Dimensional Reduction of Fermions in Brane Worlds

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1. Motivation

Lattice YM theory: $\xi \gg a$ occurs naturally, based on asympt. freedom

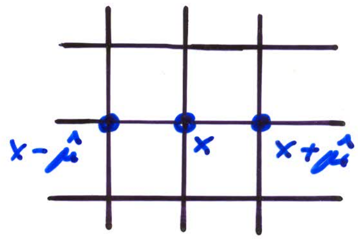
Include quarks, e.g. Wilson fermion:
naturally $\xi = O(a)$
small mass requires unnatural fine tuning.

DWF: χ limit "naturally" from higher dim.
Overlap: related, extra dim. integrated out
but: 5^{th} dim. is just technical:
(no gauge fields, separate walls at fixed ξ_{gluon} ...)
↳ non-local in x_5

Here: try to construct brane world in higher dim.
with mechanism for dim. reduction
→ even dim. with light fermions
Natural mechanism for hierarchy ?

Toy model: 3d → 2d Gross-Neveu model at large N.

Lattice Fermions

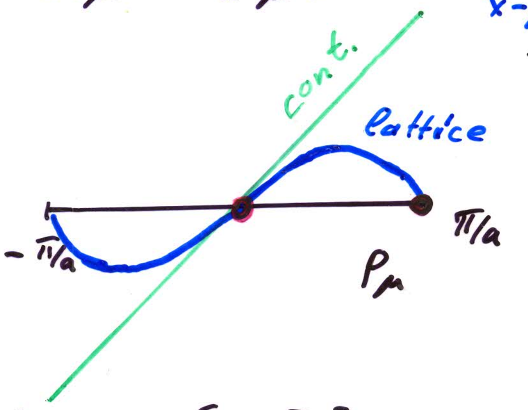


Naive formulation:

$$\partial_\mu \psi \to \frac{1}{2}\left[\psi_{x+\hat{\mu}} - \psi_{x-\hat{\mu}}\right]$$

$m=0$: free propagator

$$\left(\gamma_\mu \sin(p_\mu a)\right)^{-1}$$



(inv. prop.)

2^d fermions in Brillouin zone $\left]-\frac{\pi}{a}, \frac{\pi}{a}\right]^d$ for $p_\mu \in \{0, \frac{\pi}{a}\}$

Doubling problem related to χ sym.

Standard formulation: Wilson fermion

$$S = \sum_x \bar{\psi}_x \left\{ \frac{i}{2}\gamma_\mu \left(\psi_{x+\hat{\mu}} - \psi_{x-\hat{\mu}}\right) - \frac{1}{2}\sum_\mu \left[\psi_{x+\hat{\mu}} + \psi_{x-\hat{\mu}} - 2\psi_x\right] \right\}$$

discrete Laplacian term; removes doublers, but breaks χ sym., artifacts in $O(a^2)$

Gauge interaction:

link variables

$$U_{\mu,x} = e^{ig\int_x^{x+\hat{\mu}} A_\mu \, dy} \in SU(N)$$

(path ordered)

Wilson artifacts in $O(a)$

Additive mass renormalization:

$$\left.\begin{array}{ll} -\frac{1}{2}\bar{\psi}_x\left(U_{\mu,x}\psi_{x+\hat{\mu}} + U^\dagger_{\mu,x-\hat{\mu}}\psi_{x-\hat{\mu}}\right) & \text{suppressed} \\ +\ \bar{\psi}_x \psi_x & \text{not "} \end{array}\right\} \text{fermion mass of } O\left(\tfrac{1}{a}\right)$$

Coupling of $\bar{\psi}_x, \psi_{x\pm\hat{\mu}}$ can be amplified by "hopping parameter" to compensate mean link suppression (~ neg. bare mass).

Fine tuning by hand – depending on gauge coupling – may lead close to criticality ($m_\pi \ll \frac{1}{a}$).

2. The 2d Gross-Neveu Model

→ target theory

Euclidean action (suppress flavor indices $1 \cdots N$)

$$S[\bar{\Psi},\Psi] = \int d^2x \left[\bar{\Psi} \gamma_\mu \partial_\mu \Psi - \frac{g}{2N} (\bar{\Psi}\Psi)^2 \right]$$

g : dim'less

same magnitude as $N \to \infty$

Symmetries : $U(N)$

$Z(2)$ discrete χ sym.: $\boxed{\begin{array}{l} \Psi_L \to \pm \Psi_L ,\ \bar{\Psi}_L \to \pm \bar{\Psi}_L \\ \Psi_R \to \mp \Psi_R ,\ \bar{\Psi}_R \to \mp \bar{\Psi}_R \end{array}}$

Auxiliary scalar field $\phi = \frac{g}{N} \bar{\Psi}\Psi$:

$$\int D\bar{\Psi} D\Psi \, e^{-S[\bar{\Psi},\Psi]} = \int D\bar{\Psi} D\Psi D\phi \, e^{-S[\bar{\Psi},\Psi,\phi]}$$

$$S[\bar{\Psi},\Psi,\phi] = \int d^2x \left[\bar{\Psi} \gamma_\mu \partial_\mu \Psi - \phi \bar{\Psi}\Psi + \frac{N}{2g} \phi^2 \right]$$

$N \to \infty$: $\phi(\vec{x}) = \text{const.}$

$$\int D\bar{\Psi} D\Psi \, e^{-S[\bar{\Psi},\Psi,\phi]} = e^{-N \cdot V \cdot \underline{V_{eff}(\phi)}}$$

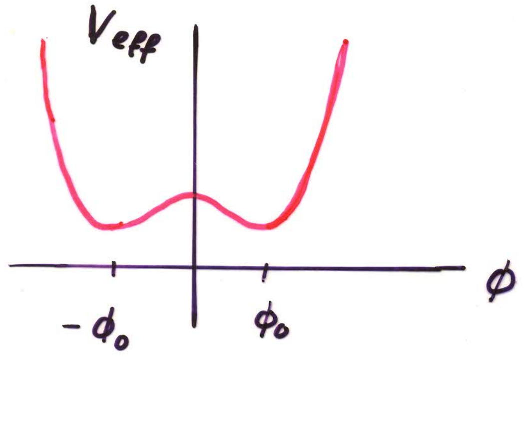
Minimum ϕ_0 of V_{eff} obeys $\underline{\text{gap eq.}}$ $(V \to \infty)$

$$\frac{1}{g} = \frac{1}{(2\pi)^2} \int d^2k \, \frac{2}{k^2 + \phi_0^2} = \frac{1}{2\pi} \int_0^{\Lambda_2} |k| \, d|k| \, \frac{2}{k^2 + \phi_0^2}$$

$g \ll 1$, $\Lambda_2 \gg \phi_0$ $\Rightarrow$ $\phi_0 = \Lambda_2 \underbrace{e^{-\pi/g}}_{\text{asympt. freedom}}$ = fermion mass m

(dyn. generated)

SSB of $Z(2)$



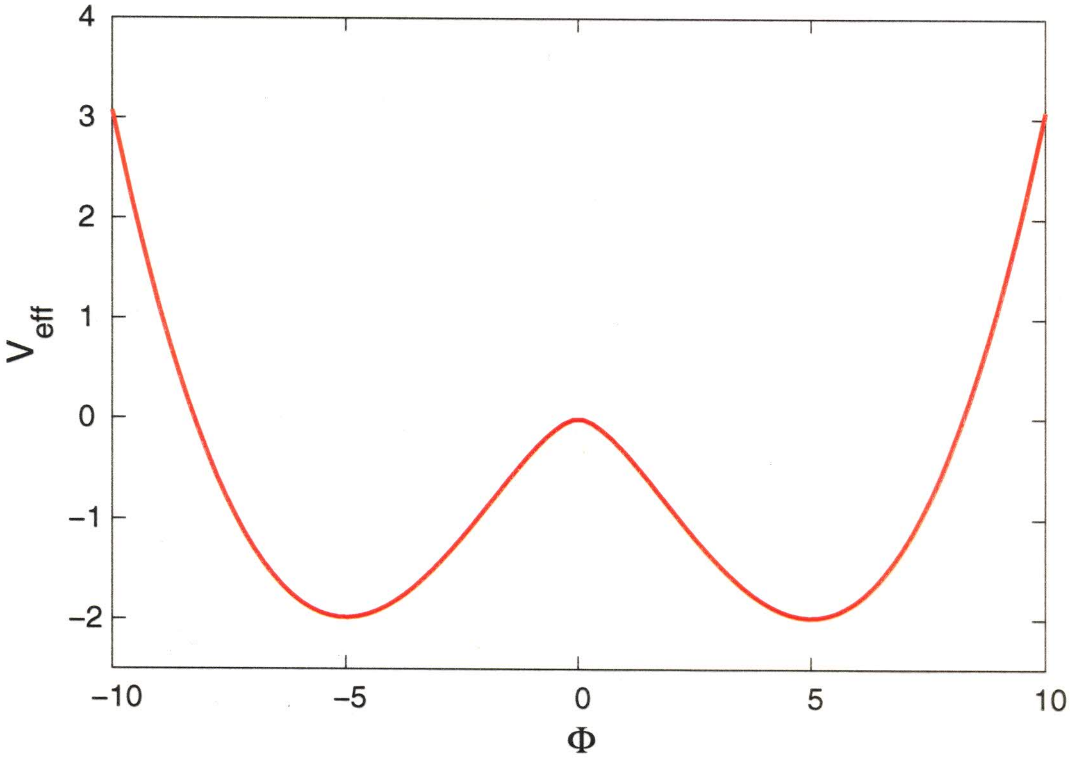


Figure 1: The effective potential in $d = 2$ ($\Phi_0 = 5$).

3. The 3d Gross-Neveu Model

$$S[\bar{\Psi},\Psi] = \int d^3x \left[\bar{\Psi}\gamma_\mu \partial_\mu \Psi + \bar{\Psi}\gamma_3 \partial_3 \Psi - \frac{G}{2N} (\bar{\Psi}\Psi)^2 \right]$$

$\dim[G] = \text{length}$

no χ sym., but still a $Z(2)$-sym.:

$$\Psi_L(\vec{x},x_3) \to \pm \Psi_L(\vec{x},-x_3)\ ,\quad \bar{\Psi}_L(\vec{x},x_3) \to \pm \bar{\Psi}_L(\vec{x},-x_3)$$

$$\Psi_R(\vec{x},x_3) \to \mp \Psi_R(\vec{x},-x_3)\ ,\quad \bar{\Psi}_R(\vec{x},x_3) \to \mp \bar{\Psi}_R(\vec{x},-x_3)$$

turns into discrete χ sym. after dim. reduction.

Auxiliary scalar field $\phi \xrightarrow{N=\infty}$ const. , $\int D\bar{\Psi} D\Psi \to V_{eff}(\phi)$

Gap eq. for minimum ϕ_0: $\quad \frac{1}{(2\pi)^3} \int d^3k \, \frac{2}{k^2+\phi_0^2} = \frac{1}{G}$

Cutoff $\Lambda_3 \gg \phi_0 \Rightarrow$ critical coupling $G_c = \frac{\pi^2}{\Lambda_3}$

$$V_{eff}(\phi) = \frac{1}{6\pi} |\phi|^3 - \frac{1}{2}\left(\frac{1}{G_c} - \frac{1}{G}\right)\phi^2 + \text{const.}$$

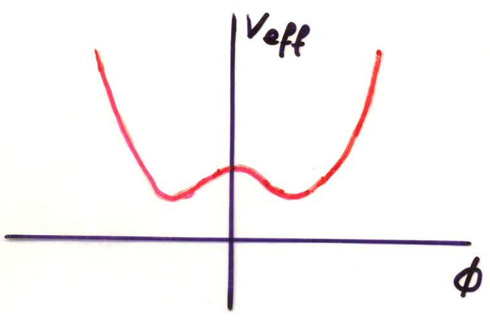
(bulk)

$$\phi_0 = \begin{cases} 2\pi\left(\frac{1}{G_c} - \frac{1}{G}\right) \\ 0 \end{cases}$$

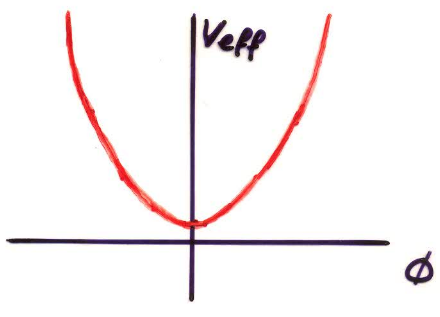
$G > G_c$: broken phase

$G \leq G_c$: sym. phase

(unlike cont. symmetries)



broken



symmetric

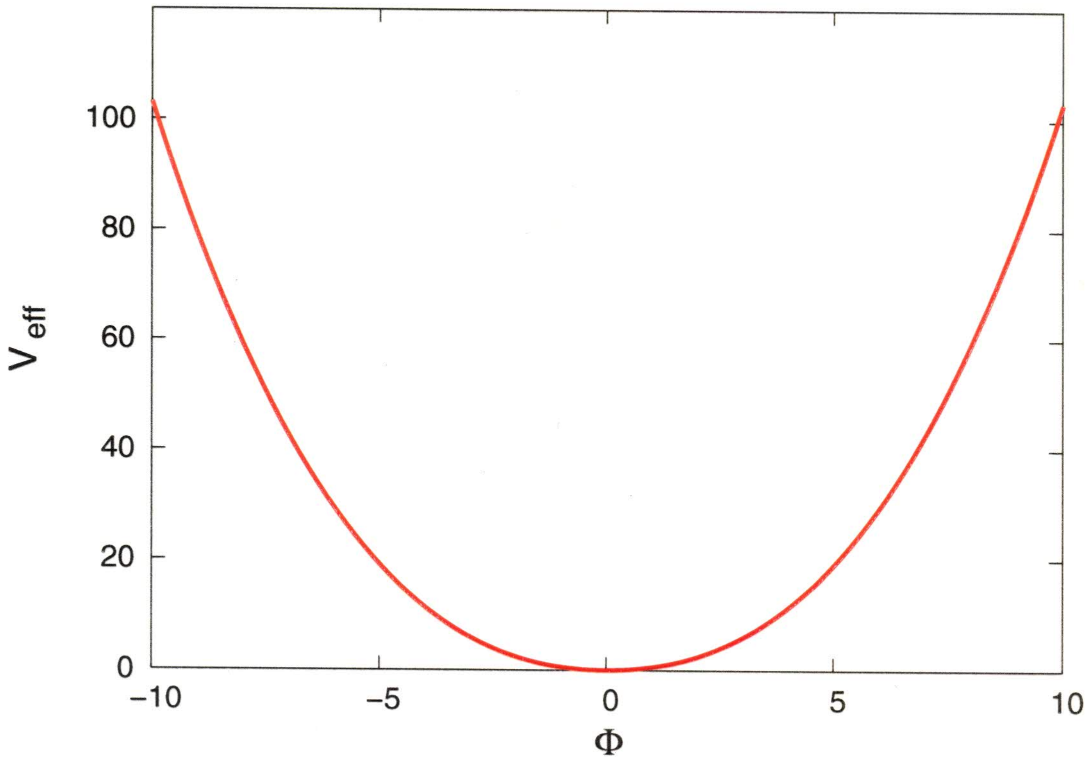


Figure 2: The effective potential in $d = 3$ in the symmetric phase ($G_c = 1$, $G = 0.5$).

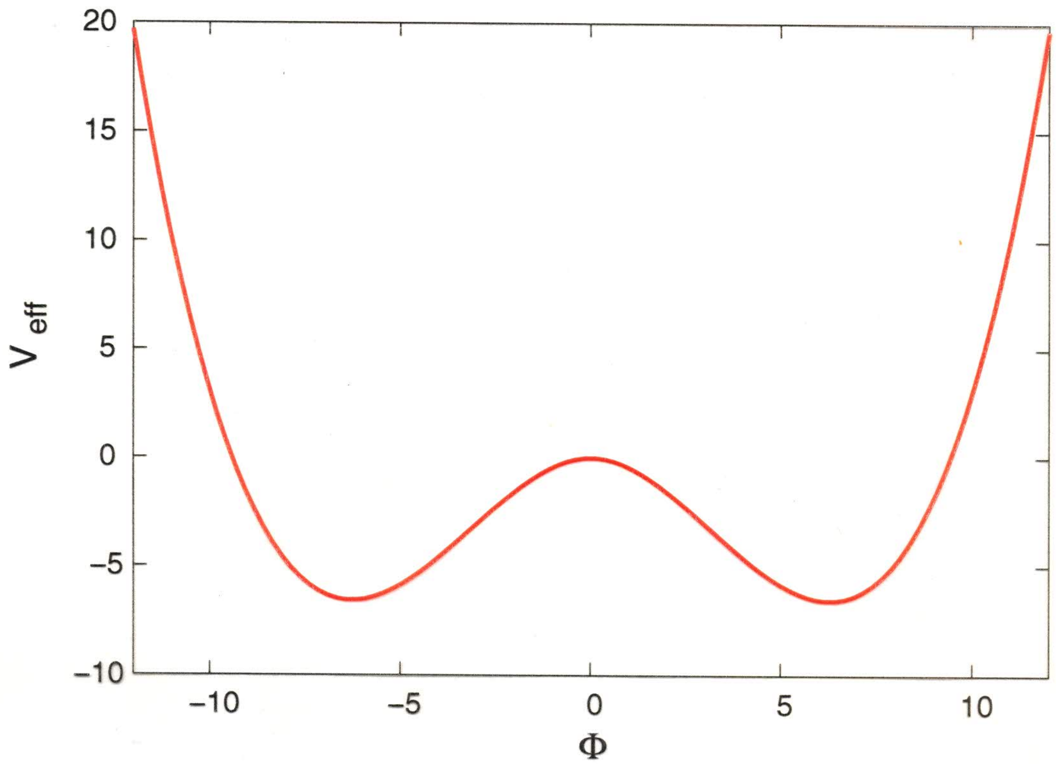


Figure 3: The effective potential in $d = 3$ in the broken phase ($G_c = 1$, $G = 2$).

4. Dim. Reduction from Symmetric Phase

Compactify x_3 with periodicity β, $k_3 = \frac{2\pi n}{\beta}$, $n \in \mathbb{Z}$

Gap eq.
$$\frac{1}{G} = \frac{1}{(2\pi)^3} \int d^2k \, \frac{2\pi}{\beta} \sum_n \frac{2}{\vec{k}^2 + \left(\frac{2\pi n}{\beta}\right)^2 + \phi_0^2}$$
$$= \frac{1}{(2\pi)^2} \int d^2k \, \frac{\coth\left(\frac{\beta}{2}\sqrt{\vec{k}^2 + \phi_0^2}\right)}{\sqrt{\vec{k}^2 + \phi_0^2}}$$

Cutoff matching: $\frac{\Lambda_2}{2\pi} = \frac{1}{G_c} = \frac{\Lambda_3}{\pi^2}$ $\Rightarrow$ $\frac{\beta\phi_0}{2} \simeq e^{-\pi\beta \underbrace{\left(\frac{1}{G} - \frac{1}{G_c}\right)}_{1/g}}$

$$\Rightarrow \quad m = \frac{2}{\beta} e^{-\pi/g}$$

Recover 2d asympt. freedom with $\Lambda_2 \hat{=} \frac{2}{\beta}$: eff. cutoff

Massless 3d fermion cannot remain massless in $d=2$ (only broken phase), but $\xi = \frac{1}{m} = \frac{\beta}{2} e^{\pi\beta \underbrace{\left(\frac{1}{G} - \frac{1}{G_c}\right)}_{>0}}$

$\Rightarrow$ $\xi \gg \beta$ for $\beta \to \infty$

We have to make β large to make it negligible.

Looks okay, but not satisfactory in view of our motivation:

non-pert. treatment at finite N (lattice);

start from massless phase just shifts fine tuning problem to $d=3$.

$\Rightarrow$ Consider dim. reduction from the broken phase.

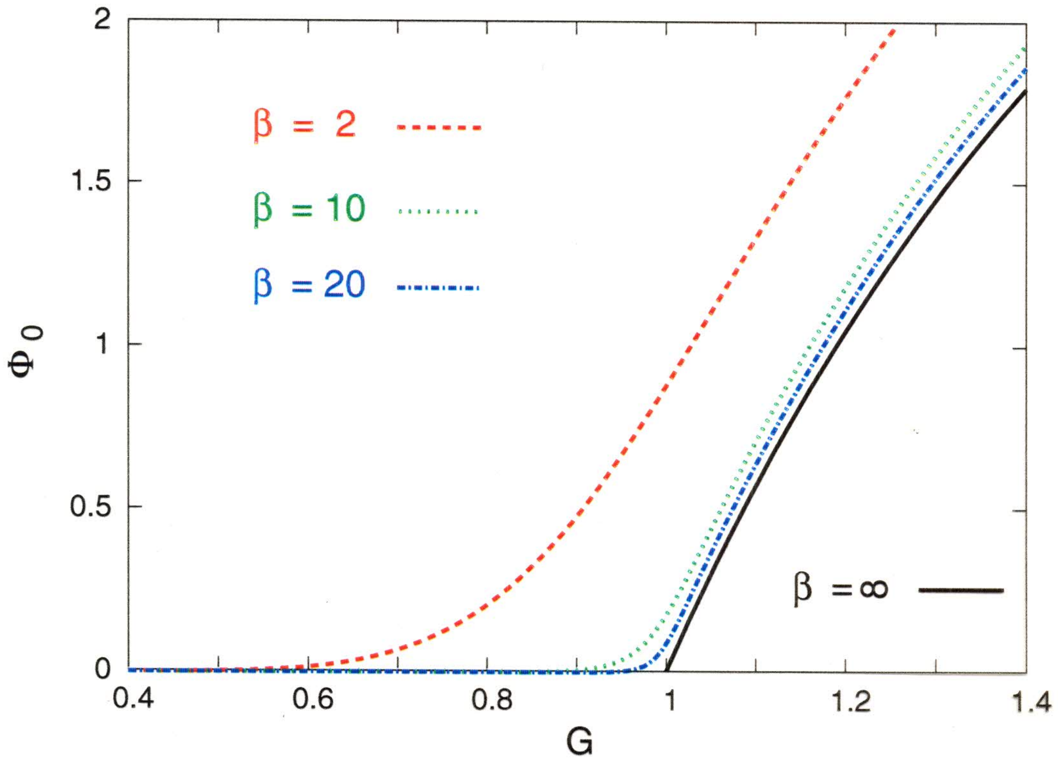


Figure 4: The mass gap in $d = 3$ as a function of G at various values of β ($G_c = 1$).

Fix $G < G_c$, β large $\rightarrow$ ξ very large

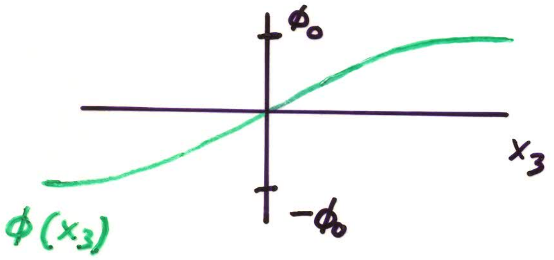
[Consider also broken phase, $G > G_c$:

here ξ does not become large, $\beta \to \infty$ doesn't help]

5. A Single Domain Wall

Broken phase in $d=3$: $\frac{\beta}{\xi} \xrightarrow{\beta \to 0} 2 \ln(1+\sqrt{2})$, no light fermion

$\Rightarrow$ generate light 2d fermion as $k_3 = 0$-mode on a domain wall in $d=3$:



Conf. $\phi(x_3)$, $\phi(\pm\infty) = \pm\phi_0$

Ansatz:

$$\phi(x_3) = \phi_0 \, \mathrm{tgh}(\phi_0 x_3)$$

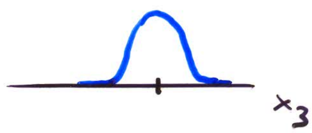
Hamiltonian (x_2: time)

$$H = \gamma_2 \left[\gamma_1 \partial_1 + \gamma_3 \partial_3 - \phi(x_3) \right] \qquad \gamma_i = \sigma_i$$

Separation ansatz: $\Psi = \psi(x_3)\, e^{ik_1 x_1}\, e^{-iEt}$ transl. inv.

H has a localized eigenstate ($k_3 = 0$)

$$\Psi_0(x_3) = \sqrt{\frac{\phi_0}{2}} \begin{pmatrix} 0 \\ \cosh^{-1}(\phi_0 x_3) \end{pmatrix}$$



with $E_0 = -k_1$: left mover

Anti-Wall: $\phi(x_3) = \underline{-}\,\phi_0 \, \mathrm{tgh}(\phi_0 x_3)$ $\rightarrow$ right mover with $E_0 = k_1$

In addition: bulk states (not localized in x_3)

$$\Psi_{k_3}(x_3) = \frac{e^{ik_3 x_3}}{\sqrt{2E(E+k_1)}} \begin{pmatrix} i(E+k_1) \\ \phi_0 \, \mathrm{tgh}(\phi_0 x_3) - ik_3 \end{pmatrix}$$

$$E = \pm\sqrt{\vec{k}^2 + \phi_0^2}$$

$\{\Psi_0, \Psi_k\}$: orthonormal basis for 1-particle Hilbert space.

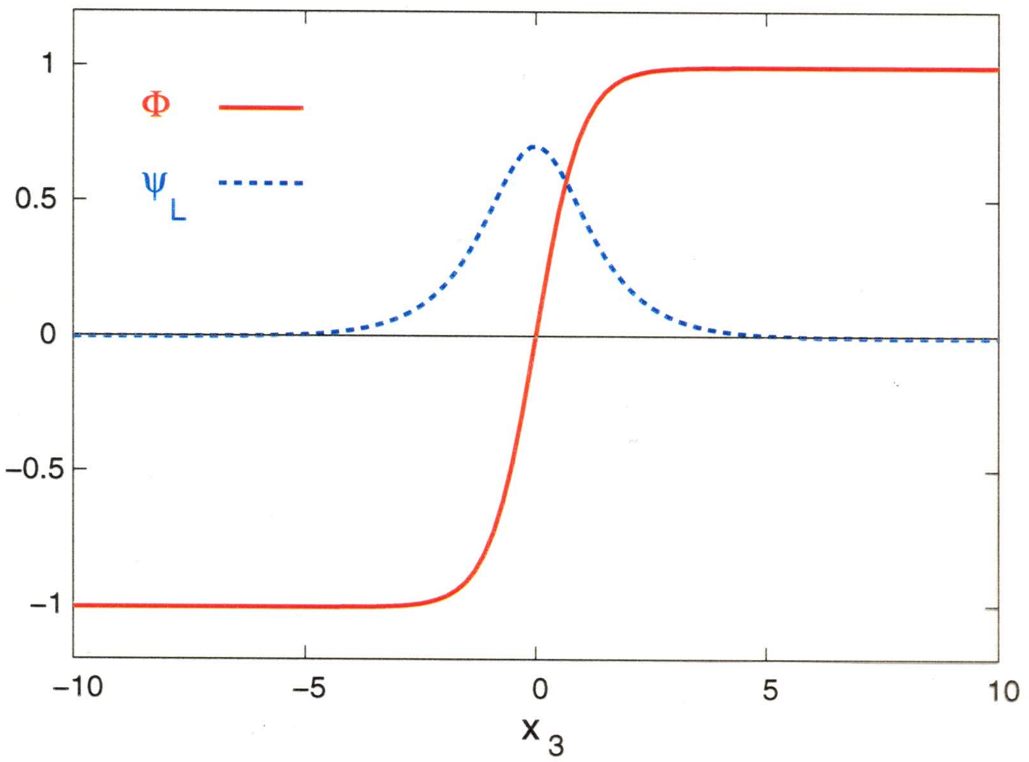


Figure 5: The profile of a single domain wall and the left-handed component of the state Ψ_0 localized on this wall. ($\Phi_0 = 1$)

Verification of ansatz for DW profile:
evaluate $\bar{\Psi}\Psi$

$$\bar{\Psi}_0(x_3)\,\Psi_0(x_3) = 0$$

$$\bar{\Psi}_{k_3}(x_3)\,\Psi_{k_3}(x_3) = \dots = -\frac{1}{E}\,\phi(x_3)$$

Summing up $E<0$ - states:

$$\frac{G}{N}\,\bar{\Psi}(x_3)\,\Psi(x_3) = \phi(x_3)\,\underbrace{\frac{G}{2\pi}\int_0^{\Lambda_2}\frac{k\,dk}{\sqrt{k^2+\phi_0^2}}}_{1 \text{ from cutoff matching}} \overset{!}{=} \phi(x_3)$$

$\Rightarrow$ Profile is indeed reproduced by condensate on this background.

$\bar{\Psi}_0\Psi_0 = 0 \Rightarrow$ localized states may also be filled, no effect on $\bar{\Psi}\Psi$.

$E_0 \ll \phi_0$: sharp localization, fermion only feels $d = 1+1$

$E_0 \gtrsim \phi_0$: fermion can escape in 3-direction (bulk: $E \gtrsim \phi_0$)

For low energy observer on the brane:
<u>fermion number violation</u>

We have constructed a topol. stable brane, on which a massless left-handed fermion propagates.

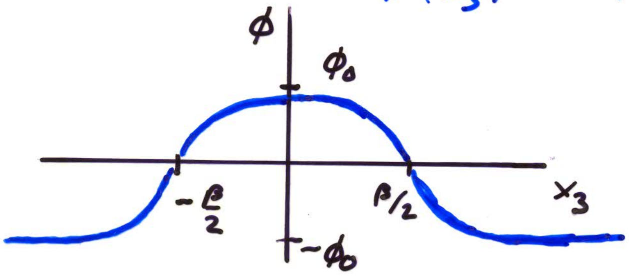
Dim. reduction more tricky for Wall - Antiwall pair:

6. A Brane World with Wall-Antiwall (W-$\bar{W}$)

We try to include Ψ_L, Ψ_R, localized on $W, \bar{W}$, separated by β.

Ansatz for profile:

$$\phi(x_3) = \phi_0 \left\{ a \left[tgh_+ - tgh_- \right] - 1 \right\}$$



$$tgh_\pm := tgh\left(a\phi_0 \left[x_3 \pm \frac{\beta}{2}\right]\right)$$

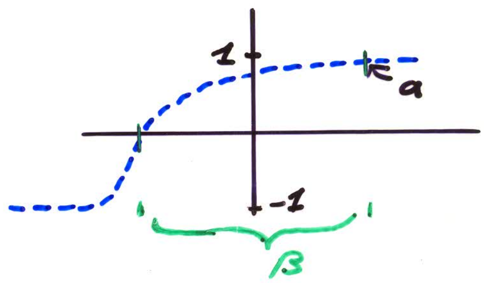
$a \in [0,1]$ controls brane separation:

Ansatz for bound state $\Psi_0(x_3) = \begin{pmatrix} A \cosh_-^{-1} \\ B \cosh_+^{-1} \end{pmatrix}$

$$\cosh_\pm := \cosh\left(a\phi_0 \left[x_3 \pm \frac{\beta}{2}\right]\right)$$

leads to condition:

$$\underline{tgh(a\phi_0\beta) = a}$$



one branch:

$a \to 0$: $\phi(x_3) \equiv -\phi_0$ $(\beta \to 0)$

$a \to 1$: $\phi(x_3) \equiv \phi_0$ $(\beta \to \infty)$

$0 < a < 1$ interpolates between the two 3d vacua.

Remarkably, Dirac eq. in this background still has an analytic solution:

$$k_3 = 0 : \quad \Psi_0(x_3) = \frac{1}{2}\sqrt{\frac{a\phi_0}{E_0(E_0+k_1)}} \begin{pmatrix} -i(E_0+k_1)\cosh_-^{-1} \\ m \cosh_+^{-1} \end{pmatrix}$$

$$m = \sqrt{1-a^2}\,\phi_0 \,, \quad E_0 = \pm\sqrt{k_1^2 + m^2}$$

Ψ_0 : Dirac fermion with components ψ_L, ψ_R, localized on Wall, $\overline{\text{Wall}}$.

Fast motion to $\begin{matrix}\text{left}\\ \text{right}\end{matrix}$ $\Rightarrow$ $0 < E_0 \simeq \begin{cases} -k_1 & \text{lower comp. dominates} \\ k_1 & \text{upper " "} \end{cases}$

Mass $m = \sqrt{1-a^2}\,\phi_0$: extent of L, R mixing

$a \to 0$, $\beta \to 0$, $m = \phi_0$ = 3d fermion mass, dim. red. but no light fermion in d=2

$a \to 1$, $\beta \to \infty$, $m \cong 2\phi_0\, e^{-\beta\phi_0}$

dim. reduction with $\xi \gg \beta$, as for sym. phase ..

- Low energy observer in $d = 1+1$:

  scale $= m$; L-, R-modes compose a point-like Dirac fermion.

- High energy observer in $d = 2+1$:

  scale ϕ_0 (3d fermion mass), $\frac{1}{\phi_0} \ll \beta$

  Dirac fermion with separated L; R- constituents.

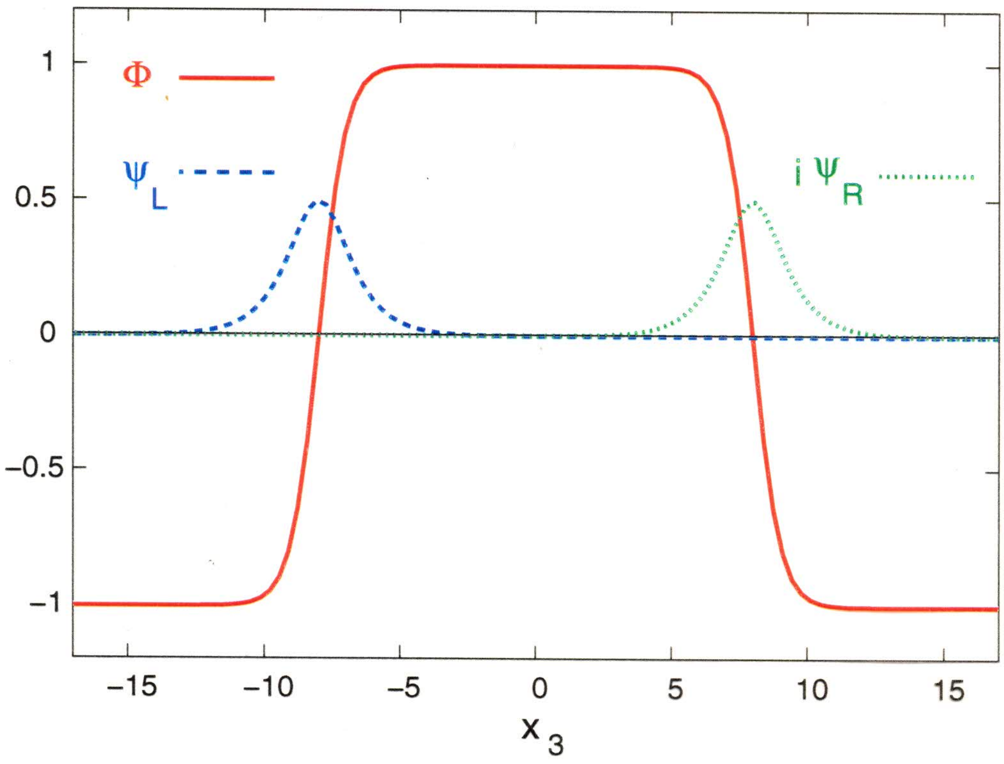


Figure 6: The profile of a wall anti-wall pair and the two components of the bound state at $k_1 = 0$. ($\Phi_0 = 1$, [redacted] $a = \tanh(16)$).

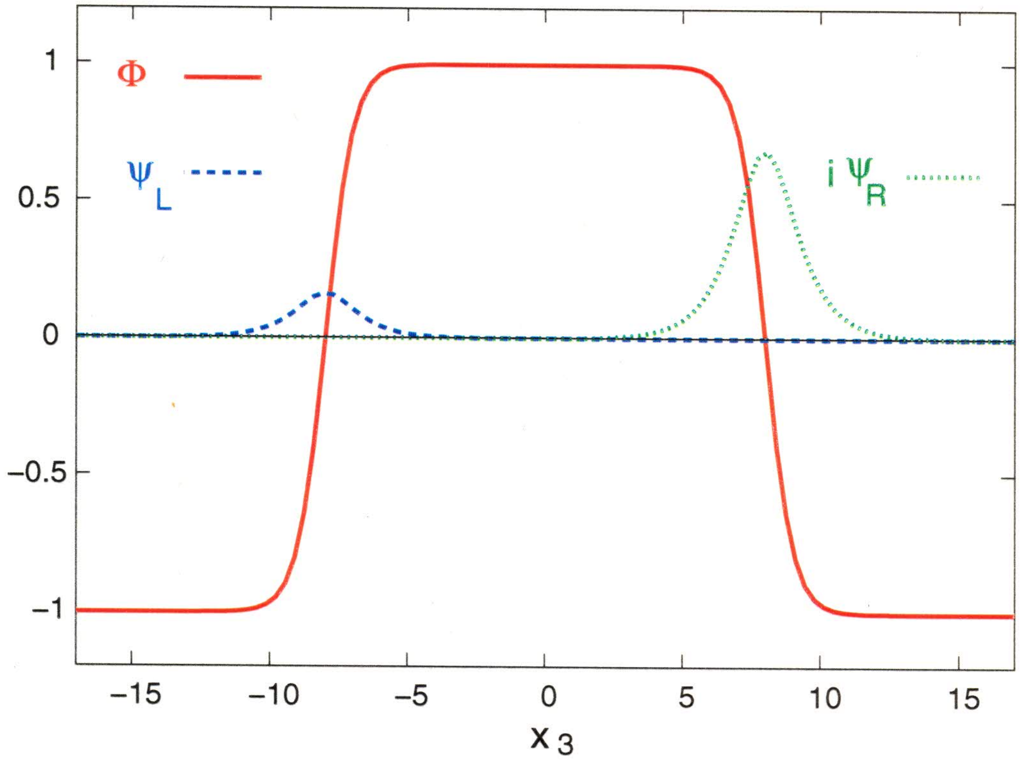


Figure 7: The profile of a wall anti-wall pair and the two components of the bound state at $k_1 = 2m$. ($\Phi_0 = 1$, [redacted] $a = \tanh(16)$).

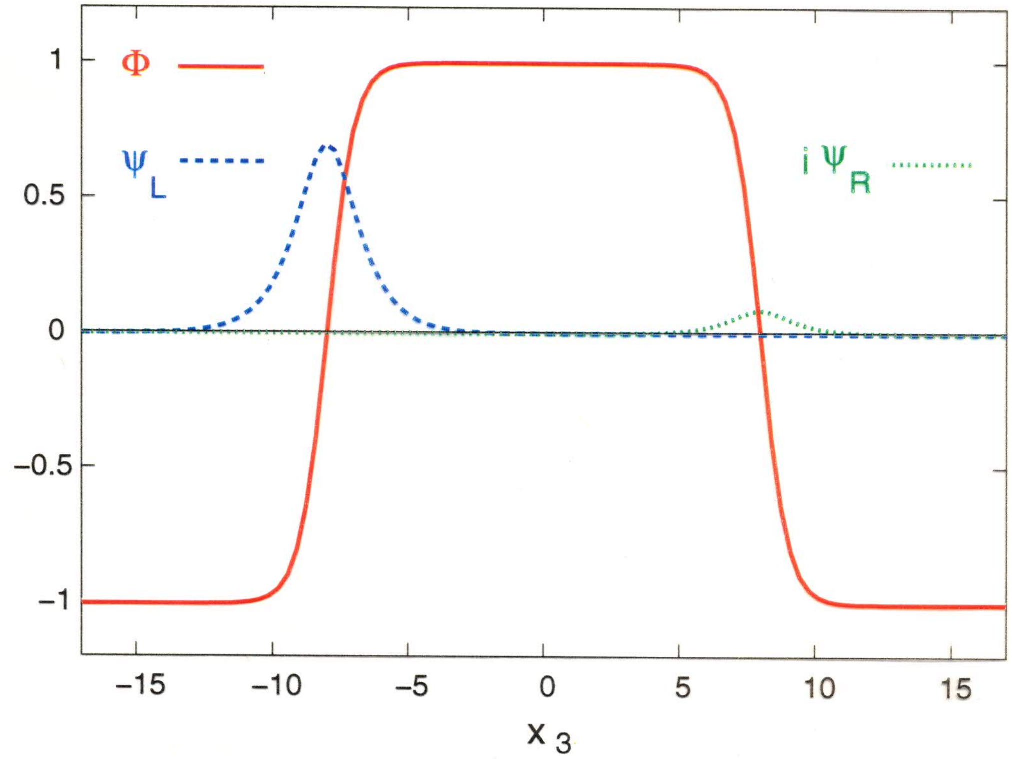


Figure 8: The profile of a wall anti-wall pair and the two components of the bound state at $k_1 = -4m$ ($\Phi_0 = 1$, $a =$ [redacted] $\tanh(16)$).

Bulk states:

$$\Psi_{k_3}(x_3) = \frac{e^{ik_3 x_3}}{\sqrt{2E\,(E+k_1)(k_3+a^2\phi_0^2)}} \begin{pmatrix} i(E+k_1)[a\phi_0 \tanh_- - ik_3] \\ -(\phi_0+ik_3)[a\phi \tanh_+ - ik_3] \end{pmatrix}$$

$$E = \pm\sqrt{k_1^2+k_3^2+\phi_0^2}$$

Check again if $\sum\limits_{\substack{\text{occupied}\\ \text{states}}} \bar{\Psi}\Psi$ reproduces the W-$\bar{W}$ profil.

$$\frac{G}{N}\int dk_3\, \bar{\Psi}_{k_3}\Psi_{k_3}\Big|_{E<0} = \phi(x_3) + \frac{1}{\sqrt{k_1^2+m^2}}\left[\underline{\pi} - 2\arcsin\left(\frac{a\phi_0}{\sqrt{k_1^2+\phi_0^2}}\right)\right]$$

just what we need

We try to cancel the rest by filling up Ψ_0-states:

i) fill all $E_0<0$ states, $E_0 = -\sqrt{k_1^2+m^2}$

cancels exactly $\underline{\pi\text{ - term}}$

ii) to cancel also $\underline{\text{arcsin - term}}$, fill up $E_0 \le \sqrt{k_F^2+m^2}$

cancelation at Fermi momentum $k_F = a\phi_0$,

$$E_F = \sqrt{k_F^2+m^2} = \sqrt{a^2\phi_0^2+(1-a^2)\phi_0^2} \overset{!}{=} \phi_0$$

$\overset{!}{=}$ minimal energy for escape in extra dim.

$\Rightarrow$ Any fermion still added will escape.

W-$\bar{W}$ brane does contain naturally light fermions, and it is completely packed by them; all physics is unfortunately Pauli-blocked (in contrast to single wall).

Origin of the 2d Fermion Mass

Remember 3d $Z(2)$ sym.:

$$\Psi_L(\vec{x},x_3) \to \pm \Psi_L(\vec{x},-x_3), \quad \bar{\Psi}_L(\vec{x},x_3) \to \pm \bar{\Psi}_L(\vec{x},-x_3)$$

$$\Psi_R(\vec{x},x_3) \to \mp \Psi_R(\vec{x},-x_3), \quad \bar{\Psi}_R(\vec{x},x_3) \to \mp \bar{\Psi}_R(\vec{x},-x_3)$$

1) Single Wall:

Profile $\phi(\vec{x},x_3) = \phi_0 \,\mathrm{tgh}(\phi_0 x_3) \overset{!}{=} -\phi_0(\vec{x},-x_3)$

$Z(2)$ sym intact

For 2d observer on brane: χ sym. intact, $m = 0$

2) Wall-$\overline{\text{Wall}}$

$\phi(\vec{x},x_3) \neq -\phi(\vec{x},-x_3)$, (height Φ_0)

no exact χ sym. for 2d observer;
breaking explicit or spontaneous ?

Consider 2d fermion fields:

$$\Psi_{L,R}(\vec{x}) = \int dx_3 \;\Psi_{L,R}(\vec{x},x_3)\sqrt{\frac{a\Phi_0}{2}}\cosh^{-1}(a\Phi_0 x_3 \pm c)$$

↑ smearing with localized modes

Build now 2d
χ condensate (= order parameter): only loc. modes contribute

$$\bar{\Psi}\Psi = \bar{\Psi}_0\Psi_0 = \frac{m}{E_0}$$

$m = \sqrt{1-a^2}\,\phi_0$

$E_0 = \pm\sqrt{k_1^2 + m^2}$

ϕ_0: cutoff for 2d physics

Occupy localized states with $E_0 \in [-\phi_0, 0]$
(vacuum, though not stable)

$$\Rightarrow \left|\frac{g}{N}\bar{\psi}\psi\right| = \underset{\text{large }\beta}{\overset{\text{cutoff matching}}{\cdots\cdots}} \overset{!}{=} m$$

$\Rightarrow$ χ condensate at large β $\overset{!}{=}$ dyn. generated mass in $d=2$

$\Rightarrow$ m results from SSB of emergent $Z(2)$ sym.

7. Stability of the Branes

Single Wall: topol. stable, but $W-\bar{W}$ not:

Conceivable disaster:

- $W, \bar{W}$ attraction, annihilation, $\phi(x_3) = -\phi_0$ tension energy -> heavy 3d fermion
- or: $W, \bar{W}$ repel -> simply drift apart

Check:

- Fermion density (filling up to $k_F = a\phi_0$): $\frac{F}{L} = \frac{N}{2\pi}\int_{-k_F}^{k_F} dk_1 = \frac{1}{\pi} N a \phi_0$
- Brane tension: $\frac{\text{Energy}}{L} = \sigma_1 + \sigma_2 + \sigma_3$

  - ϕ^2-term: $\sigma_1 = \frac{N}{2G}\int d^3x \left[\phi^2(x_3) - \phi_0^2\right] = -\frac{2Na\phi_0}{G}$
  - filled localized states:

$$\sigma_2 = \frac{N}{2\pi}\Big[\underbrace{-\int dk_1\sqrt{k_1^2+m^2}}_{(E_0<0)} + \underbrace{\int_{-k_F}^{k_F} dk_1\sqrt{k_1^2+m^2}}_{(E_0>0)}\Big]$$

  - σ_3 from bulk states, more complicated ...

$\Rightarrow$ $\sigma = \frac{1}{\pi} N a \phi_0^2$, $\frac{\text{Energy}}{\text{Fermion}} = \phi_0$ indep. of $W.\bar{W}$ separation (i.e. of a)

$W, \bar{W}$ do not attract, nor repel.

ϕ_0 = mass of bulk fermions =>

Fermions can be freely distributed between bulk and brane.

8. Dim. Reduction without $Z(2)$-Sym. ?

Non-pert. (lattice) formulation ($N<\infty$) in $d=3$:

$Z(2)$-sym. is explicitly broken.

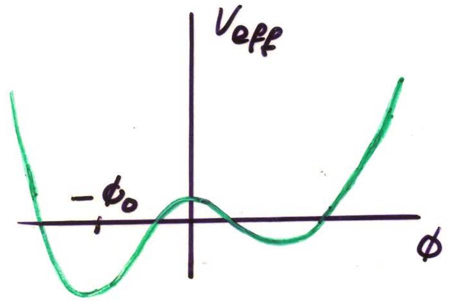
To mimic this situation we add a mass term :

$$S[\bar{\Psi},\Psi] = \int d^3x \left[\bar{\Psi}\gamma_\mu \partial_\mu \Psi + \bar{\Psi}\gamma_3 \partial_3 \Psi + M\bar{\Psi}\Psi - \frac{G}{2N}(\bar{\Psi}\Psi)^2 \right]$$

$$S[\bar{\Psi},\Psi,\phi] = \int d^3x \left[\bar{\Psi}\gamma_\mu \partial_\mu \Psi + \bar{\Psi}\gamma_3 \partial_3 \Psi - \bar{\Psi}\Psi\phi + \frac{N}{G}M\phi + \frac{N}{2G}\phi^2 \right]$$

$$\left(\phi = \frac{G}{N}\bar{\Psi}\Psi - M \right)$$

$$\int D\bar{\Psi}D\Psi \rightarrow V_{eff}(\phi) = \frac{1}{6\pi}|\phi|^3 - \frac{1}{2}\left(\frac{1}{G_c} - \frac{1}{G}\right)\phi^2 + \frac{M}{G}\phi$$



$M>0$:

Non-degenerate global min. at

$$-\phi_0 = -\pi\left[\frac{1}{G_c} - \frac{1}{G} + \sqrt{\left(\frac{1}{G_c} - \frac{1}{G}\right)^2 + \frac{2M}{\pi G}} \right]$$

How about W-$\bar{W}$ conf. $\phi(x_3) = \phi_0\left[a(tgh_+ - tgh_-) - 1\right]$?

$$\frac{G}{N}\bar{\Psi}\Psi = (\phi_0 - M)\left[a(tgh_+ - tgh_-) - 1\right]$$

$$+ \frac{Gm^2}{4\pi}(tgh_+ - tgh_-)\left\{ \int_{-k_F}^{k_F} \frac{dk_1}{\sqrt{k_1^2+m^2}} - \frac{2}{\pi}\int \frac{dk_1}{\sqrt{k_1^2+m^2}} \arcsin\left(\frac{a\phi_0}{\sqrt{k_1^2+\phi_0^2}}\right) \right\}$$

$$\stackrel{!}{=} \phi(x_3) + M \qquad \Rightarrow \quad \{ \text{"} \ \text{"} \} = \frac{4\pi M a}{G m^2}$$

Condition fixes k_F even larger than for $M=0$, beyond threshold for localization on W, $\bar{W}$.

$\Rightarrow$ At $M>0$ there is no stable W-$\bar{W}$ configuration.

$\left[\begin{array}{l} M<0 \text{ : global min. at } +\phi_0 \text{, but same filling condition} \\ \quad \Rightarrow \text{still no phys. degree of freedom.} \end{array}\right.$

Summary

We studied dim. reduction $3d \to 2d$ in GN model at large N.

$d=3$: $\nearrow$ sym. phase, $\phi_0 = 0$ = 3d fermion mass
$\searrow$ broken phase, $\phi_0 > 0$

$d=2$: only broken phase, mass m, asympt. freedom

Can a "natural" setting in $d=3$ reduce to $d=2$ $(\frac{1}{m} \equiv \xi \gg \beta)$
with $m \ll \Lambda_2$?

▷ Reduction from sym. phase : $\beta \to \infty$ implies $\xi \gg \beta$
$m \ll \Lambda_2 = \frac{2}{\beta}$
fine, but "unnatural" starting point.

▷ Start from 3d broken phase, build brane world:

- single domain wall, profil $\phi(x_3) = \phi_0 \, tgh(\phi_0 x_3)$
$\to$ localized state $\Psi_0 = \binom{0}{\psi_L}$ + bulk
Summing up $\bar{\Psi}\Psi$ reproduces $\phi(x_3) \to$ consistent. ☺

- $\omega - \bar{\omega}$: simple profile $\phi(x_3)$ still solves Dirac eq., stable
$\Psi_0(x_3)$ has components ψ_L, ψ_R, localized on $\omega, \bar{\omega}$,
separated by β.
Communication through $x_3 \to m$; $\beta \to \infty : m \to 0$, $\xi \gg \beta$
dim. reduction to a point-like Dirac fermion. ☺

But: $\bar{\Psi}\Psi$ reproduces $\phi(x_3)$ only if Ψ_0 is filled up to
ϕ_0 = threshold for escape in extra dim. ☹
$\to$ reduced world has only one state. [Self-organized system realises intended properties only in part.]
(Explicit $M \neq 0$ in $d=3$ $\to$ requires filling even beyond ϕ_0
$\to$ no stable conf's localized on $\omega, \bar{\omega}$.)

Outlook

Possible ways to vitalise such a brane world:

1. Consider finite N, e.g. one flavour:

→ $V_{eff}(\phi)$ depends on a fluctuating scalar field

$$Z = \int D\bar{\Psi} D\Psi D\phi \exp\Big[-\int d^dx \big(\underbrace{\bar{\Psi}(\partial\!\!\!/ - \phi)\Psi}_{\text{→ fermion determinant, depends on } \phi\text{-configuration}} + \frac{N}{2G}\phi^2\big)\Big]$$

$$= \int D\phi \exp\Big[-\int d^dx\, V_{eff}(\phi(x))\Big]$$

System with V_{eff} could be simulated
→ measure $\langle\bar{\Psi}\Psi\rangle = \frac{1}{G}\langle\phi\rangle$, verify agreement with brane profile; required filling below threshold for escape in extra dimension ?

2. In Gross-Neveu model at finite fermion density, the x_1-translation sym. can break spontaneously
⇒ crystal phase (Theis/Urlichs '03)

So far we assumed x_1-invariance for all states localised on the branes;
crystal states could come to the rescue to free the fermions from Pauli blocking.