Exploring New Geometries in String Theory: Non-Abelian Orbifolds

In collaboration with S. Ramos-Sánchez [2506.08370] and R. Pérez-Martínez [2601.XXXX]

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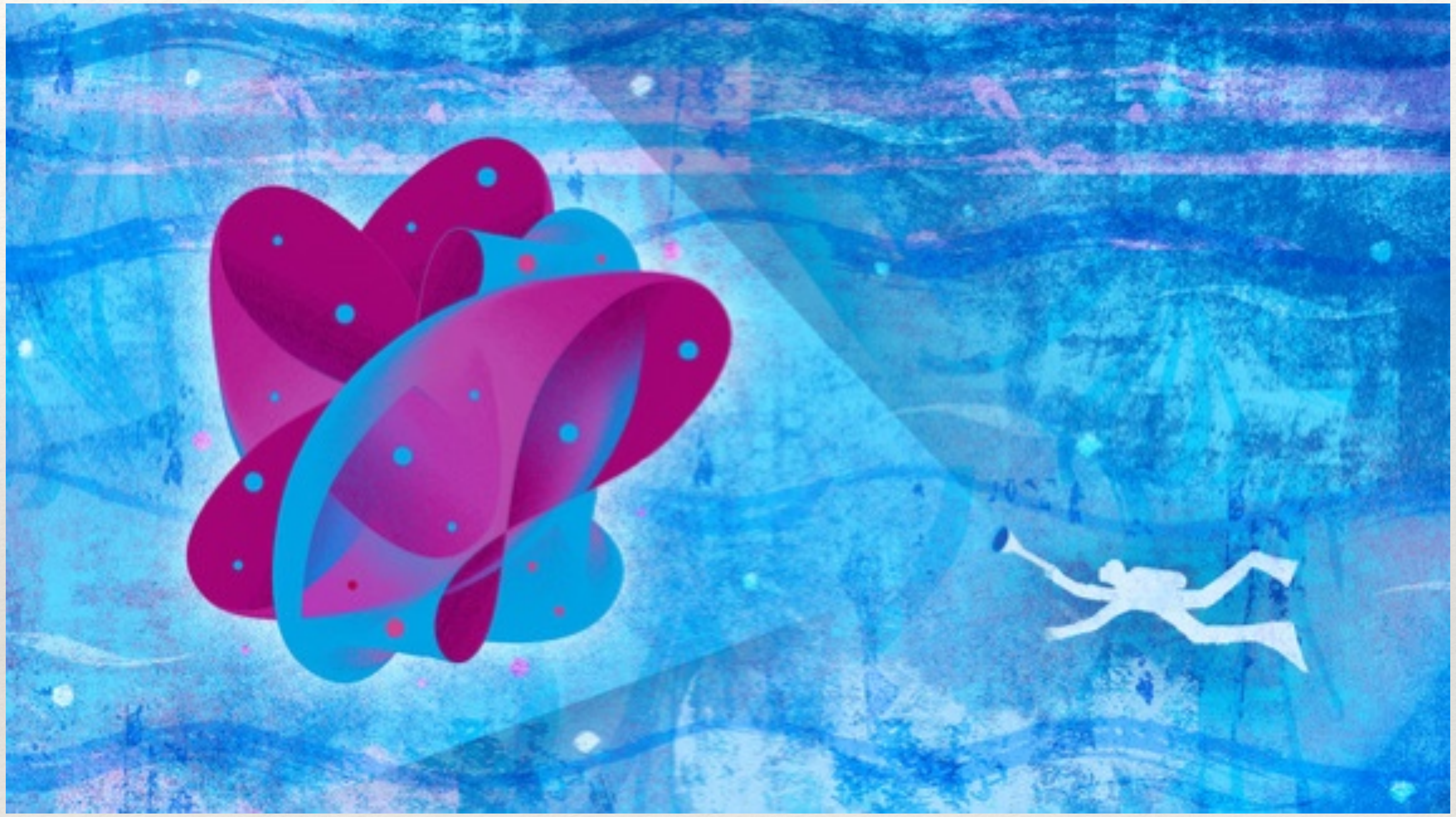
XIX Mexican Workshop on Particles and Fields

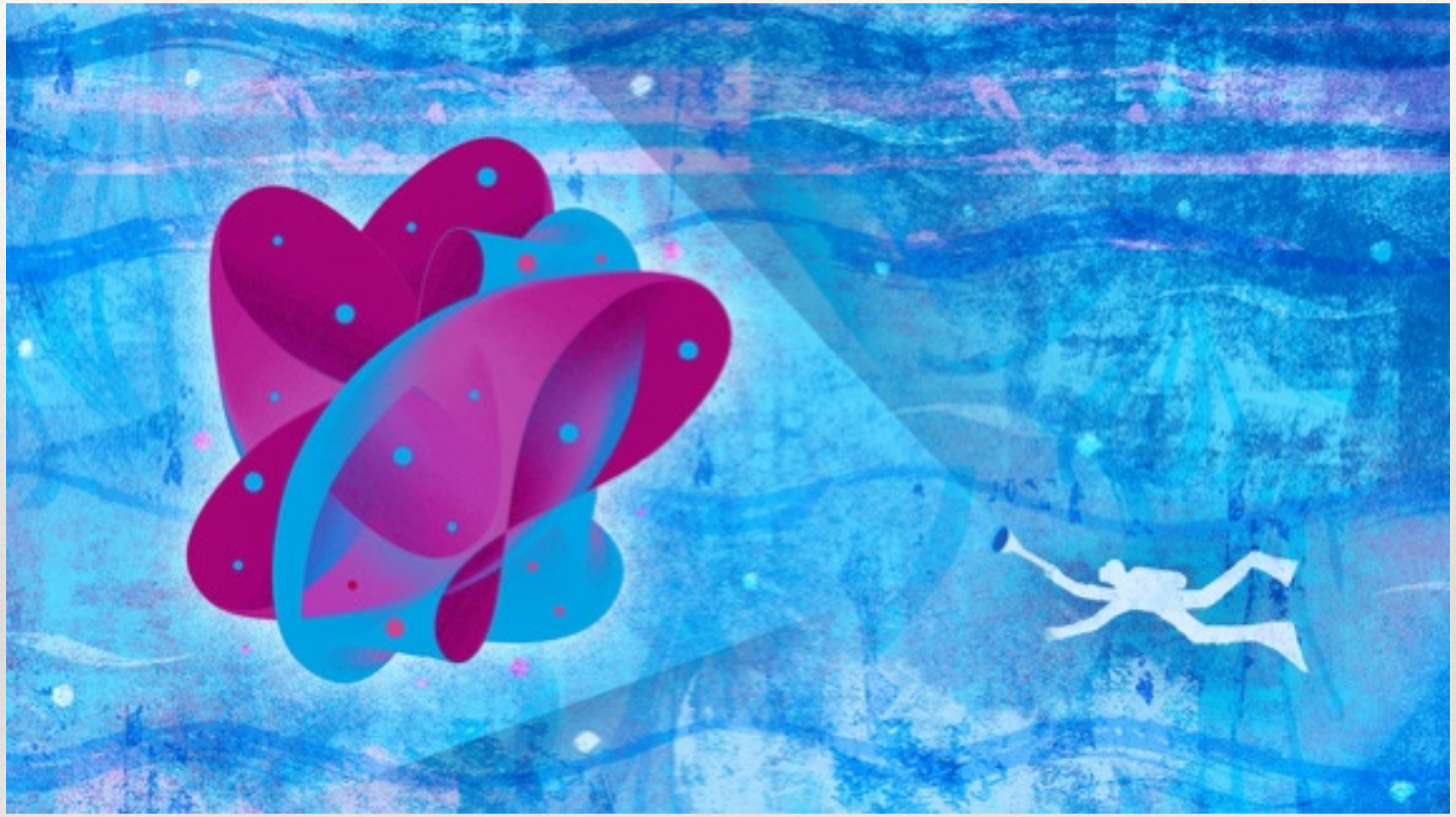
Motivation



[The power of mirror symmetry, Robbert Dijkgraaf]

Motivation

Framework for obtaining B-U ingredients:
-Bigger gauge groups SU(5), SO(10), $SU(3)^3$, etc
-Non SM fields (axions, extra Higgs, leptoquarks, RHN …)
-SUSY(?) and extra symmetries (R sym, $\mathbb{Z}_2$, Q_6 , S_3 , A_5 , $\Delta(96)$ …)
-Extra dimensions
-Modular symmetries



[The power of mirror symmetry, Robbert Dijkgraaf]

As nicely presented in talks by Myriam, Catalina, Selim, León, Alexander, Lorenzo, Melina, Hansel, Saúl… and many more (:

Framework: Heterotic String Theory

We have:

- 10 dimensional theory
- Supersymmetric theory
- Fixed gauge group:
  $E_8 \times E_8$ or $SO(32)$

We wish:

- 4 dimensional theory
- Supersymmetric theory (?)
- SM gauge group:
  $SU(3) \times SU(2) \times U(1)$

The spacetime dimension and the rank of the gauge groups does not match!

Dimensional Reduction

- Need to go from a 10 dim theory to a 4 dim one
- Ansatz $\mathscr{M}_{10} \rightarrow \mathscr{M}_4 \times X_6$
- In general X_6 should be a Calabi-Yau manifold
- In this talk, X_6 is a toroidal orbifold

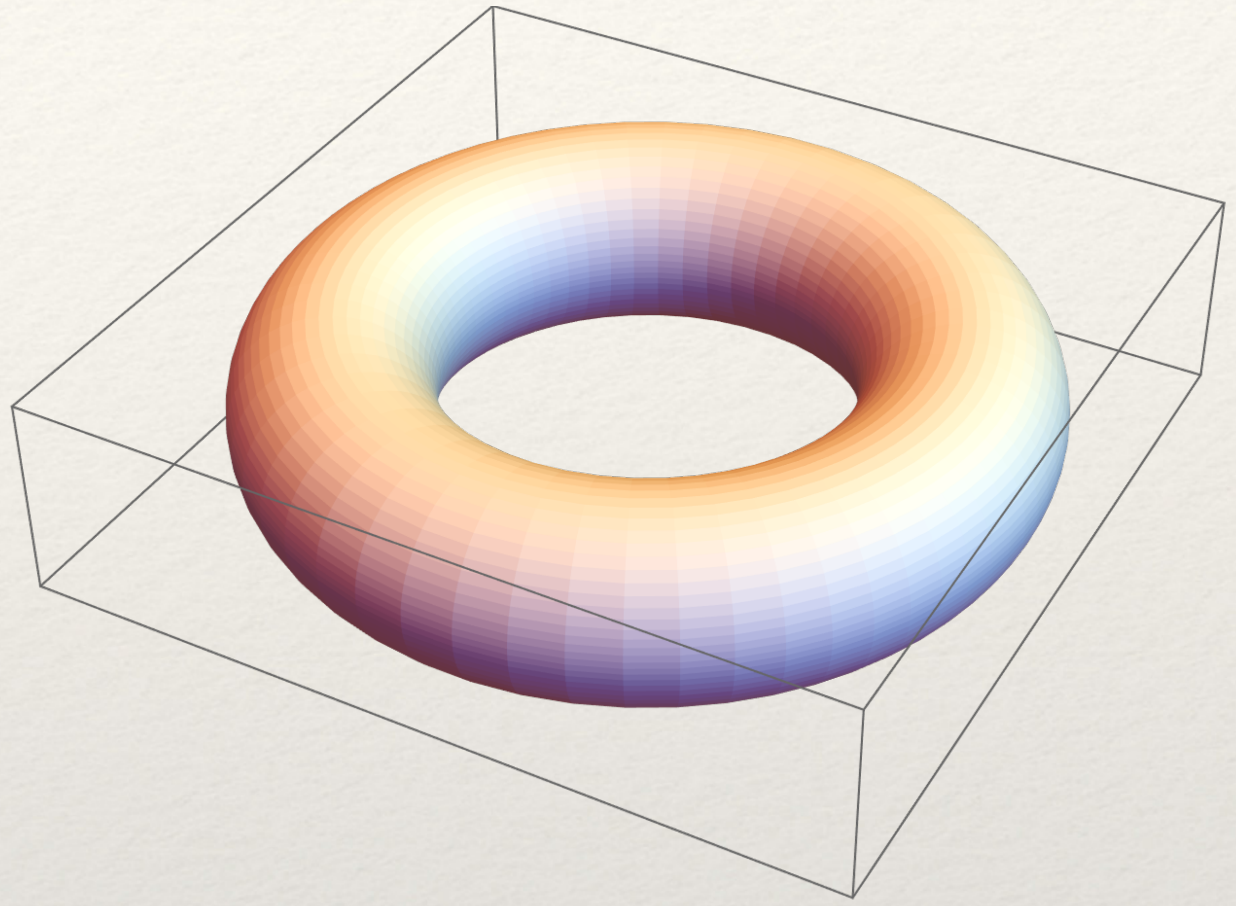
Orbifolds: an Invitation

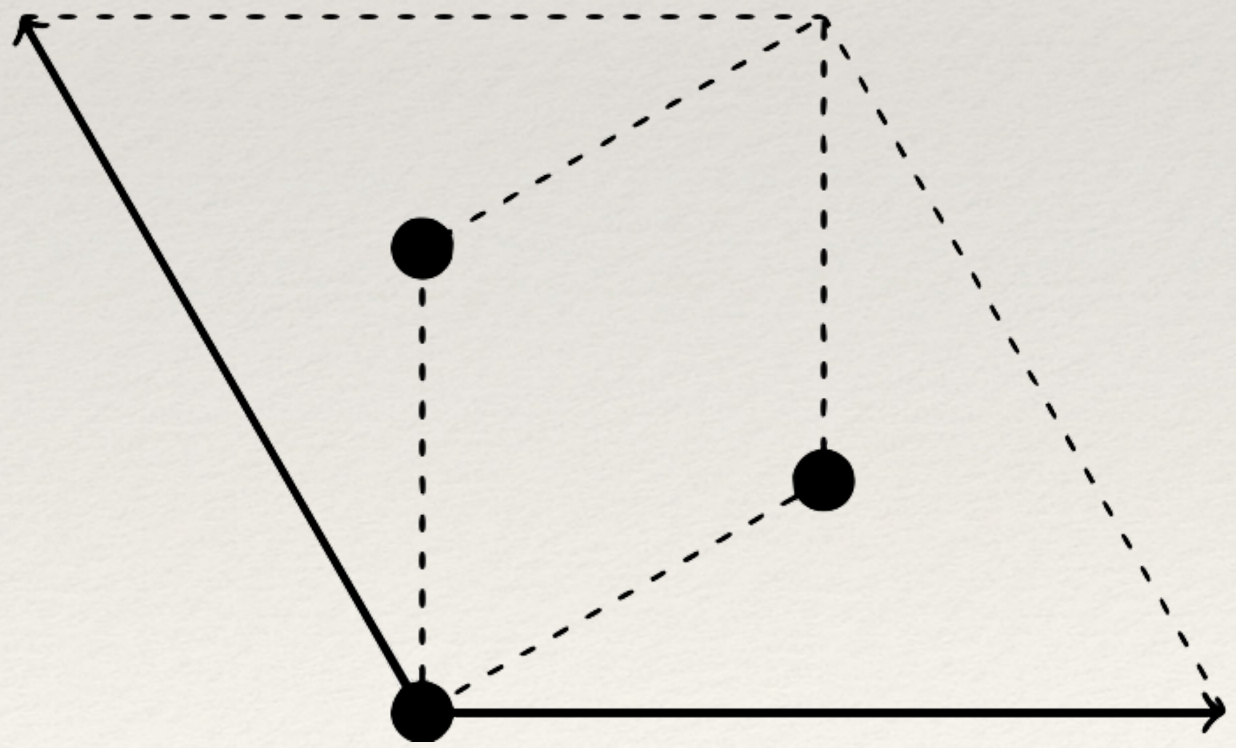
- An orbifold is defined as the quotient $\mathcal{O} = M/P$

- Specially interesting if $M = \mathbb{T}^6, P \subset \mathrm{SU}(3)$

- $\mathbb{T}^6 = \mathbb{R}^6/\Gamma$

- Hence, $\mathcal{O} = \mathbb{R}^6/S, \quad S = P \ltimes \Gamma$





Why Non-Abelian P?

- In Abelian orbifolds, $\text{rank}(G_{4\text{D}}) = 16 \; = \; 4 \times \text{rank}(G_{\text{SM}})$
- From the bottom-up perspective, it is possible to obtain rank reduction from non-Abelian twists [Hebecker-Ratz 0306049]
- Rank reduction evidence from the top-down approach [Konopka 1210.5040]

Which Non-Abelian Point Groups?

- There are 35 inequivalent point groups compatibles with 4 dim. SUSY $\mathcal{N} = 1$,

S_3	$\mathbb{Z}_3 \times S_3$	$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$	$\mathbb{Z}_3 \times S_4$
D_4	$\Delta(27)$	$SL(2,3) - I$	$\mathbb{Z}_3 \times A_4$	$\Delta(96)$
A_4	$\mathbb{Z}_4 \times S_3$	$\mathbb{Z}_3 \times SL(2,3)$	$\mathbb{Z}_6 \times S_3$	$SL(2,3) \rtimes \mathbb{Z}_4$
D_6	$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$	$\Delta(48)$	$\Sigma(36\phi)$
$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$	$\mathbb{Z}_3 \times D_4$	$\mathbb{Z}_3 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$	$GL(2,3)$	$\Delta(108)$
QD_{16}	$\mathbb{Z}_3 \rtimes Q_8$	$\Delta(216)$	$SL(2,3) \rtimes \mathbb{Z}_2$	$PSL(3,2)$
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	Frobenius T_7	S_4	$\Delta(54)$	$\Sigma(72\phi)$

[Fischer, Ratz, Torrado, Vaudrevange 1209.3906]

- 331 non equivalent geometries arise from them (twice as many as Abelian ones)

Many other options for Non SUSY orbifolds!
To be discussed on a poster by Isaac Castañeda on Friday

Abelian vs Non-Abelian

- In the Abelian case, P is a subgroup of the SO(6) Cartan subalgebra
- Not the case in the non-Abelian scenario, *i.e.* needs a new formalism 🫠
- This fact gives rise to rank reduction 😬

Generalization to the Non-Abelian Case

- Tasks:
  Embed P in the geometric degrees of freedom $SO(6) \subset SO(8)$ and in the gauge degrees of freedom $SO(32)$
  Compute the 4 dim spectrum

- Difficulties:
  Write P elements as rotations (block diagonal matrices, needs an algorithm)
  Deal simultaneously with different choices for the Cartan basis (different roots systems)

$P \hookrightarrow \mathrm{SO}(6)$

- To achieve the embedding, we have to assign a twist vector $v = (0, v_1, v_2, v_3) \in \Lambda_{\mathrm{SO}(8)}$ to each $[g] \in P$ $\quad$ v in the SO(8) Cartan basis
- The components of the twist vector of a given g, are such that $g = \exp\left[2\pi i v_k J_k\right]$, with J_k SO(6) generators
- So, we look for a basis β_g, such that $g = \exp\left[2\pi i v_k J_k\right]$
- Successfully done for S_3, D_4 and $\left(Z_4 \times Z_2\right) \rtimes Z_2$

4 D Gauge Group G

- To achieve $P \hookrightarrow \mathrm{SO}(32)$, we have to assign a shift vector $V = (V_1, V_2, \ldots, V_{16}) \in \Lambda_{\mathrm{SO}(32)}$ to each $g \in P$ V is in the SO(32) Cartan basis
- Solution: $V = (v_1, v_2, v_3, 0, \ldots, 0)$ [Standard embedding]
- This gives
  $\mathrm{SO}(32) \to \mathrm{SO}(6) \times \mathrm{SO}(26) \to G \times \mathrm{SO}(26)$
- With G, such that $\mathrm{rank}(G) < 3$.
  Therefore
  $\mathrm{rank}(G \times \mathrm{SO}(26)) < 16$ 😁

Spectrum: Untwisted Sector

- Work with SO(8) weights, $|q\rangle$ and SO(32) roots, $|p\rangle$ that
  $$0 = \frac{q^2}{2} + N - 1/2, \quad 0 = \frac{p^2}{2} + \tilde{N} - 1.$$
- We build states $|q\rangle \otimes |p\rangle$, such that
  $$p \cdot V_g - q \cdot v_g = 0, \mod 1 \quad \forall g \in S$$
- This require us to work with different Cartan bases simultaneously
  Trouble!

Spectrum: Twisted Sectors

- We look for $|q\rangle$ in the SO(8) weight lattice, and $|p\rangle$ in the SO(32) root lattice such that, $\frac{q_{sh}^2}{2} - \frac{1}{2} + \delta_g = 0, \quad \frac{p_{sh}^2}{2} - 1 + \tilde{N} + \delta_g = 0,$ and they satisfy the physical condition for their respective equivalence class $[g]$ and its centralizer $\mathscr{C}_S(g)$
- This task reduces to the Abelian techniques (if $\mathscr{C}_S(g)$ is Abelian) Fine!

Particular Geometries

- We studied the point groups S_3, D_4 and $\left(Z_4 \times Z_2\right) \rtimes Z_2$
- These are completely inequivalent geometries.
  Therefore, their study will allow us to check our method in truly different scenarios and verify that rank reduction is indeed a generic feature of non-Abelian orbifolds

S_3 Orbifold: Results

- 4 dim gauge group: $G = SO(26) \times U(1) \times U(1)$
- 4 untwisted moduli (→ modular symmetries?) in agreement with **[Fischer, Ramos, Vaudrevange 1304.7742]**
- We found the **26** irrep with the following multiplicity in each sector
  $4 \in [e], \quad 8 \in [\theta], \quad 18 \in [\sigma]$ generalizes $E_8 \times E_8$ results **[Fischer, Ramos, Vaudrevange 1304.7742]**

S_3 Orbifold: Spectrum

$\mathrm{SO}(26)\times\mathrm{U}(1)\times\mathrm{U}(1)$ irrep.	U sector	$T_{[\vartheta]}$ sector	$T_{[\omega]}$ sector
$\mathbf{26}_{(1,0)}$	1	0	9
$\mathbf{26}_{(-1,0)}$	1	0	9
$\mathbf{26}_{(0,1)}$	1	0	0
$\mathbf{26}_{(0,-1)}$	1	0	0
$\mathbf{26}_{(-1/2,-1/2)}$	0	4	0
$\mathbf{26}_{(1/2,1/2)}$	0	4	0
$\mathbf{1}_{(0,0)}$	3	0	18
$\mathbf{1}_{(1/2,-1/2)}$	0	8	0
$\mathbf{1}_{(-1/2,1/2)}$	0	8	0
$\mathbf{1}_{(-3/2,-1/2)}$	0	4	0
$\mathbf{1}_{(3/2,1/2)}$	0	4	0
$\mathbf{1}_{(1,1)}$	1	0	9
$\mathbf{1}_{(1,-1)}$	1	0	9
$\mathbf{1}_{(-1,1)}$	1	0	9
$\mathbf{1}_{(-1,-1)}$	1	0	9

D_4 Orbifold: Results

- 4 dim gauge group $G = U(1) \times SO(26)$
- 4 untwisted moduli in agreement with [Fischer, Ramos, Vaudrevange 1304.7742]
- **26** irreps with the following multiplicity in each sector:
  $4 \in [e], \quad 4 \in [\theta], \quad 16 \in [\omega], \quad 8 \in [\theta\omega], \quad 10 \in [\theta\omega\theta\omega]$
  generalizes results for $E_8 \times E_8$ [Fischer, Ramos, Vaudrevange 1304.7742]

D_4 Orbifold: Spectrum

$\mathrm{SO}(26) \times \mathrm{U}(1)$ irrep	U	$T_{[\vartheta]}$	$T_{[\omega]}$	$T_{[\vartheta\omega]}$	$T_{[\vartheta\omega\vartheta\omega]}$
$\mathbf{26}_0$	2	4	16	8	10
$\mathbf{26}_1$	1	0	0	0	0
$\mathbf{26}_{-1}$	1	0	0	0	0
$\mathbf{1}_0$	2	16	0	0	20
$\mathbf{1}_{-1}$	2	16	16	8	20
$\mathbf{1}_1$	2	16	16	8	20

$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ Orbifold: Results

- 4 dim gauge group $G = SO(26)$ (max rank reduction from standard embedding)
- 3 untwisted moduli in agreement with **[Fischer, Ramos, Vaudrevange 1304.7742]**
- **26** irreps with the following multiplicity in each sector
  $3 \in [e], \quad 4 \in [\theta], \quad 4 \in [\omega], \quad 10 \in [\rho], \quad 2 \in [\theta\omega],$
  $2 \in [\theta\rho], \quad 2 \in [\omega\rho], \quad 6 \in [\theta\omega\rho^3], \quad 3 \in [\rho^2]$
  generalizes results for $E_8 \times E_8$ **[Fischer, Ramos, Vaudrevange 1304.7742]**]

$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ Orbifold: Spectrum

SO(26) irrep	U	$T_{[\vartheta]}$	$T_{[\omega]}$	$T_{[\rho]}$	$T_{[\vartheta\omega]}$	$T_{[\vartheta\rho]}$	$T_{[\omega\rho]}$	$T_{[\vartheta\omega\rho^3]}$	$T_{[\rho^2]}$
26	3	4	4	10	2	2	2	6	5
1	3	8	8	40	8	8	8	24	20

A Taste of Flavor

- Traditional flavor symmetries arise from the abelianization of the space group **[Ramos, Vaudrevange 1811.00580]**
- $S \to \mathcal{G}_{\text{flavor}}$
- Schematically, obtain an abelian subgroup and *mix it* (GAP) with permutations of relevant fixed points of the orbifold

A Taste of Flavor: S_3

- Through abelianization: $S_3 \to \mathbb{Z}_3^3 \times \mathbb{Z}_2^3$
- Its non-trivial conjugacy classes share 4 fixed points, then S_4 is also a symmetry
- So the traditional flavor symmetry turns out to be $\mathcal{G}_{\text{flavor}} = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_4$

Future Work

Conclusions

- We successfully extended the Abelian formalism for the non-Abelian case
- Need to be extended for non-standard embedding
- We developed an algorithm that works in certain geometries
- We found rank reduction in every case, without summoning additional mechanisms
  S_3: $\mathrm{SO}(32) \to \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SO}(26)$
  D_4: $\mathrm{SO}(32) \to \mathrm{U}(1) \times \mathrm{SO}(26)$
  $\left(\mathbb{Z}_4 \times \mathbb{Z}_2\right) \rtimes \mathbb{Z}_2$: $\mathrm{SO}(32) \to \mathrm{SO}(26)$ **(max rank reduction for standard embedding)**
- Progress in the computation of flavor symmetries in the making

Thanks for your attention!

Space Group S

- Conjugacy classes $[g] = \left\{hgh^{-1} | h \in S\right\}$
- Fixed points, for each $[g]$ there are some z such that $gz = z$
- In general, for each $g \in S$ we solve $z = (\mathbb{I} - g)^{-1} n_\alpha e_\alpha$
- Abelian case also need to embed P in $SO(6)$

Block Diagonalization

- We developed an algorithm for this task

$$\begin{pmatrix} E_p & F_{q\times p} \\ G_{p\times q} & H_q \end{pmatrix} \to \begin{pmatrix} D_p & 0_{q\times p} \\ 0_{p\times q} & D_q \end{pmatrix},$$

by solving the equations

$$R(E+FR) = G + HR, \qquad (E+FR)X - X(H-RF) = -F,$$

for R and X. **[Eisenfeld 76]**

Block Diagonalization

- If there are solutions R and X, the transformation that block diagonalize our original matrix is

$$W = \begin{pmatrix} \mathbb{1}_p & X \\ R & XR + \mathbb{1}_q \end{pmatrix}.$$ **[Eisenfeld 76]**

This W was found in 12 cases

$$\tilde{P} = \{S_3, D_4, A_4, D_6, (\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2, \mathbb{Z}_4 \rtimes S_3, S_4, \\ (\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2, \mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4), \Delta(27), \Delta(54), \Delta(96)\}\,.$$

Block Diagonalization

- One last step

$$\begin{pmatrix} E_p & F_{q\times p} \\ G_{p\times q} & H_q \end{pmatrix} \to \begin{pmatrix} D_p & 0_{q\times p} \\ 0_{p\times q} & D_q \end{pmatrix} \to \begin{pmatrix} R(\theta)_2 & 0 & 0 \\ 0 & R(-\theta)_2 & 0 \\ 0 & 0 & \mathbb{I}_2 \end{pmatrix},$$

- Say that the full transformation is Q, we restricted to the case where Q is orthogonal. This condition reduced our previous list to $P \in \{S_3, D_4, (\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2\}$.

Dealing with Different Cartan Basis Choices

- Take two different choices for the $SO(6)$ Cartan basis, $H = \{H_1, H_2, H_3\}$ and $H' = \{H'_1, H'_2, H'_3\}$ ordered bases
- Each H_i and H'_i have identical roles
- H and H' give rise to different root systems $R = \{R_1, R_2, \ldots, R_6\}$ and $R' = \{R'_1, R'_2, \ldots, R'_6\}$
- $\forall H_i$, there are R_l, $R_m \in R$ such that R_l is the raising operator and R_m is the lowering operator for H_i
- This is also true for some R'_l, $R'_m \in R'$ for each H'_i

Dealing with different Cartan bases

- Solution: We propose a *bijection* $R_l \sim R'_l \quad R_m \sim R'_m$
- With this, we can manipulate twist and shift vectors in different basis!😁

Abelian Techniques

- P is contained in the $SO(6)$ Cartan subalgebra H, i.e. every $g \in P$ is an exponential map of linear combinations of elements of H, say
$g = \exp\left[2\pi i \alpha_j H_j\right].$
- Every conjugación class is define by a twist vector
$v = (\alpha_1, \alpha_2, \alpha_3)$, v is in the $SO(6)$ Cartan basis
such that
$\alpha_1 + \alpha_2 + \alpha_3 = 0.$
- This define the embedding of P in the geometric dof

Abelian Techniques

- To embed S in the gauge dof, we map
  $\left(\theta^k, n_\alpha e_\alpha\right) \to \left(kV, n_\alpha A_\alpha\right)$,
  V is the so called shift vector, and A_α are Wilson loops
  V is such that $NV \in \Lambda, \ \Lambda$ the $SO(32)$ weight lattice

- Modular invariance requires
  $N\left(V^2 - v^2\right) = 0 \mod 2, \quad$ (no Wilson loops)

  Simplest solution: standard embedding
  $V = \left(v^1, v^2, v^3, 0^{13}\right).$ $\quad$ V is in the G Cartan basis

Abelian Techniques

- For computing the spectrum, there are two cases
- Untwisted sector $[e]$

States $|q\rangle_R \otimes |p\rangle_L$, such that $0 = \frac{q^2}{2} + N - 1/2, \quad 0 = \frac{p^2}{2} + \tilde{N} - 1.$

Solutions if $N = 0$ y $q^2 = 1$

while $\tilde{N} = 1$ y $p = (0^{16})$ (Cartan generators, sugra multiplet, modules)

or $\tilde{N} = 0$ y $p^2 = 2$ (every other gauge group generators)

Physical states, those that $p \cdot V_g - q \cdot v_g = 0, \mod 1 \quad \forall g \in S$

Abelian Techniques

- Twisted sectors $[g]$

  States $\left| q_{sh} \right\rangle_R \otimes \left| p_{sh} \right\rangle_L$ such that $\dfrac{q_{sh}^2}{2} - \dfrac{1}{2} + \delta_g = 0, \quad \dfrac{p_{sh}^2}{2} - 1 + \tilde{N} + \delta_g = 0,$

  $q_{sh} = q + v_g, \quad p_{sh} = p + V_g \, .$

  q and p in $\Lambda_{SO(8)}$ and $\Lambda_{SO(32)}$ respect.

- Physical states if

  $p_{sh} \cdot V_h - R \cdot v_h = 0, \bmod 1 \quad \forall g \in \mathscr{C}_S(g),$

  with $R^i = q_{sh}^i - \tilde{N}^i + \tilde{N}^{*\bar{i}}, \quad i \in \{0,1,\ldots,3\} \, .$

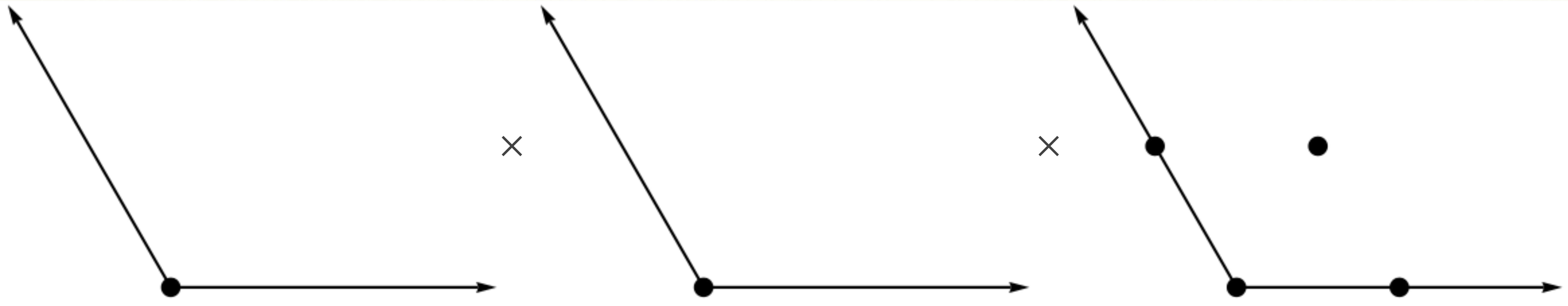
  States in this sectors are matter fields

S_3 Orbifold

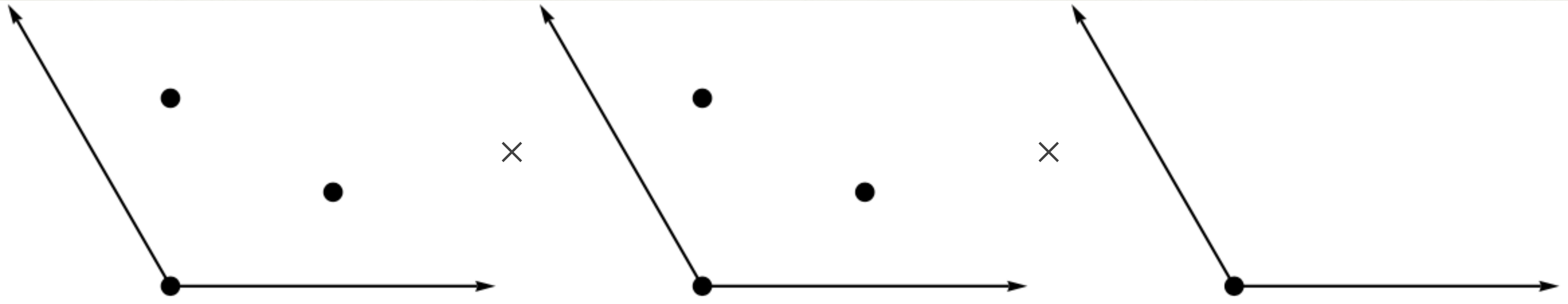
- S_3 has two generators, of order 2 and 3 respect. Say $\{\theta, \omega\}$
- S_3 is the symmetry group of an equilateral triangle
- It has two non trivial conjugation classes: $[\theta]$ y $[\omega]$
- This orbifold has 13 fixed points, 4 related to the $[\theta]$ sector and 9 for the $[\omega]$ sector

S_3 Orbifold

- Fixed points



theta sector fixed points



omega sector fixed points

-

S_3 Orbifold

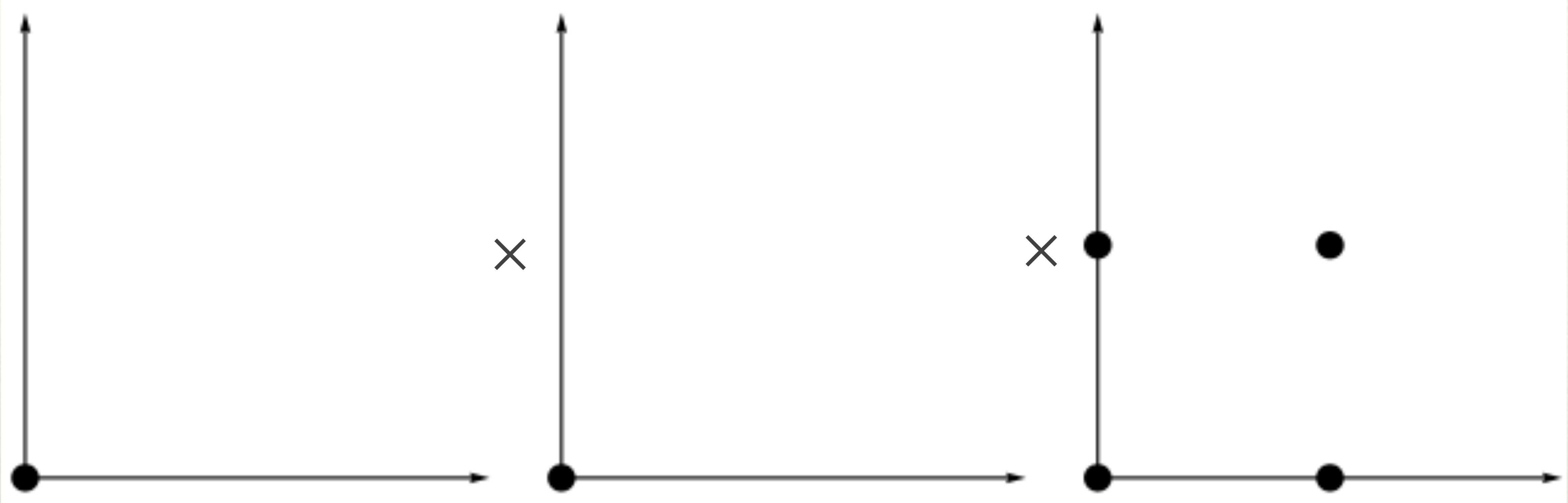
- Through one single basis transformation, we found

$$\theta = \exp\left[\frac{2\pi i}{2}\left(J_{4,6} - J_{7,8}\right)\right], \quad \omega = \exp\left[\frac{2\pi i}{3}\left(J_{3,4} - J_{5,6}\right)\right]$$

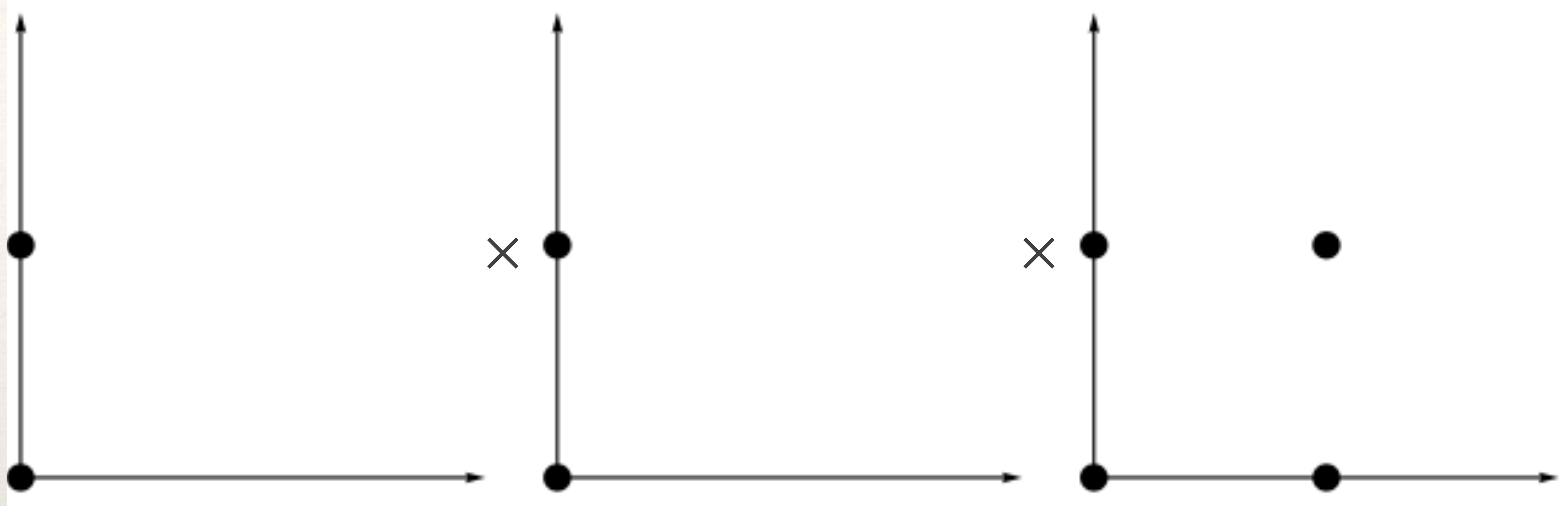
D_4 Orbifold

- D_4 has 2 order 2 generators, $\{\theta, \omega\}$
- D_4 is the symmetry group of a square
- 4 non trivial conjugacy classes: $[\theta],\ [\omega],\ [\theta\omega],\ [\theta\omega\theta\omega]$
- 34 fixed points

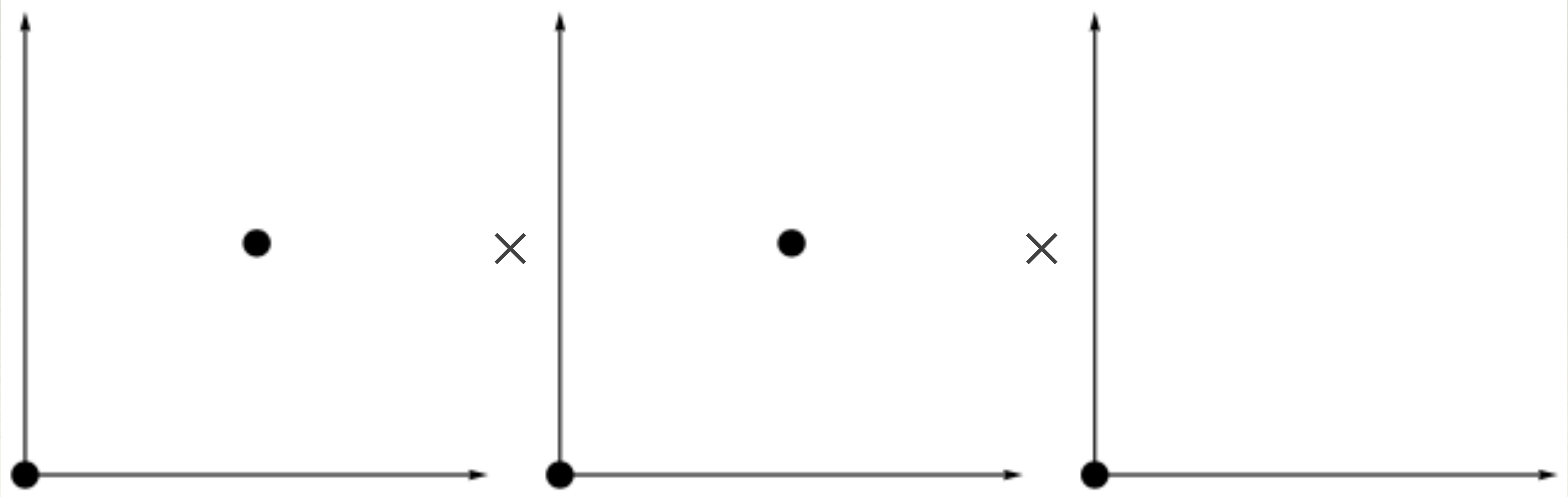
D_4 Orbifold



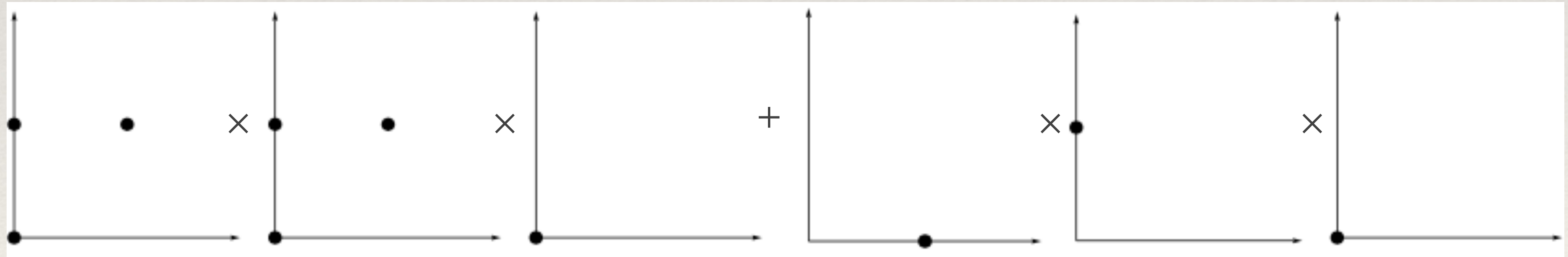
theta sector fixed points



omega sector fixed points



theta*omega sector fixed points



theta*omega*theta*omega sector fixed points

D_4 Orbifold

- We require two different transformaciones to arrive to the following expressions

$$\theta = \exp\left[\frac{2\pi i}{2}\left(J_{3,6} - J_{7,8}\right)\right], \qquad \omega = \exp\left[\frac{2\pi i}{3}\left(J_{4,6} - J_{7,8}\right)\right],$$

$$\theta\omega = \exp\left[\frac{2\pi i}{4}\left(-J_{3,4} + J_{5,6}\right)\right], \ \theta\omega\theta\omega = \exp\left[\frac{2\pi i}{2}\left(J_{3,4} - J_{5,6}\right)\right].$$

$\left(\mathbb{Z}_4 \times \mathbb{Z}_2\right) \rtimes \mathbb{Z}_2$ Orbifold

- 3 generators with order 4, 2 and 2 respect. $\{\rho, \theta, \omega\}$
- This group can be understood as a discrete version of $SU(2)$
- 8 non trivial conjugacy classes: $[\rho],\ [\theta],\ [\omega],\ [\theta\omega],\ [\theta\rho],\ [\omega\rho],\ [\theta\omega\rho^3],\ [\rho^2]$
- 35 fixed points

$\left(\mathbb{Z}_4 \times \mathbb{Z}_2\right) \rtimes \mathbb{Z}_2$ Orbifold

- We found 5 different transformations that lead us to

$$\rho = \exp\left[\frac{2\pi i}{4}\left(J_{5,6} - J_{4,7}\right)\right], \qquad \theta = \exp\left[\frac{2\pi i}{2}\left(J_{3,4} - J_{7,8}\right)\right],$$

$$\omega = \exp\left[\frac{2\pi i}{2}\left(J_{3,7} - J_{4,8}\right)\right], \qquad \theta\omega = \exp\left[\frac{2\pi i}{4}\left(-J_{4,5} - J_{6,7} + 2J_{3,8}\right)\right],$$

$$\theta\rho = \exp\left[\frac{2\pi i}{4}\left(J_{3,4} + J_{6,7} - 2J_{5,8}\right)\right], \qquad \omega\rho = \exp\left[\frac{2\pi i}{2}\left(J_{3,4} - J_{6,8}\right)\right],$$

$$\theta\omega\rho^3 = \exp\left[\frac{2\pi i}{4}\left(J_{4,5} - J_{6,7}\right)\right], \qquad \rho^2 = \exp\left[\frac{2\pi i}{2}\left(J_{5,6} - J_{4,7}\right)\right].$$