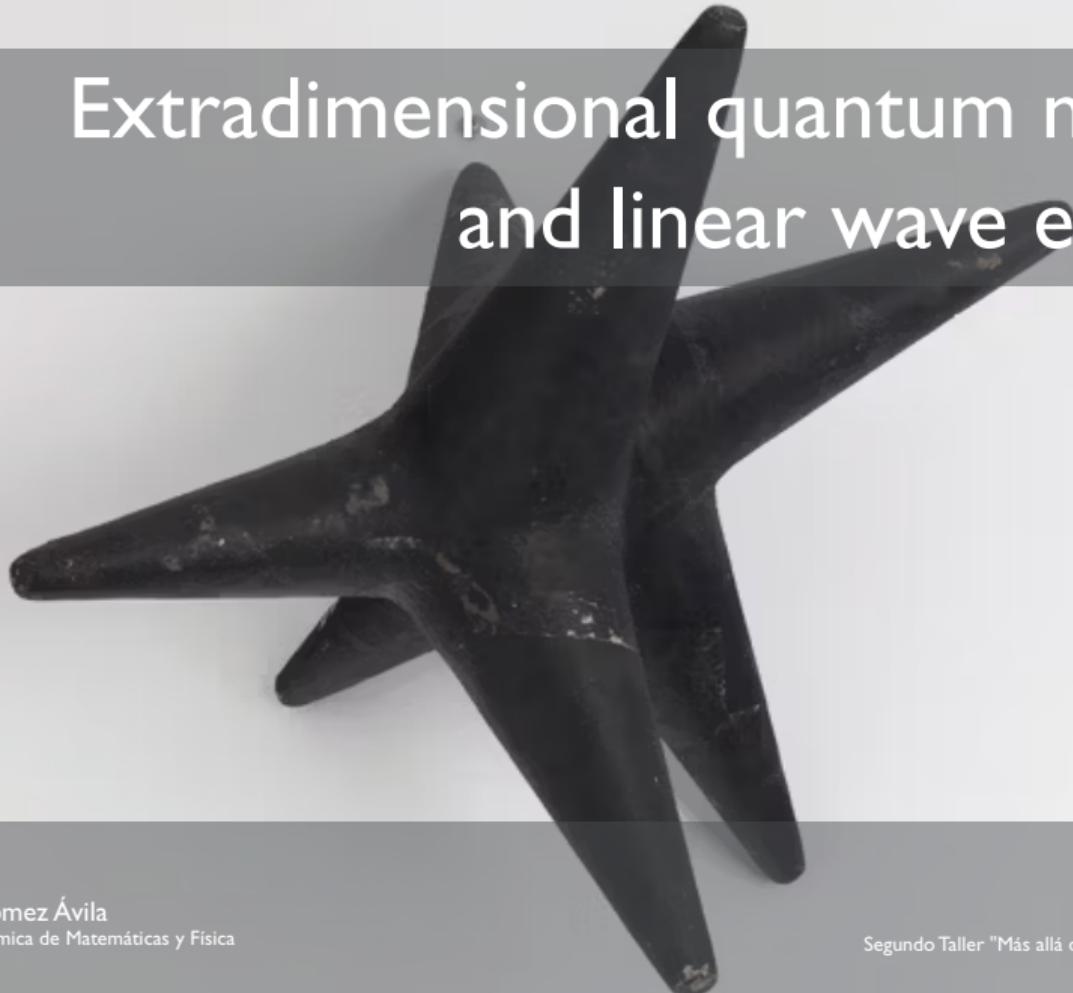


# Extradimensional quantum numbers and linear wave equations



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Segundo Taller "Más allá del Modelo Estándar y Astropartículas"

## The set of all QFTs is vast

Without a complete classification of all relativistic field theories, it is impossible to perform a systematic search in them of solutions to open problems. For this reason I believe that a crucial responsibility of theorists is to push the boundaries of quantum field theory to eventually achieve such a classification.

## Outline

- ▶ Fundamental ingredients of QFT
- ▶ Field theories and covariant bases
- ▶ Extradimensional excursions

# Fundamental ingredients of QFT

## Beyond the Standard Model can mean two things

1. The ingredients of the **Standard Model**, arranged in novel ways, or
2. entirely new ingredients.

What are the possibilities for new ingredients? That's constrained by known physics, which gives several no-go theorems and obstructions:

- ▶ Weinberg–Witten no-go theorem .
- ▶ Haag–Lopuszanski–Sohnius theorem.
- ▶ Ostrogradski instability.
- ▶ Vafa–Witten theorem on global symmetries.

## A (perturbative) QFT needs three ingredients

They are: a spacetime (fixing asymptotically free states), a gauge group (fixing the gauge bosons) and a matter content.

$$\mathcal{Z} = \underbrace{\int \prod D\Psi_i}_{\text{matter}} \exp \underbrace{\int_{\mathcal{M}} d^D x}_{\text{spacetime}} \mathcal{L} \left( \underbrace{D^\mu}_{\text{gauge}} \Psi_j, \mathcal{V}(\Psi_j) \right)$$

Historically: we build a noninteracting theory of matter fields and then we add interactions.

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Anomaly cancellation

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Gauge charges

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Kinematical algebra

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GUTS, LR, etc.

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NHDM, High Spin

$$\mathcal{Z} = \underbrace{\int \prod D\Psi_i}_{\text{matter}} \exp \underbrace{\int_{\mathcal{M}} d^D x}_{\text{spacetime}} \mathcal{L} \left( \underbrace{D^\mu}_{\text{gauge}} \Psi_j, \mathcal{V}(\Psi_j) \right)$$

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String theory

Historically: we build a noninteracting theory of matter fields and then we add interactions.

## The matter sector is the least systematically understood

$$\mathcal{L} = \underbrace{\mathcal{L}_{\text{gauge}}(\Psi, A^\mu)}_{\text{3 coupling constants + 1 phase}} + \underbrace{\mathcal{L}_{\text{Higgs}}(\Phi, A^\mu)}_{\text{1 mass & 1 vev}} + \underbrace{\mathcal{L}_{\text{flavor}}(\Psi, \Phi)}_{\text{(9) 12 masses, (3) 6 angles & (1) 2 phases}} .$$

- ▶ The gauge Lagrangian is very economical.
- ▶ Most SM parameters come from the flavor Lagrangian .
- ▶ Formal simplicity is achieved by specifying the gauge group and the matter multiplets. There is nothing similar for the matter Lagrangian.

# One-particle states are Poincaré irreps

The Poincaré algebra has two algebraic invariants

$$\mathcal{C}_2 = P^\mu P_\mu, \quad \mathcal{C}_4 = W^\mu W_\mu \quad \text{with} \quad W_\mu = \frac{1}{2} \varepsilon_{\mu\sigma\tau\rho} M^{\sigma\tau} P^\rho.$$

One-particle states satisfy

$$\mathcal{C}_2 |\Psi\rangle = m^2 |\Psi\rangle, \quad \mathcal{C}_4 |\Psi\rangle = -m^2 j(j+1) |\Psi\rangle$$

We call  $m$  the mass and  $j$  the spin of  $\Psi$ .

The Lorentz algebra is the homogeneous part of the Poincaré algebra. This algebra  $\mathfrak{so}(1, 3) \cong \mathfrak{su}(2)_A \oplus \mathfrak{su}(2)_B$  is generated by the commuting sets

$$\mathbb{A} = \frac{1}{2}(\mathbb{J} - i\mathbb{K}), \quad \mathbb{B} = \frac{1}{2}(\mathbb{J} + i\mathbb{K}).$$

We label the Lorentz irreps with the  $\mathfrak{su}(2)$  numbers  $(a, b)$ .

## The SM includes a few possibilities

- ▶ Klein-Gordon's equation is the eigenvalue condition for  $P^2$  in the scalar representation:

$$(0, 0).$$

- ▶ Dirac's equation is the parity projection for the representation

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right).$$

- ▶ Proca's equation for a massive vector field is the spin (i.e., Pauli-Lubanski) projection for  $j = 1$  in the representation

$$\left(\frac{1}{2}, \frac{1}{2}\right).$$

## Even if we are conservative, others are theoretically possible

$(0, 0)$

$(\frac{1}{2}, 0)$      $(0, \frac{1}{2})$

$(1, 0)$      $(\frac{1}{2}, \frac{1}{2})$      $(0, 1)$

$(\frac{3}{2}, 0)$      $(1, \frac{1}{2})$      $(\frac{1}{2}, 1)$      $(0, \frac{3}{2})$

$(2, 0)$      $(\frac{3}{2}, \frac{1}{2})$      $(1, 1)$      $(\frac{1}{2}, \frac{3}{2})$      $(0, 2)$

Even if we are conservative, others are theoretically possible

Higgs

$(0, 0)$

$(\frac{1}{2}, 0)$      $(0, \frac{1}{2})$

$(1, 0)$      $(\frac{1}{2}, \frac{1}{2})$      $(0, 1)$

$(\frac{3}{2}, 0)$      $(1, \frac{1}{2})$      $(\frac{1}{2}, 1)$      $(0, \frac{3}{2})$

$(2, 0)$      $(\frac{3}{2}, \frac{1}{2})$      $(1, 1)$      $(\frac{1}{2}, \frac{3}{2})$      $(0, 2)$

## Even if we are conservative, others are theoretically possible

		(0, 0)			
Quarks & Leptons	$\left(\frac{1}{2}, 0\right)$	$\left(0, \frac{1}{2}\right)$			
	(1, 0)	$\left(\frac{1}{2}, \frac{1}{2}\right)$	(0, 1)		
	$\left(\frac{3}{2}, 0\right)$	$\left(1, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 1\right)$	$\left(0, \frac{3}{2}\right)$	
	(2, 0)	$\left(\frac{3}{2}, \frac{1}{2}\right)$	(1, 1)	$\left(\frac{1}{2}, \frac{3}{2}\right)$	(0, 2)

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What about these?

Are they allowed?

If not, what forbids them?

# Field theories and covariant bases

## General constructions run into consistency obstructions

1. A general  $(a, b)$  field contains spins  $|a - b| \dots (a + b)$ ; theory must include constraints to remove unwanted spins. These may propagate as ghosts or produce other acausalities when interactions are introduced<sup>1</sup>.
2. In some cases, the only kinematical terms available produce a higher-derivative theory, which may run into the Ostrogradski instability<sup>2</sup>.
3. Massless  $j > 2$  fields imply a trivial S-matrix, rendering interactions inconsistent<sup>3</sup>.

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<sup>1</sup>Velo & Zwanziger, [Noncausality and Other Defects of Interaction Lagrangians for Particles with Spin One and Higher](#), 1969

<sup>2</sup>Woodard, [Avoiding Dark Energy with 1/R Modifications of Gravity](#), 2006

<sup>3</sup>Weinberg & Witten, [Limits on Massless Particles](#), 10.1016/0370-2693(80)90212-9, 1980

## We conventionally use parity-invariant representations

Parity exchanges the  $(a, b)$  Lorentz labels; parity-invariant representations are:

- ▶ Non-chiral  $(a, a)$  irreps, which correspond to integer spin Fierz-Pauli.
- ▶ Chiral, reducible  $(a, b) \oplus (b, a)$  with  $a \neq b$ .

Those with only two spin sectors (including the three in the SM, plus others) can be enumerated:

1. The **non-chiral**  $(0, 0)$  and  $\left(\frac{1}{2}, \frac{1}{2}\right)$ .
2. The **single spin chiral representations**  $(SS\chi R)$   $(j, 0) \oplus (0, j)$  with  $j \geq \frac{1}{2}$  (Joos-Weinberg representations).
3. The **double spin chiral representations**  $(DS\chi R)$   $\left(j - \frac{1}{2}, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, j - \frac{1}{2}\right)$  for  $j > 1$ .

## Duffin-Kemmer-Petiau and Rarita-Schwinger are exceptions

Other formalisms have been used to study the properties of hadrons, which don't fit in this scheme. One is the Rarita-Schwinger field, transforming as

$$\left(1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 1\right) \oplus \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right).$$

Historically this came about from the product of a spinor and a vector. The other one is the Duffin-Kemmer-Petiau theory, using the 10-dimensional vector-tensor representation

$$(1, 0) \oplus (0, 1) \oplus \left(\frac{1}{2}, \frac{1}{2}\right).$$

This is a **puzzling construction**: the propagating dofs comprise a mixture from different irreps of the Lorentz algebra including both chiral and non-chiral representations. Let us try to understand it!

## The defining property of DKP is the meson algebra

The Duffin–Kemmer–Petiau relativistic wave equation was used in the XX century to describe the properties of spin zero and spin one mesons. At the heart of the DKP theory is the meson algebra

$$S_\rho S_\mu S_\nu + S_\nu S_\mu S_\rho = g_{\mu\nu} S_\rho + g_{\mu\rho} S_\nu, \quad (1)$$

which has nontrivial 5 and 10 dimensional representations. This algebra permits the construction of a Dirac-like first order wave equation

$$(S_\mu \partial^\mu - m)\Psi = 0. \quad (2)$$

It is not very popular anymore.

## Lie representations are Jordan-Lie algebras

Given a Lie algebra  $\mathfrak{g}$  over  $\mathbb{F}$ , the universal enveloping algebra  $\mathfrak{U}(\mathfrak{g})$  is the infinite sum

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} T^n(\mathfrak{g}),$$

where  $T^0 = \mathbb{F}$ , and  $T^n$  is the  $n$ -fold product  $\mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ , modded by the Lie product  $x \diamond y = [x, y]$ . This means that every representation of  $\mathfrak{g}$  can be uniquely extended to a representation of  $\mathfrak{U}(\mathfrak{g})$ !

This unfaithful representation carries a dual algebraic structure, called a Jordan-Lie algebra, where the products are the anticommutator and commutator. By constructing this algebra, we can find all possible covariant operators acting on the representation space<sup>4</sup>.

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<sup>4</sup>S. Gómez-Ávila, M. Napsuciale, Covariant basis induced by parity for the  $(j, 0) \oplus (0, j)$  representation, 10.1103/PhysRevD.88.096012, 2013

## From Jordan-Lie we get covariant kinetic terms

Given a Lorentz irrep  $V$ , the associated Jordan-Lie algebra provides a basis for  $End(V)$ . We build a wave equation for each **kinematical term**, that is, every covariant operator that can be contracted with momenta.

A very well-known example is the Dirac representation:

$$\left[ \left( \frac{1}{2}, 0 \right) \oplus \left( 0, \frac{1}{2} \right) \right]^2 = (0, 0)_2 \oplus (1, 0) \oplus (0, 1) \oplus \left( \frac{1}{2}, \frac{1}{2} \right)_2.$$

This corresponds to the following basis for  $End(V)$ :

$$\{ \mathbb{1}, \gamma_5, \gamma_\mu, \gamma_5 \gamma_\mu, \sigma_{\mu\nu} \}.$$

The kinematical terms available are:  $\gamma_\mu P^\mu$ ,  $\gamma_5 \gamma_\mu P^\mu$  and  $P_\mu P^\mu$ . The other terms are forbidden by symmetry<sup>5</sup>.

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<sup>5</sup>Ferro, Olmos, Peinado & Vaquera, [Quantization of second-order fermions](#), doi:10.1103/PhysRevD.109.085003.

## From kinetic terms we get field theories

1. We start with some representation of the Lorentz algebra  $\mathfrak{g}$ .
2. We define new elements through the Jordan product  $\mathfrak{g} \bullet \mathfrak{g}$ .
3. We evaluate the Jordan and Lie products of these new elements with  $\mathfrak{g}$  and itself.
4. We continue this process until the algebra closes and we have a complete basis for  $\text{End}(V)$ .
5. We classify all kinematical terms (symmetric operators), and construct the corresponding wave equations.
6. The remaining operators classify all possible self-interactions, and bilinear couplings to other fields.

## Kinetic terms for the $SS\chi R$

In the single-spin chiral representations there are two possible kinetic operators. One comes from

$$(j, 0) \otimes (j, 0) \oplus (0, j) \otimes (0, j) \simeq (0, 0)_2 \oplus \bigoplus_{i=1}^{2j} (i, 0) \oplus (0, i). \quad (3)$$

This scalar kinetic term will produce a Klein-Gordon equation of motion for every spinor component, typically double the number of degrees of freedom desired. We also have anti block diagonal operators:

$$2((j, 0) \otimes (0, j)) \simeq (j, j)_2. \quad (4)$$

This correspond to another possible kinetic term, formed from a symmetric traceless tensor with  $2j$  indices. This is of order  $2j$  in the momenta; for  $j > 1$ , it corresponds to a higher-derivative theory.

## ...and for the $DS\chi R$

In the double spin chiral representation block diagonal operators in the direct-sum basis are

$$\begin{aligned} \left(j - \frac{1}{2}, \frac{1}{2}\right)^{\otimes 2} &= \bigoplus_{r=0}^{2j-1} [(r, 0) \oplus (r, 1)], \\ \left(\frac{1}{2}, j - \frac{1}{2}\right)^{\otimes 2} &= \bigoplus_{r=0}^{2j-1} [(0, r) \oplus (1, r)]. \end{aligned} \tag{5}$$

Since  $j \geq \frac{3}{2}$ , this will always include a pair of symmetric tensors  $(1, 1)$ .

Anti-block diagonal operators come from the cross product

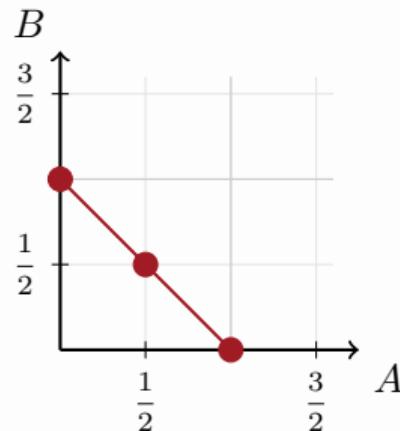
$$2 \left(j - \frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, j - \frac{1}{2}\right) = 2 \bigoplus_{r,s=j-1}^j (r, s). \tag{6}$$

The lowest order kinetic term comes from the anti diagonal product  $(j - 1, j - 1)$ , which means that we have a first-order wave equation for  $j = 3/2$ , and a second-order wave equation for  $j = 2$ .

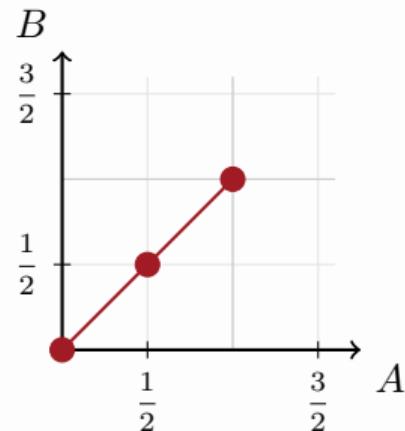
# Extradimensional excursions

## Lorentz in 4+1 dimensions

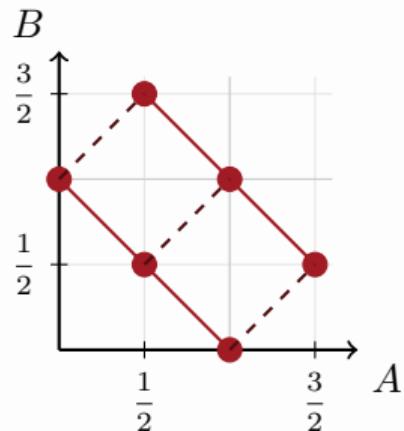
Representations of the 4+1 Lorentz algebra are classified by two numbers  $m/n$ , defining the shape of a figure in the  $(a, b)$  space.



3/1 representation



1/3 representation



3/2 representation

4+1 irreps are in correspondence with sequences of 3+1 Lorentz irreps.

## Poincaré in 4+1 dimensions

The 4+1 Poincaré algebra has rank 3 and therefore its irreducible representations are indexed by three Casimir operators. Two are well-known:

$$\mathcal{C}_2 = P^\mu P_\mu, \quad \mathcal{C}_4 = W^{\rho\mu\nu} W_{\rho\mu\nu}, \quad \text{with} \quad W_{\rho\mu\nu} = \frac{1}{2} \sum_{(\rho\mu\nu)} M_{\mu\nu} P_\rho.$$

The new Casimir is lineal in the momenta:

$$\mathcal{C}_3 = \frac{1}{2^3} \epsilon^{\alpha\beta\mu\nu\rho} M_{\alpha\beta} M_{\mu\nu} P_\rho. \quad (7)$$

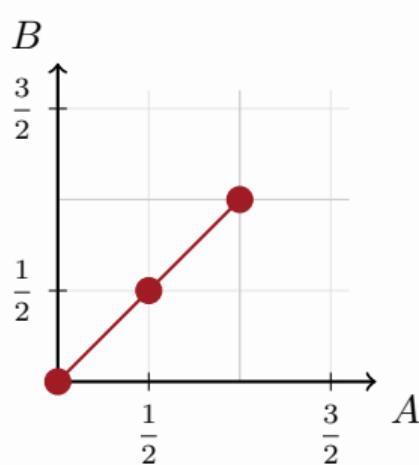
The problem of building free QFTs for the extradimensional theory amounts to constructing the induced representations with good  $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$  quantum numbers.

## The linear Casimir projects linear equations

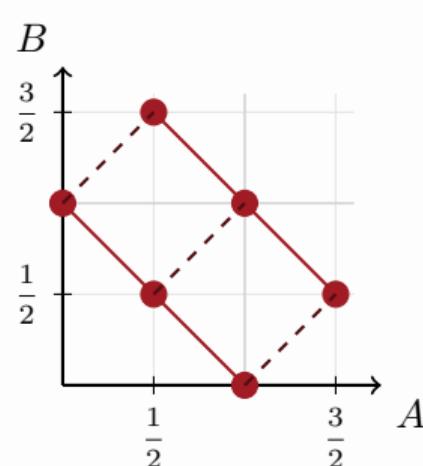
1+4 Irrep	Lorentz Projection	$C_3$
1/1	Scalar Field	0
1/2	Spin Zero DKP	0
2/1	Dirac Field	$\gamma_\mu P^\mu$
2/2	Rarita-Schwinger	$\gamma'_\mu P^\mu$
1/3	Spin 2 Symmetric + Vector + Scalar	0
3/1	Spin 1 DKP	$\beta_\mu P^\mu$

All of the three known linear equations can be understood as the  $3 + 1$  projection of the  $4 + 1$   $C_3$  eigenvalue equation.

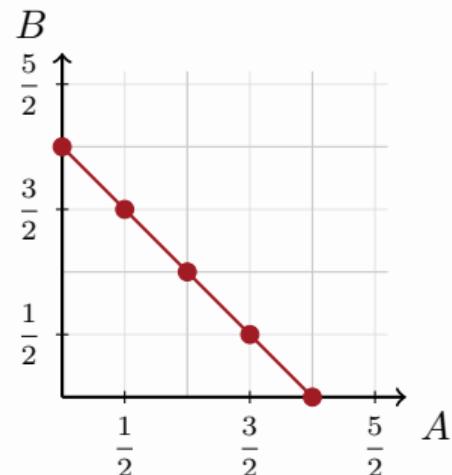
## In the bosonic case, two spin 2 linear equations exist



1/3 TeVeS representation  
with  $\mathcal{C}_3 = 0$ .



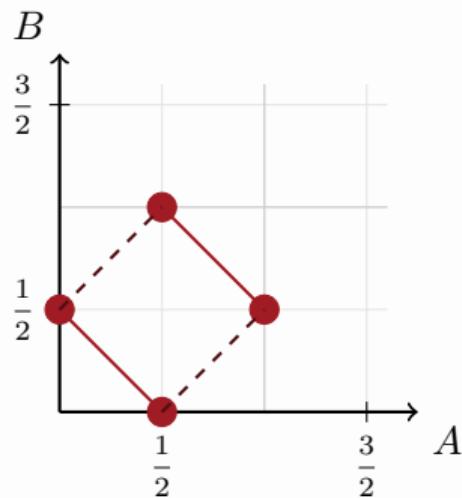
3/2 Connection + Tensor +  
DKP.



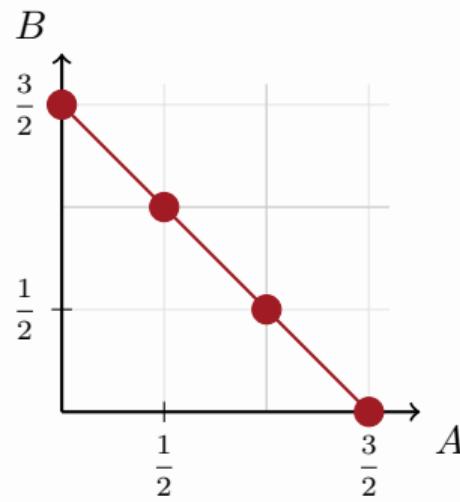
5/1 Weyl + Connection +  
Tensor.

Both 3/2 and 5/1 carry three spin sectors  $\Rightarrow$  constraints might be bad.

For spin  $\frac{3}{2}$ , there is a novel possibility



2/2 Rarita-Schwinger representation



4/1 Novel spin  $\frac{3}{2}$  representation

The  $4/1$  corresponds to the

$$\left(\frac{3}{2}, 0\right) \oplus \left(\frac{3}{2}, 0\right) \oplus \left(1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 1\right)$$

a 20-dimensional representation. The covariant vector operator is

$$S_\mu \equiv \frac{1}{2^3} \eta_{\mu\nu} \epsilon^{\nu\alpha\beta\rho\sigma} \{M_{\alpha\beta}, M_{\rho\sigma}\},$$

which we can use to write a linear equation for this field.

# Conclusions

- ▶ We linked quantum numbers in  $4 + 1$  Minkowski to field theories in  $3 + 1$ .
- ▶ The three known relativistic linear theories can be projected from the eigenvalue equation for  $\mathcal{L}_3$ .
- ▶ In this sense, Dirac's, Rarita-Schwinger and Kemmer-Duffin-Petiau are related.
- ▶ An unexplored (as far as I know) alternative to Rarita-Schwinger was identified.
- ▶ There is also a pair of spin 2 linear equations mixing Weyl's, Levi-Civita and Proca representations.

Thank you for your attention!