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Can non-linear boundary conditions lead to new non-perturbative physics?

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Based on

- F. Canfora, D. Dudal, T. Oosthuyse, P. Pais, L. Rosa and S. Stouten, “Non-perturbative corrections to the Casimir energy for a toy scalar field theory with non-linear boundary conditions”, to appear (2025 or 2026).

- But the answer is YES! ☺

Overview

Motivation for this work

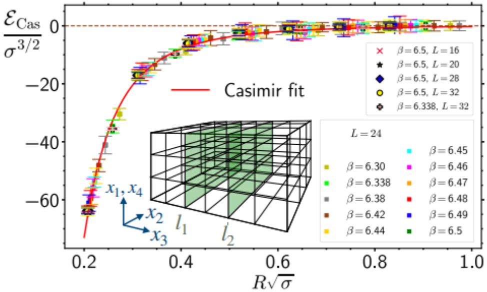
A scalar toy model

Non-linear Dirichlet boundary conditions

Non-linear Neumann boundary conditions

Non-Abelian Casimir energy: lattice results

Consider pure $4D$ $SU(2)$ Yang-Mills, 2 parallel plates at $z = 0$, $z = R$ and impose $n_\mu \tilde{F}_{\mu\nu}\big|_{bdy} = 0$, with $n_\mu = (0, 0, 0, 1)$.



$$\mathcal{E}_{\rm Cas} = -C_0 \frac{2(N_c^2-1)m_{\rm gt}^2}{8\pi^2 R} \sum_{n=1}^{\infty} \frac{K_2(2n m_{\rm gt} R)}{n^2} .$$

Figure taken from Chernodub et al, Phys.Rev. D **108** (2023) 1, 014515. The Casimir energy can be fitted with a gluon mass scale $m_{gt} = 0.49(5)$ GeV. This is *not* a glueball mass scale! Interestingly, in the very same ballpark of dynamical gluon masses appearing in a pleiad of other works, see other talks during this meeting, in particular that of Sebbe Stouten on Wednesday! $C_0 = 5.60(7)$ is a phenomenological fitting constant.
Similar results reported for $SU(3)$ and/or $3D$, see Ngwenya et al, 2507.21333 [hep-lat].

Role non-linear boundary conditions

Notice that

$$n_\mu \tilde{F}^a_{\mu\nu}\Big|_{bdy} = n_\mu \varepsilon_{\mu\nu\rho\sigma}(\partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho - gf^{abc} A^b_\rho A^c_\sigma F)\Big|_{bdy} = 0.$$

These conditions are gauge invariant (albeit that $n_\mu \tilde{F}^a_{\mu\nu}$ is not), but also non-linear.

Non-linear terms usually mean interaction terms, but how to take into consideration these boundary interactions in a path integral approach? And what are the potential consequences?

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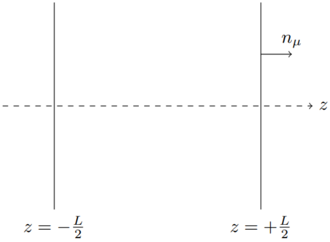
Non-linear boundary conditions

Consider a free massive scalar in the following geometry



We impose non-linear boundary conditions

$$\text{Dirichlet: } \varphi + \frac{g}{2}\varphi^2\Big|_{z=\pm L/2} = 0\,, \quad \text{or} \quad \text{Neumann: } \partial_z\varphi + \frac{g}{2}\varphi^2\Big|_{z=\pm L/2} = 0$$

via

$$\int d^4x \left[-\frac{1}{2}\varphi(\partial^2 - m^2)\varphi - b\delta\left(z + \frac{L}{2}\right)\left((\partial_z)\varphi + \frac{g}{2}\varphi^2\right) - \bar{b}\delta\left(z - \frac{L}{2}\right)\left((\partial_z)\varphi + \frac{g}{2}\varphi^2\right)\right].$$

$b(\tilde{x})$ and $\bar{b}(\tilde{x})$ are auxiliary fields imposing the boundary conditions. We set $\tilde{x} = (t, x, y)$.

What happens at the quantum level?

Dynamical boundary mass

We study 2 classes of boundary conditions:

$$\text{(Dirichlet)}\ \varphi + \frac{g}{2}\varphi^2\Big|_{z=\pm L/2} = 0 \quad \text{or} \quad \text{(Neumann)}\ \partial_z\varphi + \frac{g}{2}\varphi^2\Big|_{z=\pm L/2} = 0$$

By integrating out the $4D$ scalar field, a $3D$ effective (dynamical!) theory for the boundary dynamics is built in terms of the $(b, \bar{b})$-fields.

We allow for a dynamically generated boundary mass, via non-trivial VEVs

$$\begin{cases} b = b_0 + \beta & ; \quad b_0 = \langle b \rangle \\ \bar{b} = \bar{b}_0 + \bar{\beta} & ; \quad \bar{b}_0 = \langle \bar{b} \rangle \end{cases}$$

to be extracted from the quantum effective potential.

Two-point Green's function, first without boundary conditions

Relevant quadratic part of the action:

$$S_2 = \int d^4x \left[-\frac{1}{2}\varphi(\partial^2 - m^2)\varphi + \Delta(z)\varphi^2 \right],$$

with

$$\Delta(z) = -\frac{g}{2}\left(b_0\delta_+ + \bar{b}_0\delta_-\right), \qquad \delta_\pm \equiv \delta(z \pm L/2)$$

the boundary mass part!

The Green's function obeys, in mixed coordinates,

$$\left(-\partial_z^2 + k^2 + m^2 + \Delta(z)\right) G(\vec{k}, z, z') \equiv \mathcal{M}(z) G(\vec{k}, z, z') = \delta(z - z').$$

For $\Delta(z) = 0$:

$$G_0(\vec{k}, z, z') = \frac{1}{2\sqrt{k^2 + m^2}} e^{-|z-z'|\sqrt{k^2+m^2}}.$$

Two-point Green's function, first without boundary conditions

For $\Delta(z) \neq 0$; Dyson-Schwinger equation:

$$G(z,z') = G_0(z,z') - \int dx G_0(z,x)\Delta(x)G(x,z')$$

or, concretely

$$G(z,z') = G_0(z,z') + gb_0 G_0(z,-L/2)G(-L/2,z') + g\bar{b}_0 G_0(z,L/2)G(L/2,z')$$

After some manipulations, exact solution:

$$\begin{cases} G(-L/2,z') &= -\dfrac{g\bar{b}_0 e^{-\sqrt{k^2+m^2}(L+|L/2-z'|)} + e^{-\sqrt{k^2+m^2}|L/2+z'|}(-g\bar{b}_0 + 2\sqrt{k^2+m^2})}{g^2 b_0 \bar{b}_0 e^{-2L\sqrt{k^2+m^2}} - (gb_0 - 2\sqrt{k^2+m^2})(g\bar{b}_0 - 2\sqrt{k^2+m^2})} \\ G(+L/2,z') &= -\dfrac{gb_0 e^{-\sqrt{k^2+m^2}(L+|L/2+z'|)} + e^{-\sqrt{k^2+m^2}|L/2-z'|}(-gb_0 + 2\sqrt{k^2+m^2})}{g^2 b_0 \bar{b}_0 e^{-2L\sqrt{k^2+m^2}} - (gb_0 - 2\sqrt{k^2+m^2})(g\bar{b}_0 - 2\sqrt{k^2+m^2})} \end{cases} .$$

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The effective potential, now with boundary conditions

We must compute

$$
\begin{aligned}
& \int [\mathcal{D}\varphi][\mathcal{D}\beta][\mathcal{D}\bar{\beta}] \exp\left[-S_2 - \int d^4x \varphi\left((b_0+\beta)\delta_+ + (\bar{b}_0+\bar{\beta})\delta_-\right)\right]) \\
= \; & \int [\mathcal{D}\beta][\mathcal{D}\bar{\beta}] \exp\left[\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3} dzdz' \left(J(\vec{k},z)G(z,z')(2\pi)^3\delta^3(\vec{k}-\vec{k}')J(\vec{k}',z')\right)\right] \\
& \times \exp\left(-\frac{1}{2}\ln\det\mathcal{M}\right)
\end{aligned}
$$

with

$$
J(\vec{k},z) = \left(b_0(2\pi)^3\delta^{(3)}(\vec{k}) + \beta(\vec{k})\right)\delta_+ + \left(\bar{b}_0(2\pi)^3\delta^{(3)}(\vec{k}) + \bar{\eta}(\vec{k})\right)\delta_-
$$

the “source term”.

The det $\mathcal{M}$

Consider the "Schrödinger equation"

$$\mathcal{M} u_n = \lambda_n u_n,$$

in particular, its zero mode $\lambda_0 = 0$. As normalizable solution, we find

$$u_0(z) = \begin{cases} a e^{\sqrt{k^2+m^2} z} & \text{for} \quad z < -L/2 \\ b e^{-\sqrt{k^2+m^2} z} + c e^{\sqrt{k^2+m^2} z} & \text{for} \quad -L/2 < z < L/2 \\ d e^{-\sqrt{k^2+m^2} z} & \text{for} \quad L/2 < z \end{cases}$$

"Gluing" at the boundaries: $u_0(z)$ continuous, and jumps in its derivatives:

$$\begin{cases} u_0(-L/2-\varepsilon) - u_0(-L/2+\varepsilon) = 0 \\ u_0(+L/2+\varepsilon) - u_0(+L/2-\varepsilon) = 0 \\ u_0'(-L/2+\varepsilon) - u_0'(-L/2-\varepsilon) + g b_0 u(-L/2) = 0 \\ u_0'(+L/2+\varepsilon) - u_0'(+L/2-\varepsilon) + g \bar{b}_0 u(+L/2) = 0 \end{cases}$$

Its coefficient matrix $\mathcal{N} \Rightarrow \det \mathcal{N} = 0$ for a non-trivial solution!

The det $\mathcal{M}$

Generally true, also $\lambda_n \neq 0 \Rightarrow$ normalizable eigensolutions will exist iff $\det \mathcal{N}(\lambda) = 0$ for $\lambda = \lambda_n$.

Said otherwise, we have found a function $\det \mathcal{N}(\lambda)$ that (only) vanishes at the eigenvalues of the original Schrödinger problem.

$\Rightarrow$ Gel'fand-Yaglom approach to compute the *functional* $\det \mathcal{M}$ in terms of an *ordinary* $\det \mathcal{N}$, see Dunne, J. Phys. A **41**, 304006 (2008) for details.

Gel'fand-Yaglom and ζ-function regularization

Skipping all details:

$$\det \mathcal{M} = e^{-\zeta'(0)}, \qquad \zeta(s) = \sum_n \frac{1}{\lambda_n^s}.$$

Using complex analysis tools and the properties of $\det \mathcal{N}(\lambda)$, it can be shown that

$$\ln \det \mathcal{M} = \ln \det \mathcal{N}(0)$$

up to an irrelevant (divergent) constant.

In concreto, for Dirichlet case:

$$\ln \det \mathcal{M} = \ln \left(g^2 b_0 \bar{b}_0 (1 - e^{-2L\sqrt{k^2+m^2}}) + 4k^2 + 4m^2 - 2g(b_0 + \bar{b}_0)\sqrt{k^2+m^2} \right).$$

The full effective potential

It remains to compute

$$\mathcal{Z}=\int[\mathcal{D}\beta][\mathcal{D}\bar{\beta}]\exp\left(\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}dzdz'\left[J(\vec{k},z)G(z,z')(2\pi)^3\delta^3(\vec{k}-\vec{k}')J(\vec{k}',z')\right]\right).$$

The terms linear in β or $\bar{\beta}$ vanish upon minimization. Defining

$$\mathcal{P}=\begin{pmatrix} G(-L/2,-L/2) & G(-L/2,+L/2) \\ G(+L/2,-L/2) & G(+L/2,+L/2)\end{pmatrix}$$

we eventually get

$$\mathcal{Z}=\exp-\left(\frac{1}{2}\ln\det(\mathcal{M}\cdot\mathcal{P})-\frac{V_3}{2}\lim_{\vec{k}\to 0}\begin{pmatrix} b_0 & \bar{b}_0\end{pmatrix}\cdot\mathcal{P}\cdot\begin{pmatrix} b_0 \\ \bar{b}_0\end{pmatrix}\right).$$

The full effective potential

As

$$\frac{1}{2}\ln\det(\mathcal{M}\cdot\mathcal{P}) = \frac{1}{2}\ln\det\left(1-e^{-2L\sqrt{k^2+m^2}}\right)$$

$\Rightarrow$

$$\begin{aligned} Z &= \exp\left(\frac{V_3}{2}\lim_{\overleftarrow{k}\to 0}\begin{pmatrix} b_0 & \bar{b}_0 \end{pmatrix}\cdot\mathcal{P}\cdot\begin{pmatrix} b_0 \\ \bar{b}_0 \end{pmatrix}\right)\exp-\frac{1}{2}\mathrm{tr}\ln\left(1-e^{-2L\sqrt{k^2+m^2}}\right) \\ &= \exp(-V_3 E_{\text{Cas, D}}) \end{aligned}$$

from which we can read off the Casimir energy in the Dirichlet case,

$$\begin{aligned} E_{\text{Cas, D}} &= -\frac{1}{2}\lim_{\overleftarrow{k}\to 0}\begin{pmatrix} b_0 & \bar{b}_0 \end{pmatrix}\cdot\mathcal{P}\cdot\begin{pmatrix} b_0 \\ \bar{b}_0 \end{pmatrix}+\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\left(1-e^{-2L\sqrt{k^2+m^2}}\right) \\ &\equiv E^{(1)}_{\text{Cas, D}}+E^{(2)}_{\text{Cas, D}}. \end{aligned}$$

Interpretation

The second part gives the well-known (usual) Casimir energy

$$E^{(2)}_{\text{Cas, D}} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-2L\sqrt{k^2+m^2}}) = -\frac{m^2}{4\pi^2 L}\sum_{n=1}^{+\infty} \frac{K_2(2nmL)}{n^2}$$
$$= \frac{-\pi^2}{1440L^3} \quad \text{for } m \to 0.$$

More interestingly, let us look at the new piece,

$$E^{(1)}_{\text{Cas, D}} = -\frac{e^{2Lm}\left(2m\left(b_0^2+\bar{b}_0^2\right) - b_0\bar{b}_0 g(b_0+\bar{b}_0)\right) + b_0\bar{b}_0 g(b_0+\bar{b}_0) + 4b_0\bar{b}_0 m e^{Lm}}{2\left(e^{2Lm}(gb_0-2m)(g\bar{b}_0-2m) - g^2 b_0\bar{b}_0\right)}.$$

Interpretation

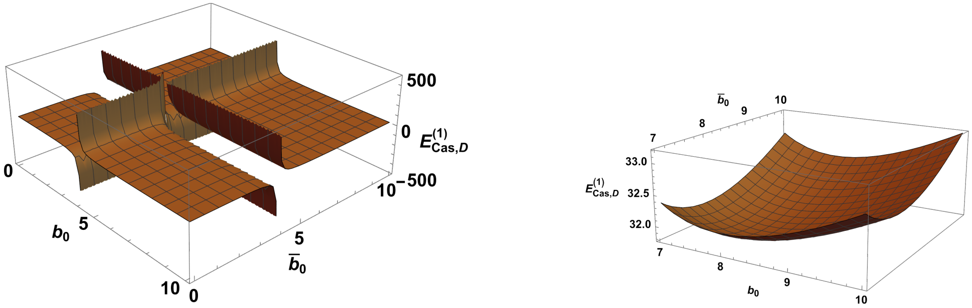

Figure: (top) $E^{(1)}_{\text{Cas, D}}$ for $g = \frac{1}{2}$, $L = 5$, $m = 1$; (bottom) a zoom-in near the minimum of $E^{(1)}_{\text{Cas, D}}$.

Interpretation

Despite regions where energy is unbounded from below, physically under control! Indeed, (only) for $m \neq 0$, non-trivial extremum at

$$b_0 = \bar{b}_0 = \frac{4me^{Lm}}{g(1+e^{Lm})}$$

with Hessian eigenvalues $\left\{\frac{1+e^{-Lm}}{2m}, \frac{\sinh(Lm)}{m(-3+e^{Lm})}\right\} \Rightarrow$ non-perturbative local minimum if

$$L > \frac{\ln 3}{m}$$

$\Rightarrow$ new dynamics for sufficiently large L, with non-perturbative boundary mass generation.

Importantly, even for large fields, no way to tunnel through the infinite (positive energy) walls to reach the regions where the potential drops to $-\infty$.

The trivial vacuum $b_0 = \bar{b}_0 = 0$ is always a local maximum, hence unstable.
For $L < \frac{\ln 3}{m}$, no minimum to be found.

The Casimir energy/force itself

If $L > \frac{\ln 3}{m}$:

$$E_{\text{Cas, D}}(L) = \frac{e^{Lm}}{1+e^{Lm}}\frac{m}{8g^2} - \frac{m^2}{4\pi^2 L}\sum_{n=1}^{+\infty}\frac{K_2(2nmL)}{n^2}$$

next to

$$F_{\text{Cas, D}} = -\partial_L E^{(1)}_{\text{Cas, D}} - \partial_L E^{(2)}_{\text{Cas, D}} \equiv F^{(1)}_{\text{Cas, D}} + F^{(2)}_{\text{Cas, D}}.$$

In the region $L < \frac{\ln 3}{m}$, we cannot make definite statements.

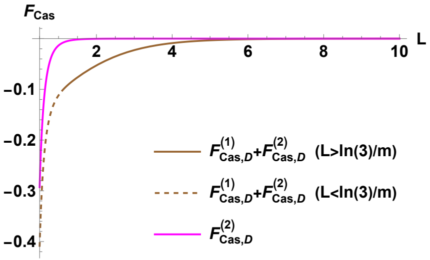

Figure: The non-linear Dirichlet Casimir force for $g = \frac{1}{2}$, $L = 5$, $m = 1$.

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Massless case: two distinct groundstates

In the Neumann case, $m \to 0$ already gives very interesting dynamics. Perhaps most interesting case as “toy model” for YM.

Using similar tools, we arrive at

$$E^{(1)}_{\text{Cas, N}} = \frac{g b_0^2 \bar{b}_0^2}{2gLb_0\bar{b}_0 - 2(b_0 + \bar{b}_0)}$$

with a non-trivial groundstate at $b_0 = \bar{b}_0 = \frac{3}{gL}$ for all L, leading to

$$E^{(1)}_{\text{Cas, N}} = \frac{27}{2g^2L^3} \Rightarrow F^{(1)}_{\text{Cas, N}} = \frac{81}{2g^2L^4}.$$

This state has higher energy than the trivial one at $b_0 = \bar{b}_0 = 0$, also minimum now! BUT: no tunneling due to infinite energy walls, so it is a another physical groundstate/vacuum!

The Casimir energy/force itself

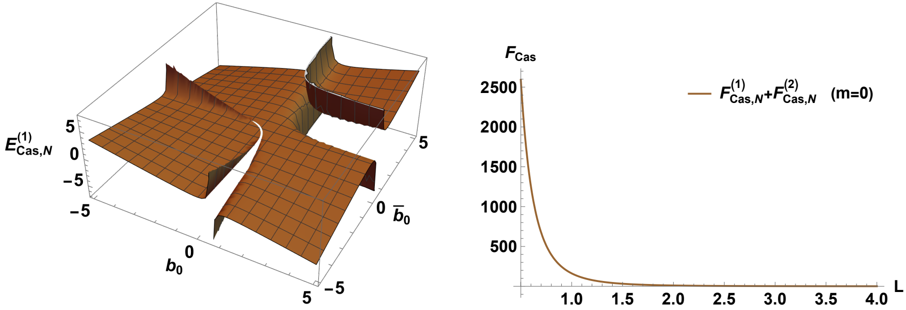


Figure: (Left) The non-linear Neumann Casimir energy for $g = \frac{1}{2}$, (Right) The associated force.

We get a non-perturbative *positive* shift of the standard negative Neumann ($=$ Dirichlet) Casimir force!

Outlook

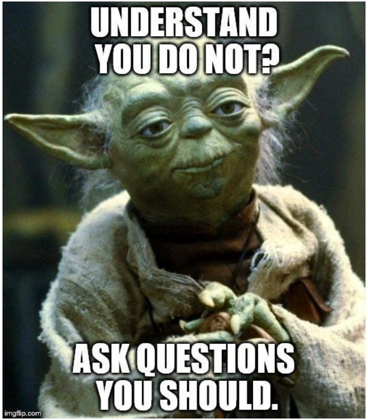
A few pertinent questions

- We introduced a scalar toy model with non-linear (Dirichlet or Neumann) boundary conditions.
- We have given strong indications of “boundary mass generation” with a rich structure of non-perturbative groundstates.

$\rightarrow$ non-perturbative corrections to the Casimir energy & force!

- Strong incentive to investigate whether such non-perturbative boundary physics also happens in Yang-Mills gauge theories, in relation to the non-Abelian Casimir effect! (cf. lattice results)
  Interplay with dynamical gluon mass in the bulk? See other talks during this meeting, in particular that of Sebbe Stouten.
- ...

¡El Fin!



¡Gracias!