$

Since one-loop amplitudes decompose into tadpoles, bubbles, triangles, boxes, and rational terms, we propose the following ansatz

$$\int_0^1 \prod_{k=1}^{N} du_k \, \frac{Q(\dot{G})}{\Big[m^2 - \frac{1}{2}\sum_{i,j=1}^{N} p_i \cdot p_j \, G_{ij}\Big]^M}$$

$$= \text{ r.t } + \text{ r.t } \times \text{tadpoles} + \text{ r.t } \times \text{bubbles} + \text{ r.t } \times \text{triangles} + \text{ r.t } \times \text{boxes}$$

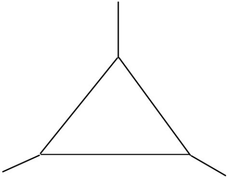
The transcendental functions associated with tadpole, bubble, triangle, and box integrals are distinguished by their discontinuities in specific kinematic invariants



$$\int \frac{d^D k}{(\pi)^{D/2}} \frac{1}{(k^2+m^2)[(k+p)^2+m^2]^2}$$

One-mass triangle integral

The transcendental functions associated with tadpole, bubble, triangle, and box integrals are distinguished by their discontinuities in specific kinematic invariants



$$\int \frac{d^D k}{(\pi)^{D/2}} \frac{1}{(k^2+m^2)[(k+p)^2+m^2]^2}$$

One-mass triangle integral

Associated transcendental function

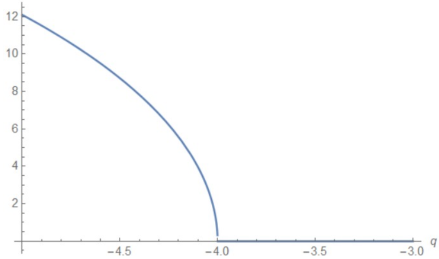
in $D = 4$ dimensions

$$\bar{T}(-q) \quad = \quad \ln^2\left(\frac{\beta_{-\hat{q}}-1}{\beta_{-\hat{q}}+1}\right)$$

$$\beta_x \quad = \quad \sqrt{1-\frac{1}{x}}$$

$q = p^2/m^2$, $\hat{q} = q/4$

The transcendental functions associated with tadpole, bubble, triangle, and box integrals are distinguished by their discontinuities in specific kinematic invariants



$$\bar{T}(-q) = \ln^2\left(\frac{\beta_{-\hat{q}} - 1}{\beta_{-\hat{q}} + 1}\right)$$

$$\beta_x = \sqrt{1 - \frac{1}{x}}$$

$q = p^2/m^2$, $\hat{q} = q/4$

$\mathrm{Disc}_q \bar{T}(-q) = \lim_{\epsilon\to 0}[\bar{T}(-q + i\epsilon) - \bar{T}(-q - i\epsilon)]$

The rational terms can be extracted from the discontinuities of the worldline integrals in specific kinematic channels

$$\int_0^1 \prod_{k=1}^{N} du_k \, \frac{Q(\dot{G})}{\left[m^2 - \frac{1}{2}\sum_{i,j=1}^{N} p_i \cdot p_j \, G_{ij}\right]^M}$$

$$= \text{ r.t } + \text{ r.t } \times \text{tadpoles} + \text{ r.t } \times \text{bubbles} + \text{ r.t } \times \text{triangles} + \text{ r.t } \times \text{boxes}$$

The rational terms can be extracted from the discontinuities of the worldline integrals in specific kinematic channels

A simple example...

$$T(q,0,0) \quad = \quad \int_0^1 \prod_{k=1}^{3} du_k \, \frac{1}{m^2\,(1+q\Lambda)}$$

$$\Lambda = \frac{1}{2}\,(G_{12}+G_{13}-G_{23})$$

The rational terms can be extracted from the discontinuities of the worldline integrals in specific kinematic channels

A simple example...

$$T(q,0,0) = \frac{2}{m^2}\int_0^1 du_2 \int_0^{u_2} du_1 \, \frac{1}{1+(u_2-u_1)u_1 q}$$

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$$T(q,0,0) = \frac{2}{m^2}\int_0^1 du_2 \int_0^{u_2} du_1 \, \frac{1}{1+(u_2-u_1)u_1 q}$$

$$\mathrm{Disc}_q T(q,0,0) = -\frac{4i}{m^2}\lim_{\epsilon\to 0}\int_0^1 du_2 \int_0^{u_2} du_1 \, \frac{\Lambda_q \epsilon}{\bar{\Lambda}^2 + \Lambda_q^2\epsilon^2}$$

$$\bar{\Lambda} = 1+(u_2-u_1)u_1 q, \qquad \Lambda_q = (u_2-u_1)u_1$$

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$$\lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2} = \pi\,\delta(x)$$

The rational terms can be extracted from the discontinuities of the worldline integrals in specific kinematic channels

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$$T(q,0,0) = \frac{2}{m^2}\int_0^1 du_2 \int_0^{u_2} du_1 \, \frac{1}{1+(u_2-u_1)u_1 q}$$

$$\mathsf{Disc}_q T(q,0,0) \quad = \quad -\frac{4i\pi}{m^2}\int_0^1 du_2 \int_0^{u_2} du_1 \, \frac{1}{\Lambda_q}\,\delta\left(\frac{\bar{\Lambda}}{\Lambda_q}\right)$$

The rational terms can be extracted from the discontinuities of the worldline integrals in specific kinematic channels

A simple example...

$$T(q,0,0) = \frac{2}{m^2}\int_0^1 du_2 \int_0^{u_2} du_1 \, \frac{1}{1+(u_2-u_1)u_1 q}$$

$$\mathsf{Disc}_q T(q,0,0) = \frac{4i\pi}{m^2 q}\ln(-z)$$

$$\begin{aligned} z &= -r_+(q)/r_-(q) \\ r_\pm(q) &= \frac{1}{2} \pm \frac{\sqrt{1+\frac{4}{q}}}{2} \end{aligned}$$

The rational terms can be extracted from the discontinuities of the worldline integrals in specific kinematic channels

A simple example...

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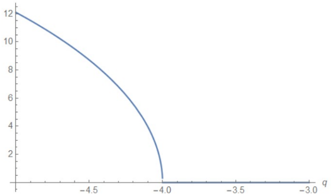

The rational terms can be extracted from the discontinuities of the worldline integrals in specific kinematic channels

$$T(q,0,0) = \frac{1}{m^2 q}\bar{T}(-q)$$

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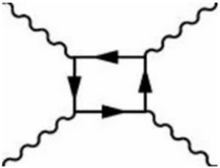
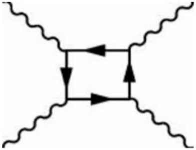


1. Discontinuity method
2. Application to photon-photon scattering

The discontinuity method enables an efficient computation of the light-by-light scattering amplitude



Scalar QED

Photon polarizations (-+++)

Kinematics

$$\begin{aligned} s &= -(p_1+p_2)^2 = -2p_1\cdot p_2 = -2p_3\cdot p_4 \\ t &= -(p_1+p_3)^2 = -2p_1\cdot p_3 = -2p_2\cdot p_4 \\ u &= -(p_1+p_4)^2 = -2p_1\cdot p_4 = -2p_2\cdot p_3 \end{aligned}$$

The discontinuity method enables an efficient computation of the light-by-light scattering amplitude

$$\Gamma_{\mathsf{scal}} = \frac{e^4}{(4\pi)^2} C^{-+++} \int_0^1 \left(\prod_{i=1}^4 du_i \right) \frac{P^{-+++}}{m^4 [1 + \Lambda^{(4)}(s,t,u)]^2}$$

$$\begin{aligned}
C^{-+++} &= -\frac{1}{4u} \langle 13 \rangle^2 [23]^2 [34]^2 \\
P^{-+++} &= (\dot{G}_{13} - \dot{G}_{12})(\dot{G}_{24} - \dot{G}_{23})(\dot{G}_{34} - \dot{G}_{32})(\dot{G}_{43} - \dot{G}_{42}) \\
\Lambda^{(4)} &= \frac{1}{2} \left[\bar{s}(G_{12} + G_{34} - G_{14} - G_{23}) + \bar{t}(G_{13} + G_{24} - G_{14} - G_{23}) \right]
\end{aligned}$$

$$\bar{s} = \frac{s}{m^2}, \qquad \bar{t} = \frac{t}{m^2}$$

The discontinuity method enables an efficient computation of the light-by-light scattering amplitude

Transcendental function related to a Box Feynman integral

$$
\begin{aligned}
\bar{B}(s,t) \quad = \quad & \frac{4}{st\beta_{\hat{s}\hat{t}}}\left(2\ln^2\left(\frac{\beta_{\hat{s}\hat{t}}+\beta_{\hat{s}}}{\beta_{\hat{s}\hat{t}}+\beta_{\hat{t}}}\right)+\ln\left(\frac{\beta_{\hat{s}\hat{t}}-\beta_{\hat{s}}}{\beta_{\hat{s}\hat{t}}+\beta_{\hat{s}}}\right)\ln\left(\frac{\beta_{\hat{s}\hat{t}}-\beta_{\hat{t}}}{\beta_{\hat{s}\hat{t}}+\beta_{\hat{t}}}\right)-\frac{\pi^2}{2}\right. \\
& \left.+\sum_{i=\hat{s},\hat{t}}\left[2\mathrm{Li}_2\left(\frac{\beta_i-1}{\beta_{\hat{s}\hat{t}}+\beta_i}\right)-2\mathrm{Li}_2\left(\frac{\beta_i-\beta_{\hat{s}\hat{t}}}{\beta_i+1}\right)-\ln^2\left(\frac{\beta_i+1}{\beta_{\hat{s}\hat{t}}+\beta_i}\right)\right]\right)
\end{aligned}
$$

$\hat{s} \equiv \frac{s}{4m^2}$, $\hat{t} \equiv \frac{t}{4m^2}$, $\hat{u} \equiv \frac{u}{4m^2}$

The discontinuity method enables an efficient computation of the light-by-light scattering amplitude

Transcendental function related to a Box Feynman integral

$$\begin{aligned}\bar{B}(s,t) \quad = \quad & \frac{4}{st\beta_{\hat{s}\hat{t}}}\Bigg(2\ln^2\left(\frac{\beta_{\hat{s}\hat{t}}+\beta_{\hat{s}}}{\beta_{\hat{s}\hat{t}}+\beta_{\hat{t}}}\right)+\ln\left(\frac{\beta_{\hat{s}\hat{t}}-\beta_{\hat{s}}}{\beta_{\hat{s}\hat{t}}+\beta_{\hat{s}}}\right)\ln\left(\frac{\beta_{\hat{s}\hat{t}}-\beta_{\hat{t}}}{\beta_{\hat{s}\hat{t}}+\beta_{\hat{t}}}\right)-\frac{\pi^2}{2} \\ & +\sum_{i=\hat{s},\hat{t}}\left[2\mathrm{Li}_2\left(\frac{\beta_i-1}{\beta_{\hat{s}\hat{t}}+\beta_i}\right)-2\mathrm{Li}_2\left(\frac{\beta_i-\beta_{\hat{s}\hat{t}}}{\beta_i+1}\right)-\ln^2\left(\frac{\beta_i+1}{\beta_{\hat{s}\hat{t}}+\beta_i}\right)\right]\Bigg)\end{aligned}$$

$$\begin{aligned}\mathrm{Disc}_{\bar{s}}\bar{B}(s,t) \quad &= \quad 0 \\ \mathrm{Disc}_{\bar{s}}\bar{B}(s,u) \quad &= \quad -\frac{8\pi i}{su\beta_{\hat{s}\hat{u}}}\ln\left(\frac{\beta_{\hat{s}\hat{u}}-\beta_{\hat{u}}}{\beta_{\hat{s}\hat{u}}+\beta_{\hat{u}}}\right) \\ \mathrm{Disc}_{\bar{s}}\bar{B}(t,u) \quad &= \quad -\frac{8\pi i}{tu\beta_{\hat{t}\hat{u}}}\ln\left(\frac{\beta_{\hat{t}\hat{u}}-\beta_{\hat{u}}}{\beta_{\hat{t}\hat{u}}+\beta_{\hat{u}}}\right)\end{aligned}$$

The discontinuity method enables an efficient computation of the light-by-light scattering amplitude

$$\begin{aligned}\Gamma_{\mathsf{scal}} \quad = \quad & \frac{16e^4}{(4\pi)^2 st} C^{-+++} \Bigg\{ 1 - m^4 \left[\bar{B}(s,t) + \bar{B}(u,s) + \bar{B}(u,t) \right] - \frac{m^2}{2stu} \Big[s^2 t^2 \bar{B}(s,t) \\ & + s^2 u^2 \bar{B}(s,u) + t^2 u^2 \bar{B}(t,u) \Big] - \frac{m^2(s^2+t^2+u^2)}{2stu} \left[\bar{T}(s) + \bar{T}(t) + \bar{T}(u) \right] \Bigg\}\end{aligned}$$

$$\int_0^1 \prod_{k=1}^{N} du_k \, \frac{Q(\dot{G})}{\left[m^2 - \frac{1}{2}\sum_{i,j=1}^{N} p_i \cdot p_j \, G_{ij}\right]^M}$$

$$= \text{ r.t } + \text{ r.t } \times \text{tadpoles} + \text{ r.t } \times \text{bubbles} + \text{ r.t } \times \text{triangles} + \text{ r.t } \times \text{boxes}$$

These examples highlight the potential of the discontinuity method as an efficient approach to evaluating one-loop worldline master integrals without direct computation

Ref. JHEP **07**, 159 (2025) doi:10.1007/JHEP07(2025)159