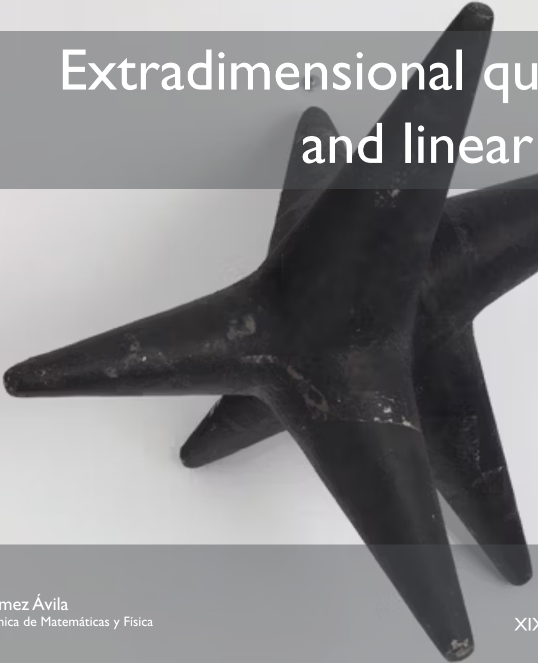
Extradimensional quantum numbers and linear wave equations



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The set of all QFTs is vast

Without a complete classification of all relativistic field theories, it is impossible to performa a systematic search in them of solutions to open problems. For this reason I believe that a crucial responsability of theorists is to push the boundaries of quantum field theory to eventually achieve such a classification.

Outline

- Fundamental ingredients of QFT
- Field theories and covariant bases
- Extradimensional excursions

Fundamental ingredients of QFT

Beyond the Standard Model can mean two things

1. The ingredients of the Standard Model, arranged in novel ways, or
2. entirely new ingredientes.

What are the possibilities for new ingredientes? That's constrained by known physics, which gives several no-go theorems and obstructions:

- Weinberg–Witten no-go theorem .
- Haag-Lopuszanski-Sohnius theorem.
- Ostrogradski unstability.
- Vafa-Witten theorem on global symmetries.

A (perturbative) QFT needs three ingredients

They are: a spacetime (fixing asymptotically free states), a gauge group (fixin the gauge bosons) and a matter content.

$$\mathcal{Z} = \underbrace{\int \prod D\Psi_i}_{\text{matter}} \; \exp \; \underbrace{\int_{\mathcal{M}} d^D x}_{\text{spacetime}} \; \mathcal{L}\Big(\underbrace{D^\mu}_{\text{gauge}} \Psi_j, \mathcal{V}(\Psi_j) \Big)$$

Historically: we build a noninteracting theory of matter fields and then we add interactions.

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Anomaly cancellation

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Gauge charges

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Kinematical algebra

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GUTS, LR, etc.

$$\mathcal{Z} = \underbrace{\int \prod D\Psi_i}_{\text{matter}} \exp \underbrace{\int_{\mathcal{M}} d^D x}_{\text{spacetime}} \, \mathcal{L}\Big(\underbrace{D^\mu}_{\text{gauge}} \Psi_j, \mathcal{V}(\Psi_j) \Big)$$

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They are: a spacetime (fixing asymptotically free states), a gauge group (fixin the gauge bosons) and a matter content.

XD, SUSY → spacetime

$$\mathcal{Z} = \underbrace{\int \prod D\Psi_i}_{\text{matter}} \; \exp \; \underbrace{\int_{\mathcal{M}} d^D x}_{\text{spacetime}} \; \mathcal{L}\Big(\underbrace{D^\mu}_{\text{gauge}} \Psi_j, \mathcal{V}(\Psi_j) \Big)$$

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NHDM, High Spin

$$\mathcal{Z} = \underbrace{\int \prod D\Psi_i}_{\text{matter}} \exp \underbrace{\int_{\mathcal{M}} d^D x}_{\text{spacetime}} \mathcal{L}\Big(\underbrace{D^\mu}_{\text{gauge}} \Psi_j, \mathcal{V}(\Psi_j) \Big)$$

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String theory

Historically: we build a noninteracting theory of matter fields and then we add interactions.

The matter sector is the least sistematically understood

$$\mathscr{L} = \underbrace{\mathscr{L}_{\text{gauge}}(\Psi, A^\mu)}_{\text{3 coupling constants + 1 phase}} + \underbrace{\mathscr{L}_{\text{Higgs}}(\Phi, A^\mu)}_{\text{1 mass \& 1 vev}} + \underbrace{\mathscr{L}_{\text{flavor}}(\Psi, \Phi)}_{\text{(9) 12 masses, (3) 6 angles \& (1) 2 phases}} .$$

- The gauge Lagrangian is very economical.
- Most SM parameters come from the flavor Lagrangian .
- Formal simplicity is achived by specifying the gauge group and the matter multiplets. There is nothing similar for the matter Lagrangian.

One-particle states are Poincaré irreps

The Poincaré algebra has two algebraic invariants

$$\mathcal{C}_2 = P^\mu P_\mu, \qquad \mathcal{C}_4 = W^\mu W_\mu \quad \text{with} \quad W_\mu = \tfrac{1}{2}\varepsilon_{\mu\sigma\tau\rho} M^{\sigma\tau} P^\rho.$$

One-particle states satisfy

$$\mathcal{C}_2 \left|\Psi\right\rangle = m^2 \left|\Psi\right\rangle, \qquad \mathcal{C}_4 \left|\Psi\right\rangle = -m^2 j(j+1) \left|\Psi\right\rangle$$

We call m the mass and j the spin of Ψ.

The Lorentz algebra is the homogeneous part of the Poincaré algebra. This algebra $\mathfrak{so}(1,3) \cong \mathfrak{su}(2)_A \oplus \mathfrak{su}(2)_B$ is generated by the commuting sets

$$\mathbb{A} = \frac{1}{2}(\mathbb{J} - i\mathbb{K}), \qquad \mathbb{B} = \frac{1}{2}(\mathbb{J} + i\mathbb{K}).$$

We label the Lorentz irreps with the $\mathfrak{su}(2)$ numbers (a, b).

The SM includes a few possibilities

- Klein-Gordon's equation is the eigenvalue condition for P^2 in the scalar representation:
$$(0,0)\,.$$
- Dirac's equation is the parity projection for the representation
$$\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\,.$$
- Proca's equation for a massive vector field is the spin (i.e., Pauli-Lubanski) projection for $j=1$ in the representation
$$\left(\frac{1}{2},\frac{1}{2}\right)\,.$$

Even if we are conservative, others are theoretically possible

$$(0,0)$$

$$(\tfrac{1}{2},0) \qquad (0,\tfrac{1}{2})$$

$$(1,0) \qquad (\tfrac{1}{2},\tfrac{1}{2}) \qquad (0,1)$$

$$(\tfrac{3}{2},0) \qquad (1,\tfrac{1}{2}) \qquad (\tfrac{1}{2},1) \qquad (0,\tfrac{3}{2})$$

$$(2,0) \qquad (\tfrac{3}{2},\tfrac{1}{2}) \qquad (1,1) \qquad (\tfrac{1}{2},\tfrac{3}{2}) \qquad (0,2)$$

Even if we are conservative, others are theoretically possible

Higgs $\boxed{(0,0)}$

$$\left(\tfrac{1}{2},0\right) \qquad \left(0,\tfrac{1}{2}\right)$$

$$(1,0) \qquad \left(\tfrac{1}{2},\tfrac{1}{2}\right) \qquad (0,1)$$

$$\left(\tfrac{3}{2},0\right) \qquad \left(1,\tfrac{1}{2}\right) \qquad \left(\tfrac{1}{2},1\right) \qquad \left(0,\tfrac{3}{2}\right)$$

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Even if we are conservative, others are theoretically possible

$$(0,0)$$

Quarks & Leptones $\quad (\tfrac{1}{2},0) \quad (0,\tfrac{1}{2})$

$$(1,0) \quad (\tfrac{1}{2},\tfrac{1}{2}) \quad (0,1)$$

$$(\tfrac{3}{2},0) \quad (1,\tfrac{1}{2}) \quad (\tfrac{1}{2},1) \quad (0,\tfrac{3}{2})$$

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Even if we are conservative, others are theoretically possible

$$(0,0)$$

$$(\tfrac{1}{2},0) \quad (0,\tfrac{1}{2})$$

Gauge Bosons $\quad (1,0) \quad \boxed{(\tfrac{1}{2},\tfrac{1}{2})} \quad (0,1)$

$$(\tfrac{3}{2},0) \quad (1,\tfrac{1}{2}) \quad (\tfrac{1}{2},1) \quad (0,\tfrac{3}{2})$$

$$(2,0) \quad (\tfrac{3}{2},\tfrac{1}{2}) \quad (1,1) \quad (\tfrac{1}{2},\tfrac{3}{2}) \quad (0,2)$$

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Graviton? $(2,0) \qquad (\tfrac{3}{2},\tfrac{1}{2}) \qquad \boxed{(1,1)} \qquad (\tfrac{1}{2},\tfrac{3}{2}) \qquad (0,2)$

Even if we are conservative, others are theoretically possible

$$(0,0)$$

$$(\tfrac{1}{2},0) \qquad (0,\tfrac{1}{2})$$

$$\boxed{(1,0)} \qquad (\tfrac{1}{2},\tfrac{1}{2}) \qquad \boxed{(0,1)}$$

$$\boxed{(\tfrac{3}{2},0)} \qquad \boxed{(1,\tfrac{1}{2})} \qquad \boxed{(\tfrac{1}{2},1)} \qquad \boxed{(0,\tfrac{3}{2})}$$

$$\boxed{(2,0)} \qquad \boxed{(\tfrac{3}{2},\tfrac{1}{2})} \qquad (1,1) \qquad \boxed{(\tfrac{1}{2},\tfrac{3}{2})} \qquad \boxed{(0,2)}$$

What about these?

Are they allowed?

If not, what forbids them?

Field theories and covariant bases

General constructions run into consistency obstructions

1. A general (a, b) field contains spins $|a - b| \dots (a + b)$; theory must include constraints to remove unwanted spins. These may propagate as ghosts or produce other acausalities when interactions are introduced [1].
2. In some cases, the only kinematical terms available produce a higher-derivative theory, which may run into the Ostrogradski unstability[2].
3. Massless $j > 2$ fields imply a trivial S-matrix, rendering interactions inconsistent[3].

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[1] Velo & Zwanziger, Noncausality and Other Defects of Interaction Lagrangians for Particles with Spin One and Higher, 1969

[2] Woodard, Avoiding Dark Energy with 1/R Modifications of Gravity, 2006

[3] Weinberg & Witten, Limits on Massless Particles, 10.1016/0370-2693(80)90212-9, 1980

We conventionally use parity-invariant representations

Parity exchanges the (a, b) Lorentz labels; parity-invariant representations are:

- Non–chiral (a, a) irreps, which correspond to integer spin Fierz-Pauli.
- Chiral, reducible $(a, b) \oplus (b, a)$ with $a \neq b$.

Those with only two spin sectors (including the three in the SM, plus others) can be enumerated:

1. The non-chiral $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$.
2. The single spin chiral representations (SSχR) $(j, 0) \oplus (0, j)$ with $j \geq \frac{1}{2}$ (Joos-Weinberg representations).
3. The double spin chiral representations (DSχR) $\left(j - \frac{1}{2}, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, j - \frac{1}{2}\right)$ for $j > 1$.

Duffin-Kemmer-Petiau and Rarita-Schwinger are exceptions

Other formalisms have been used to study the properties of hadrons, which don't fit in this scheme. One is the Rarita-Schwinger field, transforming as

$$\left(1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 1\right) \oplus \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right).$$

Historically this came about from the product of a spinor and a vector. The other one is the Duffin-Kemmer-Petiau theory, using the 10–dimensional vector–tensor representation

$$(1, 0) \oplus (0, 1) \oplus \left(\frac{1}{2}, \frac{1}{2}\right).$$

This is a puzzling construction: the propagating dofs comprise a mixture from different irreps of the Lorentz algebra including both chiral and non-chiral representations. Let us try to understand it!

The defining property of DKP is the meson algebra

The Duffin–Kemmer–Petiau relativistic wave equation was used in the XX century to describe the properties of spin zero and spin one mesons. At the heart of the DKP theory is the meson algebra

$$S_\rho S_\mu S_\nu + S_\nu S_\mu S_\rho = g_{\mu\nu} S_\rho + g_{\mu\rho} S_\nu, \tag{1}$$

which has nontrivial 5 and 10 dimensional representations. This algebra permits the construction of a Dirac-like first order wave equation

$$(S_\mu \partial^\mu - m)\Psi = 0. \tag{2}$$

It is not very popular anymore.

The representations of $\mathfrak{g}$ carry additional structure

Given a Lie algebra $\mathfrak{g}$ over F, the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is the infinite sum

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} T^n(\mathfrak{g}),$$

where $T^0 = \mathbb{F}$, and T^n is the n-fold product $\mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$, modded by the Lie product $x \diamond y = [x, y]$. It turns out that every representation $R : \mathfrak{g} \to End(V)$ of $\mathfrak{g}$ can be uniquely extended to a representation of $\mathcal{U}(\mathfrak{g})$!.

Fundamentally, this happens because composition of maps gives not just the Lie product of the generators $\{T_i\}$, but the more general matrix product

$$R(T_i \otimes T_j) = R(T_i)R(T_j).$$

This representation of the $\mathcal{U}(\mathfrak{g})$ carries a dual algebraic structure, called a Jordan-Lie algebra, where the products are the anticommutator and commutator.

Lie algebra representations are Jordan-Lie algebras

A Jordan algebra is an abelian non associative algebra with bilineal product $\bullet$:

$$x \bullet y = y \bullet x$$
$$(x \bullet y) \bullet (x \bullet x) = x \bullet [y \bullet (x \bullet x)].$$

If the Jordan product $\bullet$ coexists with a Lie product $\diamond$ fulfilling Leibnitz rule

$$(a \bullet b) \diamond c = a \bullet (b \diamond c) + (a \diamond c) \bullet b,$$

and the associator rule

$$(a \bullet b) \bullet c - a \bullet (b \bullet c) = \frac{\hbar^2}{4}(a \diamond c) \diamond b,$$

we get a Jordan-Lie algebra. By constructing this algebra, we can find all possible covariant operators acting on the representation space[4].

[4] S. Gómez-Ávila, M. Napsuciale, Covariant basis induced by parity for the $(j, 0) \oplus (0, j)$ representation, 10.1103/PhysRevD.88.096012, 2013

From Jordan-Lie we get covariant kinetic terms

Given a Lorentz irrep V, the associated Jordan-Lie algebra provides a basis for $End(V)$. We build a wave equation for each kinematical term, that is, every covariant operator that can be contracted with momenta.

A very well-known example is the Dirac representation:

$$\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]^2=(0,0)_2\oplus(1,0)\oplus(0,1)\oplus\left(\frac{1}{2},\frac{1}{2}\right)_2.$$

This corresponds to the following basis for $End(V)$:

$$\{\mathbb{1},\gamma_5,\gamma_\mu,\gamma_5\gamma_\mu,\sigma_{\mu\nu}\}.$$

The kinematical terms available are: $\gamma_\mu P^\mu, \gamma_5\gamma_\mu P^\mu$ and $P_\mu P^\mu$. The other terms are forbidden by symmetry[5].

[5] Ferro, Olmos, Peinado & Vaquera, Quantization of second-order fermions, doi:10.1103/PhysRevD.109.085003.

From kinetic terms we get field theories

1. We start with some representation of the Lorentz algebra $\mathfrak{g}$.
2. We define new elements through the Jordan product $\mathfrak{g} \bullet \mathfrak{g}$.
3. We evaluate the Jordan and Lie products of these new elements with $\mathfrak{g}$ and itself.
4. We continue this process until the algebra closes and we have a complete basis for $End(V)$.
5. We classify all kinematical terms (symmetric operators), and construct the corresponding wave equations.
6. The remaining operators classify all possible self-interactions, and bilinear couplings to other fields.

Kinetic terms for the SSχR

In the single–spin chiral representations there are two possible kinetic operators. One comes from

$$(j,0) \otimes (j,0) \oplus (0,j) \otimes (0,j) \simeq (0,0)_2 \oplus \bigoplus_{i=1}^{2j} (i,0) \oplus (0,i)\,. \qquad (3)$$

This scalar kinetic term will produce a Klein-Gordon equation of motion for every spinor component, tipically double the number of degrees of freedom desired. We also have anti block diagonal operators:

$$2((j,0) \otimes (0,j)) \simeq (j,j)_2\,. \qquad (4)$$

This correspond to another possible kinetic term, formed from a symmetric traceless tensor with $2j$ indices. This is of order $2j$ in the momenta; for $j > 1$, it corresponds to a higher–derivative theory.

...and for the DSχR

In the double spin chiral representation block diagonal operators in the direct–sum basis are

$$\begin{aligned} \left(j-\frac{1}{2},\frac{1}{2}\right)^{\otimes 2} &= \bigoplus_{r=0}^{2j-1}[(r,0)\oplus(r,1)], \\ \left(\frac{1}{2},j-\frac{1}{2}\right)^{\otimes 2} &= \bigoplus_{r=0}^{2j-1}[(0,r)\oplus(1,r)]. \end{aligned} \tag{5}$$

Since $j \geq \frac{3}{2}$, this will always include a pair of symmetric tensors $(1,1)$. Anti–block diagonal operators come from the cross product

$$2\left(j-\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},j-\frac{1}{2}\right) = 2\bigoplus_{r,s=j-1}^{j}(r,s). \tag{6}$$

The lowest order kinetic term comes from the anti diagonal product $(j-1,j-1)$, which means that we have a first-order wave equation for $j=3/2$, and a second-order wave equation for $j=2$.

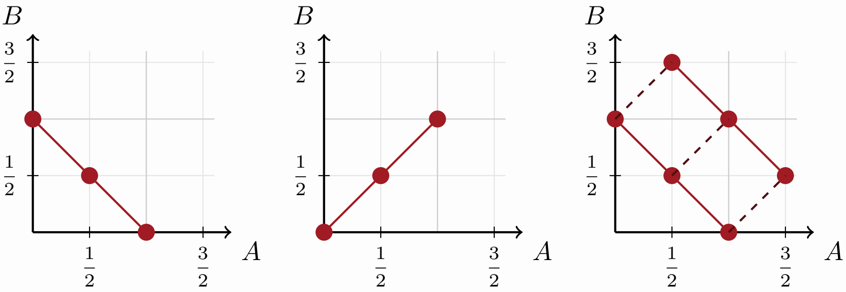
Extradimensional excursions

(This section includes results from the works of Jaqueline, Nahúm, Karina and Manuel)



Lorentz in 4+1 dimensions

Representations of the 4+1 Lorentz algebra are classified by two numbers m/n, defining the shape of a figure in the (a, b) space.



3/1 representation   1/3 representation   3/2 representation

4+1 irreps are in correspondence with sequences of 3+1 Lorentz irreps.

Poincaré in 4+1 dimensions

The 4+1 Poincaré algebra has rank 3 and therefore its irreducible representations are indexed by three Casimir operators. Two are well–known:

$$\mathcal{C}_2 = P^\mu P_\mu, \qquad \mathcal{C}_4 = W^{\rho\mu\nu} W_{\rho\mu\nu}, \quad \text{with} \quad W_{\rho\mu\nu} = \frac{1}{2} \sum_{(\rho\mu\nu)} M_{\mu\nu} P_\rho.$$

The new Casimir is lineal in the momenta:

$$\mathcal{C}_3 = \frac{1}{2^3} \epsilon^{\alpha\beta\mu\nu\rho} M_{\alpha\beta} M_{\mu\nu} P_\rho. \tag{7}$$

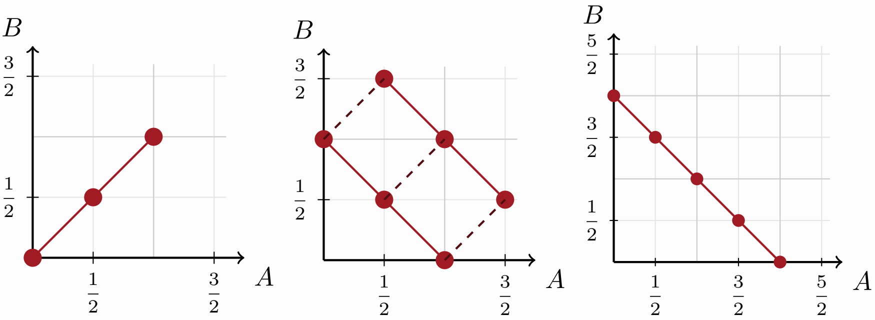
The problem of building free QFTs for the extradimensional theory amounts to constructing the induced representations with good $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ quantum numbers.

The linear Casimir projects linear equations

1+4 Irrep	Lorentz Projection	$\mathcal{C}_3$
1/1	Scalar Field	0
1/2	Spin Zero DKP	0
2/1	Dirac Field	$\gamma_\mu P^\mu$
2/2	Rarita-Schwinger	$\gamma'_\mu P^\mu$
1/3	Spin 2 Symmetric + Vector + Scalar	0
3/1	Spin 1 DKP	$\beta_\mu P^\mu$

All of the three known linear equations can be understood as the $3 + 1$ projection of the $4 + 1$ $\mathcal{C}_3$ eigenvalue equation.

In the bosonic case, two spin 2 linear equations exist



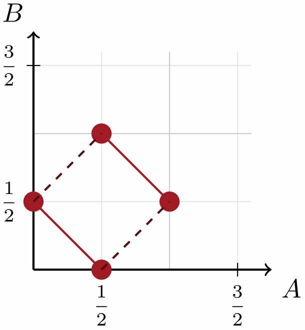
1/3 TeVeS representation with $\mathcal{C}_3 = 0$.

3/2 Connection + Tensor + DKP.

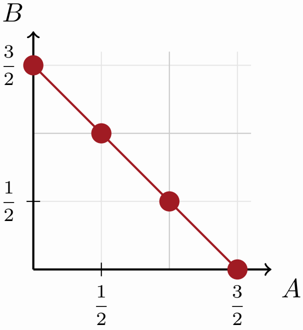
5/1 Weyl + Connection + Tensor.

Both $3/2$ and $5/1$ carry three spin sectors $\Rightarrow$ constraints might be bad.

For spin $\frac{3}{2}$, there is a novel possibility



2/2 Rarita-Schwinger representation



4/1 Novel spin $\frac{3}{2}$ representation

The 4/1 corresponds to the

$$\left(\frac{3}{2},0\right)\oplus\left(\frac{3}{2},0\right)\oplus\left(1,\frac{1}{2}\right)\oplus\left(\frac{1}{2},1\right),$$

a 20-dimensional representation. The covariant vector operator is

$$S_\mu \equiv \frac{1}{2^3}\eta_{\mu\nu}\epsilon^{\nu\alpha\beta\rho\sigma}\{M_{\alpha\beta},M_{\rho\sigma}\},$$

which we can use to write a linear equation for this field.

The $\mathfrak{so}(1,4)$ 20 representation decomposes as:

$$20 \otimes 20 = 1 \oplus 5 \oplus 10 \oplus 14 \oplus 30 \oplus 35 \oplus 35' \oplus 81 \oplus 84 \oplus 105, \tag{8}$$

corresponding to the Young diagrams

$$\begin{gathered} \mathbb{1} \oplus \ydiagram{1} \oplus \ydiagram{1,1} \oplus \ydiagram{2} \oplus \ydiagram{3} \oplus \ydiagram{2,1} \\ \oplus \ydiagram{2,2} \oplus \ydiagram{3,1} \oplus \ydiagram{3,3} \oplus \ydiagram{3,2} \,. \end{gathered} \tag{9}$$

The branching rules into $3+1$ Lorentz tell us that there is a rich structure with linear, quadratic, and higher derivative kinetical terms possible.

Dimensional reduction provides many possibilities

$$
\begin{aligned}
1 &\hookrightarrow 1 \\
5 &\hookrightarrow 1 \oplus 4 \\
.10 &\hookrightarrow 3 \oplus \bar{3} \oplus 4 \\
14 &\hookrightarrow 1 \oplus 4 \oplus 9 \\
30 &\hookrightarrow 1 \oplus 4 \oplus 9 \oplus 16 \\
35' &\hookrightarrow 3 \oplus \bar{3} \oplus 4 \oplus 8 \oplus \bar{8} \oplus 9 \\
35' &\hookrightarrow 5 \oplus \bar{5} \oplus 8 \oplus \bar{8} \oplus 9 \\
81 &\hookrightarrow 3 \oplus \bar{3} \oplus 4 \oplus 8 \oplus \bar{8} \oplus 9 \oplus 15 \oplus \bar{15} \oplus 16 \\
84 &\hookrightarrow 7 \oplus \bar{7} \oplus 12 \oplus \bar{12} \oplus 15 \oplus \bar{15} \oplus 16 \\
105 &\hookrightarrow 5 \oplus \bar{5} \oplus 9 \oplus 8 \oplus \bar{8} \oplus 12 \oplus \bar{12} \oplus 15 \oplus \bar{15} \oplus 16
\end{aligned}
\tag{10}
$$

There are multiple vectors providing linear kinetic terms, as well as second and third rank symmetric traceless tensors. The covariant parity eigenvalue equation will involve this third-rank symmetric tensor.

Conclusions

- We linked quantum numbers in $4+1$ Minkowski to field theories in $3+1$.
- The three known relativistic linear theories can be projected from the eigenvalue equation for $\mathcal{C}_3$.
- In this sense, Dirac's, Rarita-Schwinger and Kemmer-Duffin-Petiau are related.
- An unexplored (as far as I know) alternative to Rarita-Schwinger was identified.
- There is also a pair of spin 2 linear equations mixing Weyl's, Levi-Civita and Proca representations.

Thank you for your attention!