A Taste of Symmetries in Lepton Flavors

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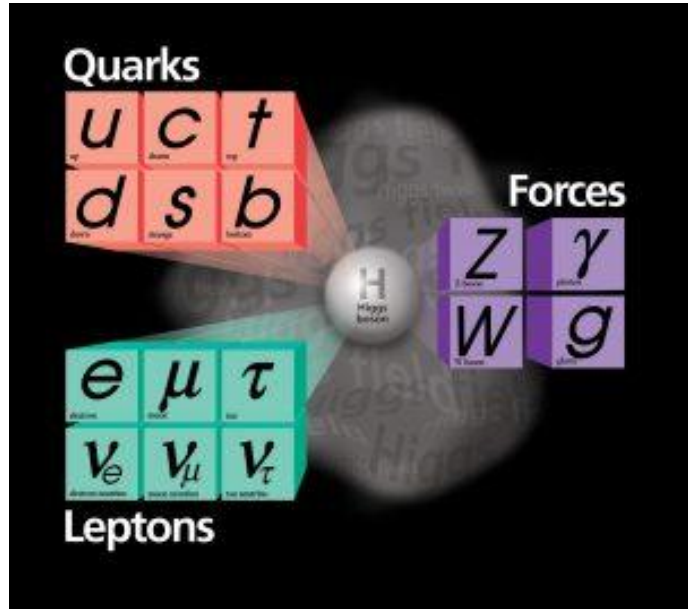
The Standard Model

http://www.particleadventure.org/standard_model.html

Triumph of modern science. It explains what particles make up the world and how these particles interact.

Contains spin-1/2 fermions (matter particles), i.e., the quarks and leptons (their antiparticles) and the spin-1 force-carrying particles , i.e., the photon (EM) , the gluons (Strong) and the $W^{\pm}$ and Z bosons (Weak)….

A More Refined View…



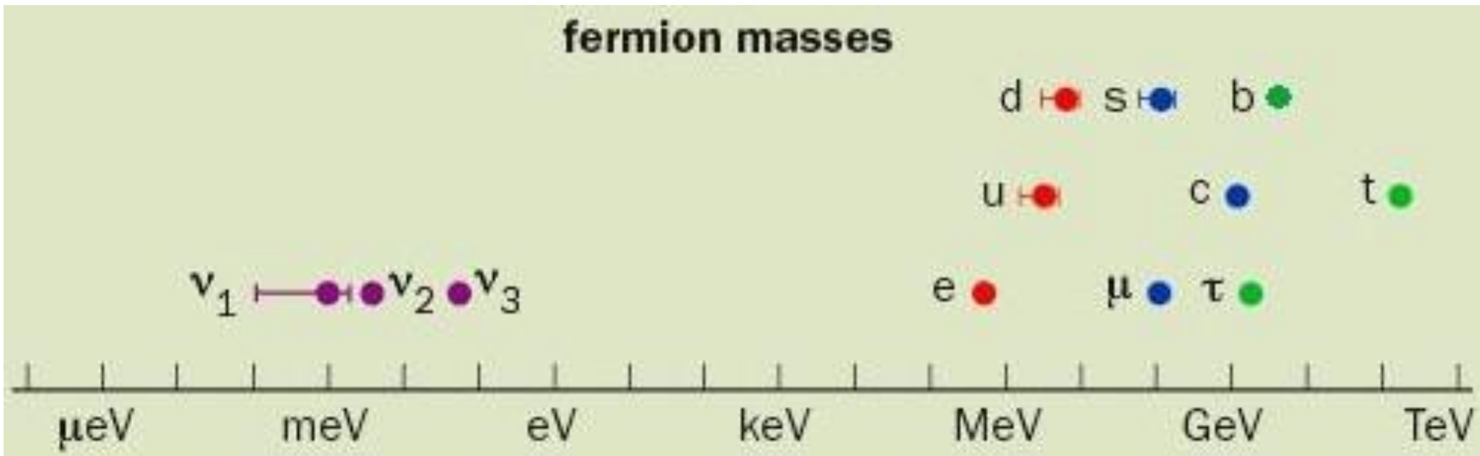
https://www.fnal.gov/pub/science/inquiring/matter/ww_discoveries/index.html

The Standard Model also contains spin-0 Higgs Boson which is associated with the Higgs Field, the field these particles interact with to gain mass.

Focus on the quarks and leptons for now, i.e., the 3 up-type quarks (+2/3 e), the 3 down-type quarks (-1/3 e), the 3 charged leptons (-1 e), and the 3 neutrinos (0 e).

Basically, the Standard Model has 3 copies/flavors of each of the 4 matter particles, identical in every way except….

Heavier Tastes



Red is the first flavor, blue is the second flavor, green is the third flavor.

Neutrinos are not labeled by their flavor states but by their mass states instead.

The three neutrino flavors have the same quantum numbers, so they mix together to form mass (eigen)states.

What about the quarks?

https://news.fnal.gov/2016/08/nova-shines-new-light-neutrinos-behave/

Experiments Reveal the Ratios

$$U=\begin{pmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & 0 & s_{13}e^{-i\delta}\\ 0 & 1 & 0\\ -s_{13}e^{i\delta} & 0 & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta}\\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13}\\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}\end{pmatrix}$$

Quarks	Leptons
$U_{CKM}=R_1(\theta_{23}^{CKM})R_2(\theta_{13}^{CKM},\delta_{CKM})R_3(\theta_{12}^{CKM})$	$U_{MNSP}=R_1(\theta_{23})R_2(\theta_{13},\delta_{CP})R_3(\theta_{12})P$
$\theta_{12}^{CKM}=13.003^\circ\pm0.039^\circ$	$\theta_{12}^{MNSP}=(33.68^\circ)^{+0.73}_{-0.70}$
(Based on PDG 2024) $\theta_{23}^{CKM}=(0.2138^\circ)^{+0.0052}_{-0.0049}$	(NuFit-6.0 (2024)) $\theta_{23}^{MNSP}=(43.3^\circ)^{+1.0}_{-0.8}$
$\theta_{13}^{CKM}=(2.397^\circ)^{+0.045}_{-0.040}$	$\theta_{13}^{MNSP}=(8.56^\circ)^{+0.11}_{-0.11}$
$\delta_{CKM}=65.72^\circ\pm1.49^\circ$	$\delta_{CP}=(212^\circ)^{+26}_{-41}$

Quarks mixing angles are all small. Perhaps we should view their mixings as perturbations around unity?

Wolfenstein Parametrization

(L. Wolfenstein (1983))

Define $s_{12} = \lambda,\ s_{23} = A\lambda^2,\ s_{13}e^{i\delta} = A\lambda^3(\rho + i\eta)$

Then,

$$U_{CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

$$\lambda = 0.22501 \pm 0.00068\,, \qquad A = 0.826^{+0.016}_{-0.015}\,,$$

$$\bar{\rho} = 0.1591 \pm 0.0094\,, \qquad \bar{\eta} = 0.3523^{+0.0073}_{-0.0071}$$

(PDG 2024)

$$\bar{\rho} = \rho(1 - \lambda^2/2 + \ldots)$$

$$If\ \lambda \to 0,\ then\ U_{CKM} \to 1.$$

Is this a meaningful limit? Can we do a similar thing in the lepton sector?

The Cabibbo Haze

(A. Datta, L. Everett, P. Ramond (2005); L. Everett (2006); L. Everett, P. Ramond (2006))

If quarks and leptons are unified (at some scale) and quark mixing has Cabibbo-sized corrections, then so should lepton mixing.
Hence, the initial lepton mixing starting point been hidden from us by a haze of Cabibbo-sized corrections/effects.

By shifting the different mixing angles in simple ways, the authors find that in some of these cases (A. Datta, L. Everett, P. Ramond (2005)):

$$\theta_{13} \approx \lambda \sin\theta_{23} \quad \text{and} \quad \theta_{13} \approx \lambda \cos\theta_{23}$$

Notice that if atmospheric mixing angle is maximal, and the initial reactor mixing angle is zero, then the LO vanishing reactor mixing angle is corrected to approximately 8.9^o at first order in the expansion!

If you look at many flavor models in the literature, you will see that most of them can be viewed within this framework.

What are 'suitable' starting points for these large lepton angles?

When Tastes Were Simpler (before 2012)

$$\tilde{U}_\nu = R_{23}\left(\theta^\nu_{23}\right) R_{12}\left(\theta^\nu_{12}\right) = \begin{pmatrix} \cos\theta^\nu_{12} & \sin\theta^\nu_{12} & 0 \\ -\frac{\sin\theta^\nu_{12}}{\sqrt{2}} & \frac{\cos\theta^\nu_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sin\theta^\nu_{12}}{\sqrt{2}} & \frac{\cos\theta^\nu_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(Marzocca, et al. (2013);
Petcov (2014);
Girardi, Petcov, Titov (2015))

TriBiMaximal (TBM) Mixing: $\theta^\nu_{12} \approx 35.26^\circ$ (P. Harrison, D. Perkins, W. Scott (2002); Z. Xing (2002); X. He, A. Zee (2003))

BiMaximal (BM) Mixing: $\theta^\nu_{12} = 45^\circ$ (F. Vissani (1997); V. Barger, S. Pakvasa, T. Weiler, K. Whisnant, (1998); A. Baltz, A. Goldhaber, M. Goldhaber (1998))

Golden Ratio 1 (GR1) Mixing: $\theta^\nu_{12} \approx 31.72^\circ$ (A. Datta, F. Ling, P. Ramond (2003); L. Everett, AS (2008))

Golden Ratio 2 (GR2) Mixing: $\theta^\nu_{12} = 36^\circ$ (W. Rodejohann (2009))

HexaGonal (HG) Mixing: $\theta^\nu_{12} = 30^\circ$ (C. Albright, A. Dueck and W. Rodejohann (2010); J. E. Kimand M. Seo (2011))

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What about mixing matrices that have a nonzero reactor mixing angle?

BiTrimaximal (BTM) Mixing

R. Toorop, F. Feruglio, C. Hagedorn (2011); G.J. Ding (2012); S. King, C. Luhn, AS (2013)

$$U^{\rm BTM} = \begin{pmatrix} \frac{1}{6}\left(3+\sqrt{3}\right) & \frac{1}{\sqrt{3}} & \frac{1}{6}\left(3-\sqrt{3}\right) \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{6}\left(-3+\sqrt{3}\right) & \frac{1}{\sqrt{3}} & \frac{1}{6}\left(-3-\sqrt{3}\right) \end{pmatrix}$$

Yielding

$$\theta_{12}^{\rm BTM} = \theta_{23}^{\rm BTM} = \tan^{-1}\left(\sqrt{3}-1\right) \approx 36.2^\circ$$

$$\theta_{13}^{\rm BTM} = \sin^{-1}\left(\frac{3-\sqrt{3}}{6}\right) \approx 12.2^\circ$$

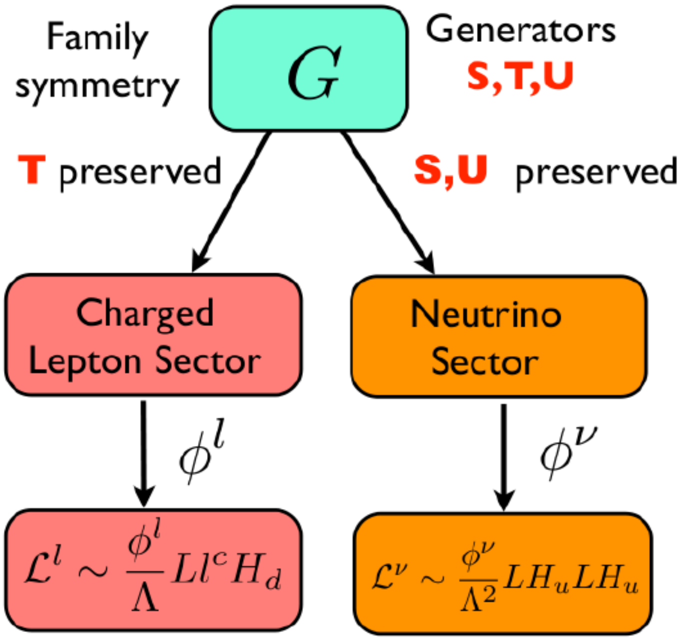
$$\delta^{\rm BTM} = 0$$

Now that we have all these pre-haze starting points, how can symmetries guide us to their recipes?

Cooking with Symmetries

Introduce a flavor symmetry *G* and a set of flavon fields (e.g. ϕ^ν and ϕ^ℓ) whose vevs break *G* to G_ν in the neutrino sector and G_e in the charged lepton sector.

$$T\langle\phi^\ell\rangle \approx \langle\phi^\ell\rangle$$



$$S\langle\phi^\nu\rangle = U\langle\phi^\nu\rangle = \langle\phi^\nu\rangle$$

(S.F. King, C. Luhn (2013))

Now that we better understand the framework, what can these residual symmetries be?

Residual Charged Lepton Symmetry

Since charged leptons are Dirac particles, consider $M_e = m_e m_e^\dagger$.
When **diagonal**, this combination is left invariant by a phase matrix

$$Q_e = \mathrm{Diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3})$$

Because $Q_e^\dagger M_e Q_e = M_e$

$$\mathrm{Det}(Q_e) = +1 \implies \beta_1 = -\beta_2 - \beta_3$$

$$\text{Assume } \beta_{2,3} = 2\pi k_{2,3}/n_{2,3} \text{ with } k_{2,3} = 0, \ldots, n_{2,3} - 1$$

Suppose we keep **all** $T = Q_e$, then

$$G_e \cong Z_{n_2} \times Z_{n_3} = Z_n \times Z_m$$

Since M_e is diagonal, all the mixing must come from the neutrino sector!

What do the residual neutrino symmetries look like?

Residual Neutrino Flavor Symmetry

Key: Assume neutrinos are Majorana particles

$$U_\nu^T M_\nu U_\nu = M_\nu^{\rm Diag} = {\rm Diag}(m_1, m_2, m_3) = {\rm Diag}(|m_1|e^{-i\alpha_1}, |m_2|e^{-i\alpha_2}, |m_3|e^{-i\alpha_3})$$

Notice $U_\nu \to U_\nu Q_\nu$ with $Q_\nu = {\rm Diag}(\pm 1, \pm 1, \pm 1)$ also diagonalizes the neutrino mass matrix. Restrict to ${\rm Det}(Q_\nu) = 1$ and define $G_0^{\rm Diag} = 1$

$$G_1^{\rm Diag} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad G_2^{\rm Diag} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, G_3^{\rm Diag} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Observe non-trivial relations:

$$(G_i^{\rm Diag})^2 = 1, \text{ for } i\text{=1, 2, and 3,}$$
$$G_i^{\rm Diag} G_j^{\rm Diag} = G_k^{\rm Diag}, \text{ for } i \neq j \neq k$$

Sometimes called *SU, S,* and *U*

Therefore, these form a $Z_2 \times Z_2$ residual (Klein) symmetry!

In non-diagonal basis: $M_\nu = G_i^T M_\nu G_i$ with $G_i = U_\nu G_i^{\rm Diag} U_\nu^\dagger$

(L. Everett, T. Garon, AS (2015))

Notice that each G_i has a +1 eigenvalue and two -1 eigenvalues…

Hinting at the Unphysical

Recall each nontrivial Klein element has one +1 eigenvalue. The eigenvector associated with this eigenvalue will be one column of the MNSP matrix (in the diagonal charged lepton basis).

As an example, consider tribimaximal mixing:

$$U^{\mathrm{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

P. F. Harrison, D. H. Perkins, W. G. Scott (2002)
P. F. Harrison, W. G. Scott (2002)
Z. -z. Xing (2002)

It an be shown to originate from the preserved Klein symmetry

$$G_1^{\mathrm{TBM}} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix} \quad G_2^{\mathrm{TBM}} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad G_3^{\mathrm{TBM}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Can we find out a way to parameterize the G_ν in terms of the mixing angles in the MNSP matrix?

Non-Diagonal Klein Elements

$$G_i = U_\nu G_i^{\mathrm{Diag}} U_\nu^\dagger \qquad U_{\mathrm{MNSP}} = U_e^\dagger U_\nu$$

$$G_1 = \begin{pmatrix} (G_1)_{11} & (G_1)_{12} & (G_1)_{13} \\ (G_1)_{12}^* & (G_1)_{22} & (G_1)_{23} \\ (G_1)_{13}^* & (G_1)_{23}^* & (G_1)_{33} \end{pmatrix} \quad G_2 = \begin{pmatrix} (G_2)_{11} & (G_2)_{12} & (G_2)_{13} \\ (G_2)_{12}^* & (G_2)_{22} & (G_2)_{23} \\ (G_2)_{13}^* & (G_2)_{23}^* & (G_2)_{33} \end{pmatrix}$$

$$G_3 = \begin{pmatrix} -c'_{13} & e^{-i\delta} s_{23} s'_{13} & -e^{-i\delta} c_{23} s'_{13} \\ e^{i\delta} s_{23} s'_{13} & s_{23}^2 c'_{13} - c_{23}^2 & -c_{13}^2 s'_{23} \\ -e^{i\delta} c_{23} s'_{13} & -c_{13}^2 s'_{23} & c_{23}^2 c'_{13} - s_{23}^2 \end{pmatrix}$$

(L.L. Everett, T. Garon, AS (2015))

$$s_{ij} = \sin(\theta_{ij}) \quad c_{ij} = \cos(\theta_{ij}) \quad s'_{ij} = \sin(2\theta_{ij}) \quad c'_{ij} = \cos(2\theta_{ij})$$

Notice that in general the Klein elements are complex and Hermitian!
Don't depend on Majorana phases because
$U_\nu \rightarrow U_\nu P_{\mathrm{Maj}}$ leaves transformation invariant.

Non-Diagonal Klein Elements (II)

$$(G_1)_{11} = c_{13}^2 c_{12}' - s_{13}^2, \quad (G_1)_{12} = -2c_{12}c_{13}\left(c_{23}s_{12} + e^{-i\delta}c_{12}s_{13}s_{23}\right)$$

$$(G_1)_{13} = 2c_{12}c_{13}\left(e^{-i\delta}c_{12}c_{23}s_{13} - s_{12}s_{23}\right)$$

$$(G_1)_{22} = -c_{23}^2 c_{12}' + s_{23}^2\left(s_{13}^2 c_{12}' - c_{13}^2\right) + \cos(\delta)s_{13}s_{12}'s_{23}'$$

$$(G_1)_{23} = c_{23}s_{23}c_{13}^2 + s_{13}\left(i\sin(\delta) - \cos(\delta)c_{23}'\right)s_{12}' + \frac{1}{4}c_{12}'\left(c_{13}' - 3\right)s_{23}'$$

$$(G_1)_{33} = \left(s_{13}^2 c_{12}' - c_{13}^2\right)c_{23}^2 - s_{23}^2 c_{12}' - \cos(\delta)s_{13}s_{12}'s_{23}'$$

$$(G_2)_{11} = -c_{12}'c_{13}^2 - s_{13}^2 \quad (G_2)_{12} = 2c_{13}s_{12}\left(c_{12}c_{23} - e^{-i\delta}s_{12}s_{13}s_{23}\right)$$

$$(G_2)_{13} = 2c_{13}s_{12}\left(e^{-i\delta}c_{23}s_{12}s_{13} + c_{12}s_{23}\right)$$

(L. Everett, T. Garon, AS (2015))

$$(G_2)_{22} = c_{12}'c_{23}^2 - s_{23}^2\left(c_{13}^2 + s_{13}^2 c_{12}'\right) - \cos(\delta)s_{13}s_{12}'s_{23}'$$

$$(G_2)_{23} = e^{-i\delta}s_{13}s_{12}'c_{23}^2 + \frac{1}{4}s_{23}'\left(2c_{13}^2 - c_{12}'\left(c_{13}' - 3\right)\right) - e^{i\delta}s_{12}'s_{13}s_{23}^2$$

$$(G_2)_{33} = -c_{23}^2\left(c_{13}^2 + s_{13}^2 c_{12}'\right) + s_{23}^2 c_{12}' + \cos(\delta)s_{13}s_{12}'s_{23}'$$

Notice if $\theta_{12} \to \theta_{12} \pm \pi/2$, then $G_2 \leftrightarrow G_1$. (C. Alvarado, J. Bautista, AS (2022))

What about the mass matrices left invariant by these symmetries?

Invariant Mass Matrix

$$M_\nu = U_\nu^* M_\nu^{\mathrm{Diag}} U_\nu^\dagger$$

$$(M_\nu)_{11} = c_{13}^2 m_2 s_{12}^2 + c_{12}^2 c_{13}^2 m_1 + e^{2i\delta} m_3 s_{13}^2$$

$$(M_\nu)_{12} = c_{13}(c_{12} m_1(-c_{23} s_{12} - c_{12} e^{-i\delta} s_{13} s_{23}) + m_2 s_{12}(c_{12} c_{23} - e^{-i\delta} s_{12} s_{13} s_{23}) + \\ + e^{i\delta} m_3 s_{13} s_{23}),$$

$$(M_\nu)_{13} = c_{13}(-c_{23} m_3 s_{13} e^{i\delta} + m_2 s_{12}(c_{12} s_{23} + c_{23} e^{-i\delta} s_{12} s_{13}) + \\ + c_{12} m_1(-s_{12} s_{23} + c_{12} c_{23} e^{-i\delta} s_{13})),$$

(L. Everett, T. Garon, AS (2015))

$$(M_\nu)_{22} = m_1(c_{23} s_{12} + c_{12} e^{-i\delta} s_{13} s_{23})^2 + m_2(c_{12} c_{23} - e^{-i\delta} s_{12} s_{13} s_{23})^2 + c_{13}^2 m_3 s_{23}^2$$

$$(M_\nu)_{23} = m_1(s_{12} s_{23} - c_{12} c_{23} e^{-i\delta} s_{13})(c_{23} s_{12} + c_{12} e^{-i\delta} s_{13} s_{23}) + \\ + m_2(c_{12} s_{23} + c_{23} e^{-i\delta} s_{12} s_{13})(c_{12} c_{23} - e^{-i\delta} s_{12} s_{13} s_{23}) - c_{13}^2 c_{23} m_3 s_{23}$$

$$(M_\nu)_{33} = m_2(c_{12} s_{23} + c_{23} e^{-i\delta} s_{12} s_{13})^2 + m_1(-s_{12} s_{23} + c_{12} c_{23} e^{-i\delta} s_{13})^2 + c_{13}^2 c_{23}^2 m_3$$

Is there a way to check to see if all of this makes any sense?

Revisiting Tribimaximal Mixing

P. F. Harrison, D. H. Perkins, W. G. Scott (2002); P. F. Harrison, W. G. Scott (2002); Z. -z. Xing (2002)

$$\theta_{12}^{\mathrm{TBM}} = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \theta_{23}^{\mathrm{TBM}} = \frac{\pi}{4} \quad \theta_{13}^{\mathrm{TBM}} = 0 \quad \delta^{\mathrm{TBM}} = 0$$

Inserting these values into the previous results yield:

$$U^{\mathrm{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad S_4$$

$$G_1^{\mathrm{TBM}} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix} \quad G_2^{\mathrm{TBM}} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad G_3^{\mathrm{TBM}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$M_\nu^{\mathrm{TBM}} = \frac{1}{3}\begin{pmatrix} (2m_1 + m_2) & (m_2 - m_1) & (m_2 - m_1) \\ (m_2 - m_1) & \frac{1}{2}(m_1 + 2m_2 + 3m_3) & \frac{1}{2}(m_1 + 2m_2 - 3m_3) \\ (m_2 - m_1) & \frac{1}{2}(m_1 + 2m_2 - 3m_3) & \frac{1}{2}(m_1 + 2m_2 + 3m_3) \end{pmatrix}$$

Golden Ratio Mixing (GR1)

A. Datta, F. Ling, P. Ramond (2003); Y. Kajiyama, M Raidal, A. Strumia (2007); L. Everett, AS (2008)

$$\theta_{12}^{\mathrm{GR1}} = \tan^{-1}\left(\frac{1}{\phi}\right) \quad \theta_{23}^{\mathrm{GR1}} = \frac{\pi}{4} \quad \theta_{13}^{\mathrm{GR1}} = 0 \quad \delta^{\mathrm{GR1}} = 0$$

$$\phi = (1+\sqrt{5})/2 \qquad U^{\mathrm{GR1}} = \begin{pmatrix} \sqrt{\frac{\phi}{\sqrt{5}}} & \sqrt{\frac{1}{\sqrt{5}\phi}} & 0 \\ -\frac{1}{\sqrt{2}}\sqrt{\frac{1}{\sqrt{5}\phi}} & \frac{1}{\sqrt{2}}\sqrt{\frac{\phi}{\sqrt{5}}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\sqrt{\frac{1}{\sqrt{5}\phi}} & \frac{1}{\sqrt{2}}\sqrt{\frac{\phi}{\sqrt{5}}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad A_5$$

(G.-J. Ding, L. Everett, AS (2012))

$$G_1^{\mathrm{GR1}} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\phi & \phi-1 \\ -\sqrt{2} & \phi-1 & -\phi \end{pmatrix} \; G_2^{\mathrm{GR1}} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1-\phi & \phi \\ \sqrt{2} & \phi & 1-\phi \end{pmatrix} \; G_3^{\mathrm{GR1}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$M_\nu^{\mathrm{GR1}} = \frac{1}{\sqrt{5}}\begin{pmatrix} \frac{m_1\phi^2+m_2}{\phi} & \frac{m_2-m_1}{\sqrt{2}} & \frac{m_2-m_1}{\sqrt{2}} \\ \frac{m_2-m_1}{\sqrt{2}} & \frac{(m_2+m_3)\phi^2+m_1+m_3}{2\phi} & \frac{m_2\phi^2-\sqrt{5}m_3\phi+m_1}{2\phi} \\ \frac{m_2-m_1}{\sqrt{2}} & \frac{m_2\phi^2-\sqrt{5}m_3\phi+m_1}{2\phi} & \frac{(m_2+m_3)\phi^2+m_1+m_3}{2\phi} \end{pmatrix}$$

What about for a mixing pattern with a nonzero reactor angle?

Bitrimaximal Mixing

R. Toorop, F. Feruglio, C. Hagedorn (2011); G.J. Ding (2012); S. King, C. Luhn, AS(2013)

$$\theta_{12}^{\mathrm{BTM}} = \theta_{23}^{\mathrm{BTM}} = \tan^{-1}(\sqrt{3}-1) \quad \theta_{13}^{\mathrm{BTM}} = \sin^{-1}(\frac{1}{6}(3-\sqrt{3})) \quad \delta^{\mathrm{BTM}} = 0$$

Yielding

$$U^{\mathrm{BTM}} = \begin{pmatrix} \frac{1}{6}\left(3+\sqrt{3}\right) & \frac{1}{\sqrt{3}} & \frac{1}{6}\left(3-\sqrt{3}\right) \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{6}\left(-3+\sqrt{3}\right) & \frac{1}{\sqrt{3}} & \frac{1}{6}\left(-3-\sqrt{3}\right) \end{pmatrix} \qquad \Delta(96)$$

$$G_1^{\mathrm{BTM}} = \begin{pmatrix} \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} & -\frac{1}{3} \\ -\frac{1}{3}-\frac{1}{\sqrt{3}} & -\frac{1}{3} & \frac{1}{\sqrt{3}}-\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} \end{pmatrix} \quad G_2^{\mathrm{BTM}} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad G_3^{\mathrm{BTM}} = \begin{pmatrix} -\frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} \\ -\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}-\frac{1}{3} \end{pmatrix}$$

And a mass matrix given by

$$(M_\nu^{\mathrm{BTM}})_{11} = \frac{1}{6}((2+\sqrt{3})m_1 + 2m_2 - (-2+\sqrt{3})m_3) \qquad (M_\nu^{\mathrm{BTM}})_{22} = \frac{1}{3}(m_1+m_2+m_3)$$

$$(M_\nu^{\mathrm{BTM}})_{13} = \frac{1}{6}(-m_1 + 2m_2 - m_3) \qquad (M_\nu^{\mathrm{BTM}})_{12} = \frac{1}{6}(-(1+\sqrt{3})m_1 + 2m_2 + (-1+\sqrt{3})m_3)$$

$$(M_\nu^{BTM})_{33} = \frac{1}{6}(-(-2+\sqrt{3})m_1 + 2m_2 + (2+\sqrt{3})m_3) \quad (M_\nu^{\mathrm{BTM}})_{23} = \frac{1}{6}((-1+\sqrt{3})m_1 + 2m_2 - (1+\sqrt{3})m_3)$$

Now that we know the parameterization works, what else can we do with it?

Adding Presentations to the Mix

A group **presentation** defines is a set of generators and a list of rules that together will generate a group. For example,

$$\mathrm{Z}_2^a \cong \langle a | a^2 = 1 \rangle \quad Z_2^a \times Z_2^b \cong \langle a, b | a^2 = b^2 = 1, ab = ba \rangle$$

Use the previously derived forms of the Klein elements as residual symmetries in the neutrino sector, i.e., (C. Alvarado, J. Bautista, AS (2022))

$$G_\nu \cong K_4 \cong Z_2^S \otimes Z_2^U$$

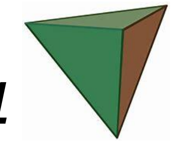
Assume residual charged lepton symmetry is the Abelian group, i.e.,

$$G_e \cong Z_n^T \qquad n \geq 3$$

$$T_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\frac{2\pi i k}{n}} & 0 \\ 0 & 0 & e^{\frac{2\pi i k}{n}} \end{pmatrix}$$

Assume *S, T*, and *U* are generators of the larger flavor symmetry group, i.e., assume this is a *direct model.* (S. King, C. Luhn (2013))

Can this actually work? Try with an "easier" group first with only 2 generators.

Proof of Principle A_4 

C. Alvarado, J. Bautista, AS (2022)

A_4 and Tribimaximal Mixing

$$S^2 = T_3^3 = (ST_3)^3 = 1$$

No *U* required!

Let $S = G_2$. Assume we are in Platonic limit where the reactor angle vanishes, and the atmospheric angle is maximal. Then,

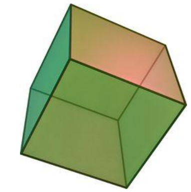
$$ST_3 = \begin{pmatrix} -c'_{12} & \sqrt{2}c_{12}s_{12}\omega^2 & \sqrt{2}c_{12}s_{12}\omega \\ \sqrt{2}c_{12}s_{12} & -s_{12}^2\omega^2 & c_{12}^2\omega \\ \sqrt{2}c_{12}s_{12} & c_{12}^2\omega^2 & -s_{12}^2\omega \end{pmatrix} \quad \omega = e^{\frac{2\pi i}{3}}$$

The first two presentation rules are fulfilled by construction, but we can solve the non-trivial presentation rule to reveal

$$\theta_{12} = \pm\cos^{-1}\left(\pm\sqrt{\frac{2}{3}}\right) \approx \pm 35.26^\circ,\ \pm 144.74^\circ$$

However, in PDG parameterization of the MNSP matrix, all angles are defined in the first quadrant, implying $\theta_{12}^{\nu} \approx 35.26^\circ$!

Will this work for a more complicated group?

Proof of Principle S_4 

(C. Alvarado, J. Bautista, AS (2022))

$\mathbf{S_4}$ and Tribimaximal and Bimaximal Mixing

$$S^2 = T_3^3 = (ST_3)^4 = 1 \qquad S^2 = T_4^4 = (ST_4)^3 = 1$$

$$T_3 = \mathrm{Diag}(1, \omega^2, \omega) \qquad T_4 = \mathrm{Diag}(1, -i, +i)$$

Has two well-known presentations with 2 generators! Again, work in Platonic Limit focusing on first presentation. ST_3 is the same as the previous case. Solving $(ST_3)^4 = 1$

$$\theta_{12} = 0, \pm\pi, \pm\cos^{-1}\left(\pm\sqrt{\frac{1}{3}}\right) \approx 0^\circ, 180^\circ,\ \pm 54.74^\circ,\ \pm 125.26^\circ$$

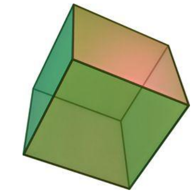
Again, restricting to first quadrant reveals

$$\theta_{12} = 0^\circ, 54.74^\circ$$

Observe 54.74° is the compliment of 35.26°. Therefore, we needed $S = G_1$ not $S = G_2$!

What happens if instead we use the second presentation?

A Different Recipe of S_4



(C. Alvarado, J. Bautista, AS (2022))

Again, work in Platonic Limit but now focusing on the second presentation, i.e.,

$$S^2 = T_4^4 = (ST_4)^3 = 1$$

With $T_4 = \mathrm{Diag}(1, -i, +i)$ and $S = G_2$ yields

$$ST_4 = \begin{pmatrix} -c'_{12} & -i\sqrt{2}c_{12}s_{12} & i\sqrt{2}c_{12}s_{12} \\ \sqrt{2}c_{12}s_{12} & is_{12}^2 & ic_{12}^2 \\ \sqrt{2}c_{12}s_{12} & -ic_{12}^2 & -is_{12}^2 \end{pmatrix}$$

Next, solve the non-trivial presentation rule to reveal

$$\theta_{12} = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$$

Then restricting to the first a maximal solar mixing angle.

This is the well-known result that S_4 can also be broken to generate bimaximal mixing where the reactor angle vanishes, but the solar and atmospheric angle are both maximal!

How about a more complicated group?

In the Test Kitchen with A_5

(C. Alvarado, J. Bautista, AS (2022))

A_5 and Golden Ratio Mixing

$$\phi = \frac{1+\sqrt{5}}{2}$$

$$S^2 = T_5^5 = (ST_5)^3 = 1$$

One last time, we work in the Platonic Limit but now with

$$S = G_2 \qquad T_5 = \mathrm{Diag}(1, \rho^4, \rho) \qquad \rho = e^{\frac{2\pi i}{5}}$$

$$ST_5 = \begin{pmatrix} -c'_{12} & \rho^4\sqrt{2}c_{12}s_{12} & \rho\sqrt{2}c_{12}s_{12} \\ \sqrt{2}c_{12}s_{12} & -\rho^4 s_{12}^2 & \rho c_{12}^2 \\ \sqrt{2}c_{12}s_{12} & \rho^4 c_{12}^2 & -\rho s_{12}^2 \end{pmatrix}$$

Solving the non-trivial presentation rule and looking for the solution in the first quadrant yields a solar angle predictions of 58.28°, the compliment of the GR1 prediction of 31.72°.

Thus, we let $G_1 \leftrightarrow G_2$. However, to still satisfy presentation rules need to instead let

$$T'_5 = \mathrm{Diag}(1, \rho^2, \rho^3)$$

What if we no longer want to work in the boring/bland Platonic limit?

Spicing It Up with BTM Mixing

(C. Alvarado, J. Bautista, AS (2022))

$\mathbf{\Delta(96)}$ and Bitrimaximal Mixing

$\Delta(96) \subset SU(3)$

$$U^2 = T^3 = (UT)^8 = (UT^{-1}UT)^3 = 1$$

Because the reactor mixing angle is mainly associated with the third column of the MNSP matrix, Let $U = G_3$. Then, to satisfy the above presentation rules,

$$T = \mathrm{Diag}(\omega^2, 1, \omega) \quad \omega = e^{\frac{2\pi i}{3}}$$

Due to the rather lengthy products, proceed numerically with to easily see the presentations rules are fulfilled with

$$\theta_{12}^{\mathrm{BTM}} = \theta_{23}^{\mathrm{BTM}} = \tan^{-1}(\sqrt{3}-1), \theta_{13}^{\mathrm{BTM}} = \sin^{-1}((3-\sqrt{3})/6), \delta^{\mathrm{BTM}} = 0$$

This leads to the BTM mixing matrix previously mentioned, i.e.,

$$U_\nu^{BTM} = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}\left(1+\sqrt{3}\right) & 1 & \frac{1}{2}\left(\sqrt{3}-1\right) \\ -1 & 1 & 1 \\ \frac{1}{2}\left(1-\sqrt{3}\right) & 1 & \frac{1}{2}\left(-1-\sqrt{3}\right) \end{pmatrix}$$

Since now the reactor mixing angle isn't zero, what can we say about CP violation?

BTM Mixing in Flavortown

(C. Alvarado, J. Bautista, AS (2022))

Use the same BTM angle values, but instead of letting the CP phase be zero let $\delta^{BTM} = \delta$.

This leads to

$$G_3^{\mathrm{BTM}}(\delta) = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 - \frac{1}{\sqrt{3}} & \left(1 - \frac{1}{\sqrt{3}}\right) e^{-i\delta} & -\frac{e^{-i\delta}}{\sqrt{3}} \\ \left(1 - \frac{1}{\sqrt{3}}\right) e^{i\delta} & -\frac{1}{\sqrt{3}} & -1 - \frac{1}{\sqrt{3}} \\ -\frac{e^{i\delta}}{\sqrt{3}} & -1 - \frac{1}{\sqrt{3}} & 1 - \frac{1}{\sqrt{3}} \end{pmatrix}$$

Perhaps even more interesting is that $S^{BTM}(\delta) = U(UT)^4 U(UT)^4 = \frac{1}{3} \begin{pmatrix} -1 & 2e^{-i\delta} & 2e^{-i\delta} \\ 2e^{i\delta} & -1 & 2 \\ 2e^{i\delta} & 2 & -1 \end{pmatrix}$

(S. King, C. Luhn, AS (2012))

Obviously,

$$S^{BTM}(\delta = 0) = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} = G_2(\theta_{ij}^{TBM})$$

Unfortunately, this phase cannot be a physical CP-violating phase because it is coming from Δ(96), a group which has a representation where all Clebsch-Gordan coefficients (the origin of this would-be phase) are real. Therefore, it is **always possible to remove the phase** from these elements.

Can CP an BTM mixing be reconciled?

Let’s proceed by noting BTM and TBM Mixing seem to be connected to each other.....

Connecting TBM and TBM Mixings

(C. Alvarado, J. Bautista, AS (2022))

They are connected by their trimaximal middle column

$$U^{\mathrm{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad U_\nu^{BTM} = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}\left(1+\sqrt{3}\right) & 1 & \frac{1}{2}\left(\sqrt{3}-1\right) \\ -1 & 1 & 1 \\ \frac{1}{2}\left(1-\sqrt{3}\right) & 1 & \frac{1}{2}\left(-1-\sqrt{3}\right) \end{pmatrix}$$

In our formalism, this is related to the preservation of the $Z_2 \times Z_2$ element

$$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

This trimaximal column is consistent to within 3σ of the data. (NuFit-6.0 (2024))

Is it possible to derive a set of conditions which must hold among the mixing to produce trimaximal mixing?

Tasting Trimaximality

(C. Alvarado, J. Bautista, AS (2022))

Recall G_2 is element associated with the middle column of the MNSP matrix. Solve for when this element is equal to the previously mentioned element to reveal

$$\cos(\theta_{23}) = \frac{1}{2}\sec(\theta_{13})\left(\sin(\theta_{13}) + \sqrt{3\cos^2(\theta_{13}) - 1}\right)$$

$$\cos(\theta_{12}) = \frac{\sec(\theta_{13})\sqrt{3\cos(2\theta_{13})+1}}{\sqrt{6}},$$

$$\delta = 0.$$

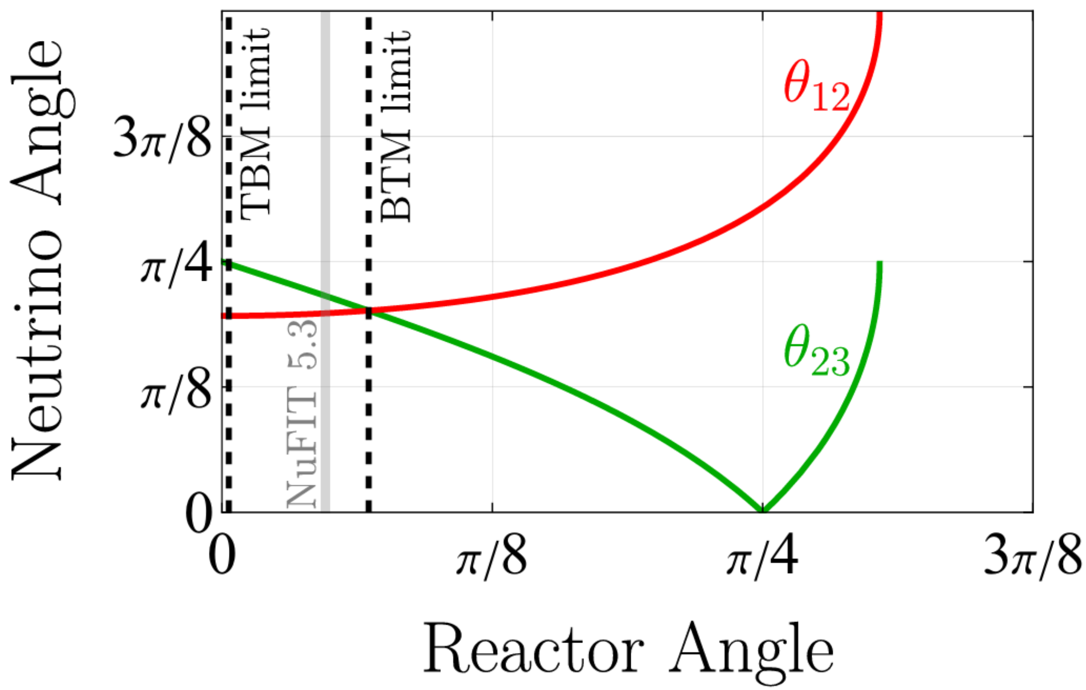
Notice that the radicals both become complex when

$$\theta_{13} > \cos^{-1}(1/\sqrt{3}) = 1/2\cos^{-1}(-1/3) \approx 0.955$$

We can use these conditions to reveal all possible angle values which can give a trimaximal middle column.....

The Mexican Plot

(C. Alvarado, J. Bautista, AS (2022))



This plot shows all possible mixing values needed to preserve a trimaximal middle column.

It also suggests that there should be a unitary matrix relating TBM and BTM.....

The TBM-BTM Connection

(C. Alvarado, J. Bautista, AS (2022))

Take advantage of the definition of the lepton mixing matrix and assume TBM mixing originates in the neutrino sector and the charged lepton sector bring us to BTM mixing:

$$U_{\mathrm{MNSP}} = U_e^\dagger U_\nu = U_e^\dagger U^{\mathrm{TBM}} \equiv U^{\mathrm{BTM}}$$

A straightforward calculation reveals that

$$U_e = U^{\mathrm{TBM}} (U^{\mathrm{BTM}})^\dagger = \frac{1}{6} \begin{pmatrix} 2+\sqrt{2}+\sqrt{6} & 2-2\sqrt{2} & 2+\sqrt{2}-\sqrt{6} \\ 2+\sqrt{2}-\sqrt{6} & 2+\sqrt{2}+\sqrt{6} & 2-2\sqrt{2} \\ 2-2\sqrt{2} & 2+\sqrt{2}-\sqrt{6} & 2+\sqrt{2}+\sqrt{6} \end{pmatrix}$$

Notice this matrix has the peculiar property that the entries of each row/column sum to 1.

$$\sum_{i=1}^{3} (U_e)_{ik} = 1,\ k = 1,2,3 \quad \text{and} \quad \sum_{j=1}^{3} (U_e)_{kj} = 1,\ k = 1,2,3$$

We can use this mixing matrix to derive the non-diagonal charged lepton flavor symmetries, but first....

A Residual Taste of Neutrinos

(C. Alvarado, J. Bautista, AS (2022))

We assume that the neutrino sector yields TBM mixing. Therefore, it must have the residual $Z_2^S \times Z_2^U$ symmetry associated with TBM mixing, i.e.,

$$G_i^{\text{Diag}} = U_\nu^\dagger G_i U_\nu$$

$$U^{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$U = G_3^{\text{TBM}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad S = G_2^{\text{TBM}} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

So, that was easy. What about the charged lepton residual symmetry now?

A Residual Taste of Charged Leptons

(C. Alvarado, J. Bautista, AS (2022))

Assume non-degenerate charged lepton masses. Thus, we must take one of the following forms for the diagonal charged lepton symmetries:

$$\omega = e^{\frac{2\pi i}{3}}$$

$$T_1^{\mathrm{Diag}} = \mathrm{Diag}(1, \omega, \omega^2),\ T_2^{\mathrm{Diag}} = \mathrm{Diag}(\omega^2, 1, \omega),\ T_3^{\mathrm{Diag}} = \mathrm{Diag}(\omega, \omega^2, 1),$$
$$(T_1^{\mathrm{Diag}})^2 = \mathrm{Diag}(1, \omega^2, \omega),\ (T_2^{\mathrm{Diag}})^2 = \mathrm{Diag}(\omega, 1, \omega^2),\ (T_3^{\mathrm{Diag}})^2 = \mathrm{Diag}(\omega^2, \omega, 1)$$

Notice that,

$$\omega^2 T_1^{\mathrm{Diag}} = T_2^{\mathrm{Diag}} = \omega T_3^{\mathrm{Diag}} \qquad \text{and} \qquad \omega (T_1^{\mathrm{Diag}})^2 = (T_2^{\mathrm{Diag}})^2 = \omega^2 (T_3^{\mathrm{Diag}})^2$$

Therefore, it is possible to transform from one of these matrices to another, but by multiplying by a power of ω. Notice, if $T_i^\dagger M_e T_i = M_e$, then $(T_i^\dagger)^2 M_e T_i^2 = M_e$

Can find all non-diagonal charged lepton symmetries with $T_i = U_e T_i^{\mathrm{Diag}} U_e^\dagger$,
(but we only need to undiagonalize one…)

Great! Now we have all the possible non-diagonal forms of the residual lepton symmetry elements, but can *S*, *U*, and *T* close to form a group?

Closing the Flavor Gate

(C. Alvarado, J. Bautista, AS (2022))

A straightforward calculation reveals that $S, U, \text{ and } all\ T_i$ obey

$$(UT_i^{-1}UT_i)^3 = 1, \qquad (SU)^2 = 1, \qquad (ST_i)^3 = 1$$

However, only T_2 satisfies

$$(UT_2)^8 = (UT_2^2)^8 = 1$$

Because 3 does not divide 8. For the same reason, it also only satisfies

$$S = U(UT_2)^4U(UT_2)^4 = U(UT_2^2)^4U(UT_2^2)^4 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

Notice that, $T_2 \leftrightarrow T_2^2$ will also satisfy these rules. Therefore, without loss of generality let

$$T = T_2$$

The presentation rules suggest that the larger flavor group is $\Delta(96)$, but......

Making Sure It Has the Right Flavor

(C. Alvarado, J. Bautista, AS (2022))

A scan over random products up to length 8 of *S, T,* and *U* was performed of $\mathcal{O}(10^4)$ was performed. This scan revealed 96 distinct elements. Additionally, the orders and characters/traces of these elements were checked and found to be all consistent with $\Delta(96)$.

Important: We did not assume a flavor symmetry to then generate BTM mixing. We assumed BTM mixing and derived the most general residual symmetry elements consistent with the mixing. We then derived the algebraic relations among them to arrive at $\Delta(96)$.

What happens if we add CP?

Adding CP to the Recipe

(C. Alvarado, J. Bautista, AS (2022))

Assume the PDG convention for the MNSP matrix and allow for the existence of a non-zero CP violating phase by including it in the group elements. Thus,

$$U_e \to U_e(\delta) = U^{\mathrm{TBM}}[U^{\mathrm{BTM}}(\delta)]^\dagger$$

$$U^{\mathrm{BTM}}(\delta) = \begin{pmatrix} \frac{1}{6}\left(3+\sqrt{3}\right) & \frac{1}{\sqrt{3}} & \frac{1}{6}\left(3-\sqrt{3}\right)e^{-i\delta} \\ \frac{3\left(\sqrt{3}-1\right)+\left(2\sqrt{3}-3\right)e^{i\delta}}{6\sqrt{3}-15} & \frac{-3+\left(9-5\sqrt{3}\right)e^{i\delta}}{6\sqrt{3}-15} & \frac{1}{\sqrt{3}} \\ \frac{1}{78}\left(12\left(\sqrt{3}-4\right)+\left(9+\sqrt{3}\right)e^{i\delta}\right) & \frac{\left(3-2\sqrt{3}\right)e^{i\delta}+3-3\sqrt{3}}{6\sqrt{3}-15} & -\frac{1}{6}\left(3+\sqrt{3}\right) \end{pmatrix}$$

The complicated expression above guarantees a complicated form for $U_e(\delta)$ (see Appendix B in **2312.15391** explicit form). The inclusion of CP violation affects the residual charged lepton symmetry as well, i.e., $T_i \to T_i(\delta) \equiv U_e(\delta)T_i^{\mathrm{Diag}}[U_e(\delta)]^\dagger$

Since, we are assuming only TBM in the neutrino sector *S* and *U* are easy, i.e.,

$$S = G_2^{\mathrm{TBM}} \text{ and } U = G_3^{\mathrm{TBM}}$$

Only thing left to do is...

How Does CP Change the Taste?

(C. Alvarado, J. Bautista, AS (2022))

A tedious check reveals that not only $T(\delta)$ is an order-3 element, but it also (surprisingly) satisfies the same relationships with *U* as before, i.e.,

$$[UT^{-1}(\delta)UT(\delta)]^3 = 1, \qquad [UT(\delta)]^8 = 1$$

Remarkably, other relationships can be found. To see them, let

$$\mathcal{S}(\delta) \equiv U\Big(UT(\delta)\Big)^4 U\Big(UT(\delta)\Big)^4$$

This element is still order 2 and satisfies

$$\Big[\mathcal{S}(\delta)T(\delta)\Big]^3 = 1, \qquad [\mathcal{S}(\delta)U]^2 = 1$$

As a result, the presentation rules with no CP violation, generalize to these presentation rules with a CP-violating phase included.

Also, $T_2(\delta) \leftrightarrow T_2^2(\delta)$ leaves the rules unchanged like before.

What's going on here?

An Unfortunate Truth of Δ(96)

(C. Alvarado, J. Bautista, AS (2022))

It is possible to *allow* for the existence of CP-violation in BTM mixing by including it as an *arbitrary* phase in the MNSP lepton mixing matrix. This in-turn leads to residual symmetry group elements which contain the phase in this paradigm.

Then, it is possible to calculate the relationships these elements satisfy. Doing so shows that they fulfill the presentations rules of Δ(96).

However, this group cannot support a non-zero CP-violating phase because it is possible to represent this group in a basis where all of the CG-coefficients are real. Thus, this cannot be real CP.

This same result was also concluded by two different groups using two different methods than ours (M. Holthausen, M. Lindner and M. Schmidt, (2012); M. Chen, M. Fallbacher, K. Mahanthappa, M. Ratz and A. Trautner(2014))

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This confirms our method works and can be used to look for a group which can generate a predictions for CP violation!

Conclusion

- Why the different particles have the different masses and mixings that they do is still an open question in particle physics.... Perhaps more importantly, 'Why do 3 flavors exist?'
- The Bottom-Up Approach introduced in **1501.04336** and further generalized in **1611.03020** puts forth a framework to better understand lepton mixing by expressing the residual lepton flavor symmetries in terms of the parameters in the PDG parametrization of the MNSP lepton mixing matrix. It also includes a way to understand Majorana phases with Generalized CP symmetries.
- Marrying this approach with the idea of using group presentations (introduced in **2211.07785**) to further understand the lepton flavor group yields a new paradigm which can reproduce all results in the literature as well as make additional predictions.
- Finally, it is possible to connect BTM and TBM mixings through their trimaximality, as shown in **2312.15391**. In doing so, a paradigm for finding a group which can predict a non-zero CP-violating phase was introduced. Additionally, this work contains a basis-independent proof showing that only the MNSP matrix matters in dictating the symmetry group.

Back-up Slides

A Caveat

If low-energy parameters are not taken as inputs for generating the possible predictions for the Klein symmetry elements, it is possible to generate them by breaking a flavor group G_f *to* $Z_2 \times Z_2$ in the neutrino sector and Z_m in the charged lepton sector, while also consistently breaking H_{CP} *to* $X_{i.}$

Then predictions for parameters can become subject to charged lepton (CL) corrections, renormalization group evolution (RGE), and canonical normalization (CN) considerations.

Although, one can expect these corrections to be sub-leading as RGE and CN effects are expected to be small in realistic models with hierarchical neutrino masses, and CL corrections are typically at most Cabibbo-sized.(J. Casa, J. Espinosa, A. Ibarra, I Navarro (2000);
S. Antusch, J Kersten, M. Lindner, M. Ratz (2003); S. King I. Peddie (2004);
S. Antusch, S. King, M. Malinsky (2009);)

Seesaw Mechanism (Type-I)

(P. Minkowski (1977); M. Gell-Mann, P. Ramond, R. Slansky (1979);
T. Yanagida (1980); R. Mohapatra, G.Senjanovic (1980)....)

Could just add right-handed neutrino fields to Standard Model.... But then neutrino Yukawa coupling is $\sim 10^{-12}$! Very small, even by comparison with electron, $y_e \sim .5\times 10^{-5}$. Instead add three heavy right-handed Majorana neutrinos with mass around GUT scale ($M_N \sim 10^{15}$ GeV) generating both Dirac and Majorana mass term for neutrinos:

$$\mathcal{L}_\nu = -Y^\nu_{ij}\overline{L}_i \epsilon H^* N_j - \frac{1}{2}(M_N)_{ij} N_i N_j + h.c.$$

Then the 6x6 neutrino mass matrix is: $M = \begin{pmatrix} 0 & M_\nu^D \\ (M_\nu^D)^T & M_N \end{pmatrix}$ $M_\nu^D = vY^\nu$

With fair assumption that scale of Dirac mass terms is less than scale of Majorana mass terms ($M_\nu^D << M_N$), we can obtain light neutrino masses by integrating out heavy right-handed states and diagonalizing:

$$M_\nu = -M_\nu^D M_N^{-1} (M_\nu^D)^T$$

To yield a light neutrino mass comparable to 1 eV.

Heavy mass states can be found by diagonalizing M_N.

How do we explain the smallness of other fermion masses?

Froggatt-Nielsen Mechanism

(C. Froggatt and H. Nielsen (1979))

Propose a flavor symmetry (originally U(1)) that is spontaneously broken by a set of additional scalar fields (flavons). Couple these fields to Yukawa terms rendering them non-renormalizable, e.g.:

$$\frac{Y_{ij}\phi\overline{Q_i}Hd_j}{\Lambda} = Y'_{ij}\overline{Q_i}Hd_j$$

After the the flavor symmetry is spontaneously broken by the flavon acquiring a vev:

$$\frac{Y_{ij} < \phi > \overline{Q_i}Hd_j}{\Lambda} = Y_{ij}\lambda\overline{Q_i}Hd_j \qquad \lambda = \frac{< \phi >}{\Lambda} \approx .22$$

Notice that 'Bare' Yukawa coupling is O(1), but effective Yukawa coupling is smaller:

$$Y'_{ij} = Y_{ij}\lambda$$

Smaller Yukawa couplings can be generated by couplings to more flavons.

So, this more or less takes care of the fermion mass hierarchy problem. But what about the different mixing angles?

Undiagonalizing the Diagonal

(C. Alvarado, J. Bautista, AS (2022))

$$M_e \equiv m_e m_e^\dagger$$

$$M_e^{\text{Diag}} = U_e^\dagger M_e U_e = \text{Diag}\{|m_e|^2, |m_\mu|^2, |m_\tau|^2\}$$

$$M_e = \begin{pmatrix} a^2|m_e|^2 + b^2|m_\tau|^2 + c^2|m_\mu|^2 & ab|m_e|^2 + ac|m_\mu|^2 + bc|m_\tau|^2 & ab|m_\tau|^2 + ac|m_e|^2 + bc|m_\mu|^2 \\ ab|m_e|^2 + ac|m_\mu|^2 + bc|m_\tau|^2 & a^2|m_\mu|^2 + b^2|m_e|^2 + c^2|m_\tau|^2 & ab|m_\mu|^2 + ac|m_\tau|^2 + bc|m_e|^2 \\ ab|m_\tau|^2 + ac|m_e|^2 + bc|m_\mu|^2 & ab|m_\mu|^2 + ac|m_\tau|^2 + bc|m_e|^2 & a^2|m_\tau|^2 + b^2|m_\mu|^2 + c^2|m_e|^2 \end{pmatrix}$$

$$a \equiv \frac{1}{6}(2+\sqrt{2}+\sqrt{6}) = 0.977 \approx \lambda^0 \quad b \equiv \frac{1}{6}(2+\sqrt{2}-\sqrt{6}) = 0.161 \approx \lambda \quad c \equiv \frac{1}{6}(2-2\sqrt{2}) = -0.138 \approx -\lambda$$

Then, with these definitions, $U_e = \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}$ such that $a + b + c = 1.$

Make things even simpler by realizing the mass matrix is dominated by m_τ^2 because

$$m_\mu^2 \approx (\lambda^2 m_\tau)^2, \quad m_e^2 \approx (\lambda^5 m_\tau)^2$$

In the limit λ→0, these masses vanish! Obviously, so do *b* and *c*. Yet now we can parameterize the charged lepton mass matrix in a perturbative/simpler manner…

Simpler Tasting Charged Leptons

(C. Alvarado, J. Bautista, AS (2022))

$$M_e = |m_\tau|^2 \begin{pmatrix} b^2 & bc & ab \\ bc & c^2 & ac \\ ab & ac & a^2 \end{pmatrix} + |m_\tau|^2 \mathcal{O}(\lambda^4)$$

This parametrization reveals some rather notable relationships:

$$\sum_{i=1}^{3} (M_e)_{i1} = \sum_{i=1}^{3} (M_e)_{1i} = |m_\tau|^2 (b^2 + bc + ab) = b\,|m_\tau|^2\,,$$

$$\sum_{i=1}^{3} (M_e)_{i2} = \sum_{i=1}^{3} (M_e)_{2i} = |m_\tau|^2 (bc + c^2 + ac) = c\,|m_\tau|^2\,, \qquad a + b + c = 1$$

$$\sum_{i=1}^{3} (M_e)_{i3} = \sum_{i=1}^{3} (M_e)_{3i} = |m_\tau|^2 (ab + ac + a^2) = a\,|m_\tau|^2\,.$$

Now that we have the charged lepton mass matrix, we can explore the flavor symmetries associated with this non-diagonal charged lepton matrix and TBM neutrino sector.