

Quantum Criticality in Non-Hermitian Dirac Matter

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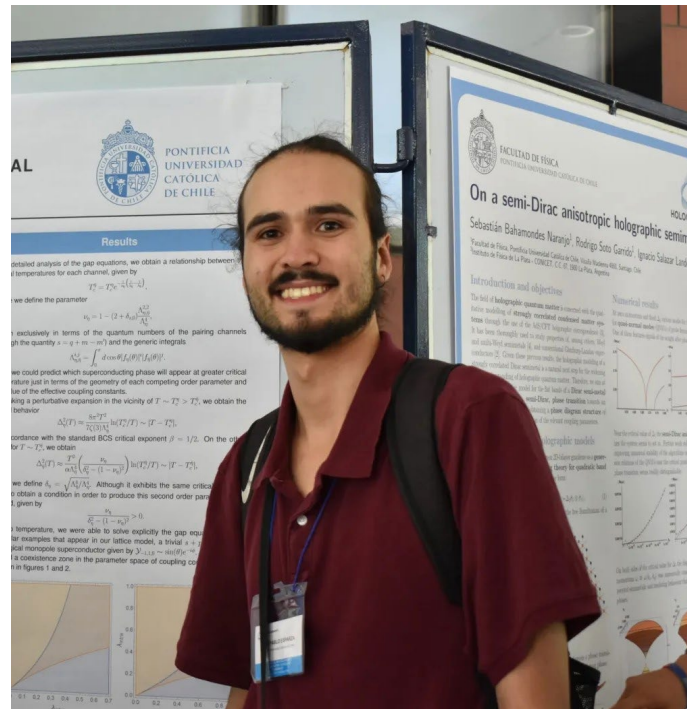
FONDECYT

Fondo Nacional de Desarrollo
Científico y Tecnológico

Collaborators



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(USM)

Outline

- Introduction
 - Non-Hermitian Dirac fermions: A general principle of construction
 - Strongly-coupled non-Hermitian Dirac semimetal: Yukawa-Lorentz symmetry
 - Tilted NH Dirac semimetals
 - Observables detecting non-Hermiticity
 - Summary
-
- V. J. & B. Roy, Communications Physics 7, 169 (2024)
 - S. Pino-Alarcón & V. J., PRB 111, 195126 (2025)
 - S. Pino-Alarcón, J. P. Esparza & V. J., in preparation.

Biorthogonal QM

Mostafazadeh, Journal of Mathematical Physics (2002)

Review on biorthogonal QM: Brody 2014

- QM: Observables are represented by Hermitian operators

$$A = A^\dagger \Rightarrow \text{Spectrum}(A) \in \mathbb{R}$$

- Hermiticity is not a necessary condition for a real spectrum.

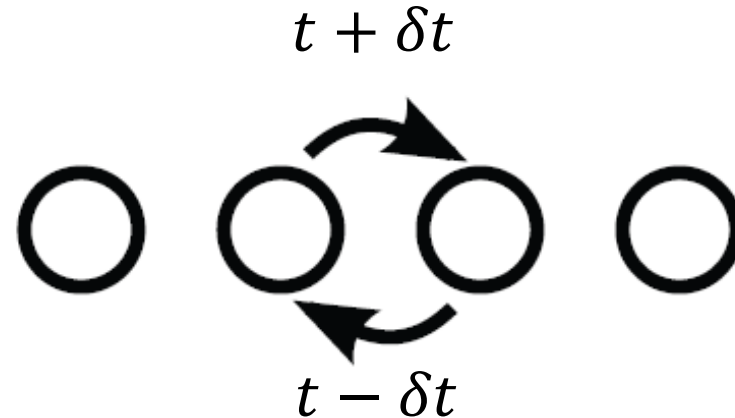
- Any pseudo-Hermitian operator $A^\dagger = \eta A \eta^\dagger$ has a real spectrum if (unitary) η is positive-definite

$$\langle \psi | \eta | \phi \rangle \equiv \langle \psi_L | \phi_R \rangle \geq 0$$

$$A|\phi_R\rangle = \lambda|\phi_R\rangle \quad A^\dagger|\phi_L\rangle = \lambda^*|\phi_L\rangle$$

Pseudo-Hermiticity generalizes PT symmetry condition introduced in Bender & Boettcher, PRL 1998

Non-Hermitian systems: Nonreciprocity



Canonical example: Hatano-Nelson model [Hatano, Nelson, PRL 1996](#)

NH skin effect: accumulation of a microscopic number of states at edge
– breaking of the usual topological bulk-boundary correspondence

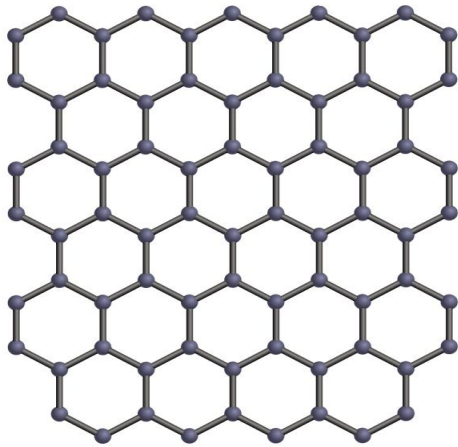
[V. M. Martinez Alvarez et al., PRB **97**, 121401\(R\) \(2018\).](#)

[S. Yao and Z. Wang, PRL **121**, 086803 \(2018\).](#)

[S. Yao, F. Song, and Z. Wang, PRL **121**, 136802 \(2018\).](#)

Hermitian Dirac(-like) fermions

Two-dimensional membrane of C atoms



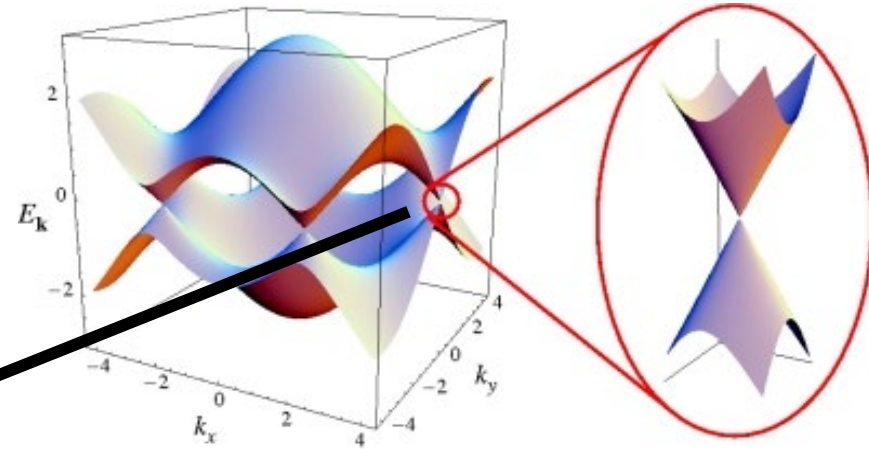
Dirac Hamiltonian

$$H(\mathbf{k}) = v_0 \boldsymbol{\sigma} \cdot \mathbf{k}$$

$$v_0 \sim c/300 \sim 10^6 \text{ m/s}$$

Density of states

$$\rho(E) \sim \frac{|E|}{v_0^2}$$



Hermitian Dirac(-like) fermions

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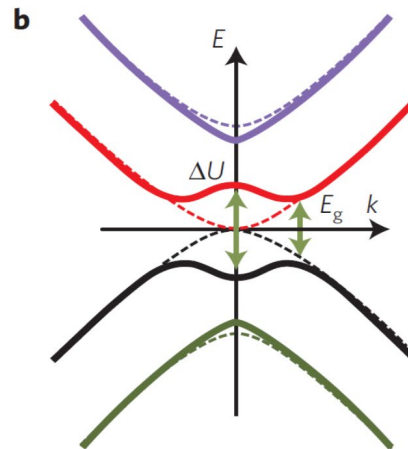
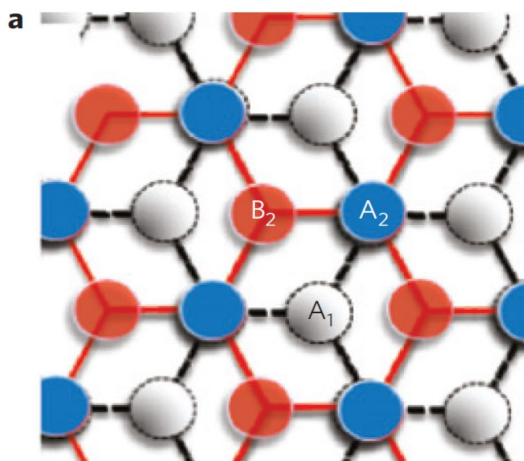
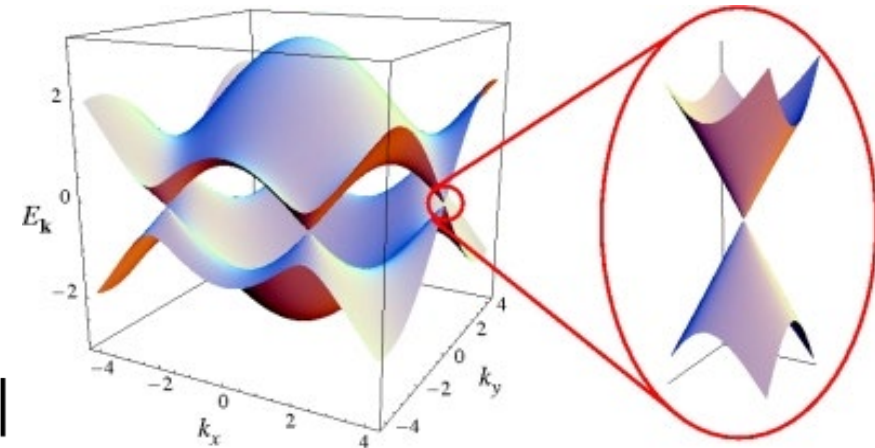
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Bilayer graphene – Bernal stacking



ABC trilayer graphene → **diverging DoS!**

$$H(\mathbf{k}) = \sigma_1 \frac{k_x^2 - k_y^2}{2m} + \sigma_2 \frac{2k_x k_y}{2m}$$

Quadratic chiral fermions!

Constant density of states

Ordering @ B=0

Velasco et al., Nat Nano 7, 156 (2012)
Freitag et al., PRB 87, 161402 (2013)

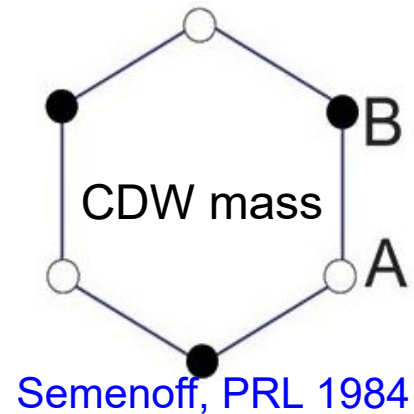
Non-Hermitian Dirac operator

- Dirac theory allows two Lorentz invariant operators: (1) Dirac kinetic energy
(2) Dirac mass

These two **Hermitian** operators mutually **anticommute**

Anti-Hermitian operator : $H_{\text{Dirac}} M$

$$\text{Proof : } (H_{\text{Dirac}} M)^\dagger = M^\dagger H_{\text{Dirac}}^\dagger = M H_{\text{Dirac}} = -H_{\text{Dirac}} M$$



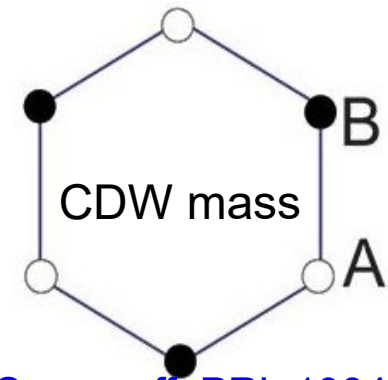
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Semenoff, PRL 1984

- General construction of Lorentz invariant NH Dirac operator :

$$H_{\text{NH}}(\mathbf{k}) = v_{\text{H}} \left(\sum_{j=1}^d \Gamma_j k_j \right) + v_{\text{NH}} \left(M \sum_{j=1}^d \Gamma_j k_j \right)$$

V. J. & B. Roy, Comm Phys 2024

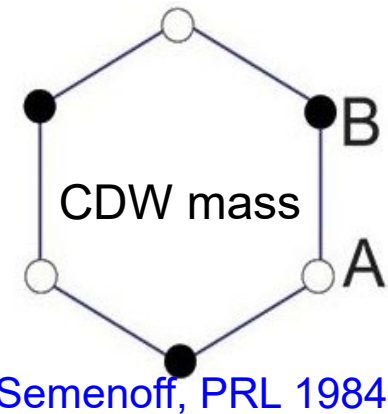
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- Eigenvalues: $E_{\text{NH}}(\mathbf{k}) = \pm \sqrt{v_{\text{H}}^2 - v_{\text{NH}}^2} |\mathbf{k}| \equiv v_{\text{F}} |\mathbf{k}|$: Real for $v_{\text{H}} > v_{\text{NH}}$

Both 2D & 3D Dirac materials host a plethora of mass orders (M)
Any mass order can be used to define NH Dirac operator

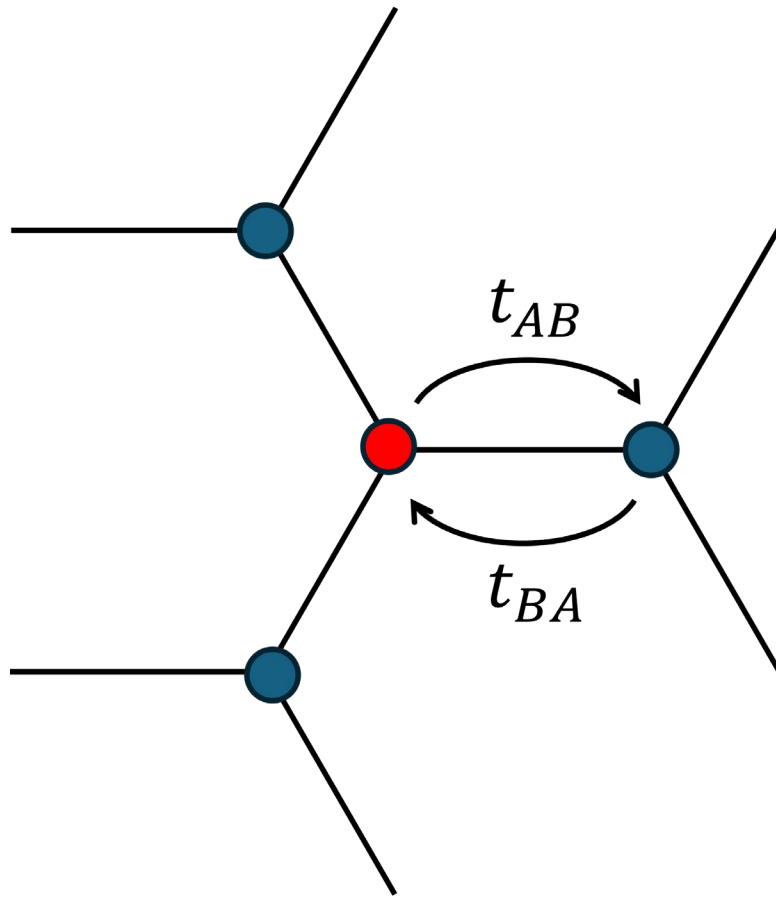
Non-Hermitian monolayer graphene

- Tight-binding Hamiltonian $\rightarrow$ Hermitian Dirac system:
$$h_0^{\text{lat}}(\mathbf{k}) = t_0 \begin{pmatrix} 0 & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^* & 0 \end{pmatrix}$$
- Sublattice symmetry breaking Dirac mass (CDW):
$$h_{\text{CDW}}^{\text{lat}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
- NH Dirac fermions on honeycomb lattice:
$$\{h_0^{\text{lat}}(\mathbf{k}), h_{\text{CDW}}^{\text{lat}}\} = 0$$

$$h_{0,\text{NH}}^{\text{lat}}(\mathbf{k}) = h_0^{\text{lat}}(\mathbf{k}) + \alpha h_0^{\text{lat}}(\mathbf{k}) h_{\text{CDW}}^{\text{lat}} = t_0 \begin{pmatrix} 0 & (1 + \alpha)\Delta_{\mathbf{k}} \\ (1 - \alpha)\Delta_{\mathbf{k}}^* & 0 \end{pmatrix}$$

Non-Hermiticity in single-layer graphene: **Hopping imbalance between sublattices**

Non-Hermitian monolayer graphene



Effective NH Hamiltonian

$$h_{\mathbf{k}} = (v_H + v_{NH}\sigma_3)\boldsymbol{\sigma} \cdot \mathbf{k}$$

Spectrum: $\epsilon_{\mathbf{k}} = v_F k$

Fermi velocity

$$v_F = v_H \sqrt{1 - \beta^2}$$

$$\beta = v_H / v_{NH}$$

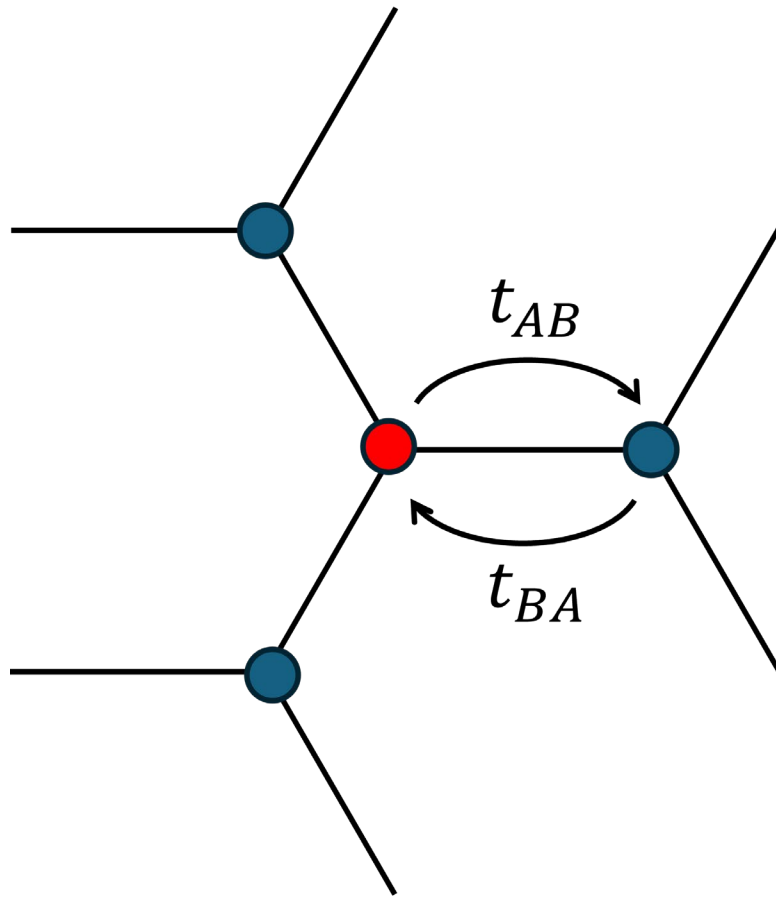
TB model with nonreciprocal hoppings: $|\beta| < 1$

v_H ~ average hopping amplitude

v_{NH} ~ hopping imbalance ($\delta t \sim t_{AB} - t_{BA}$)

Expt - ultracold atoms: NH Aharonov-Bohm chain
Liang et al., Phys. Rev. Lett. **129**, 070401 (2022).

Non-Hermitian monolayer graphene



Effective NH Hamiltonian

$$h_{\mathbf{k}} = (v_H + v_{NH}\sigma_3)\boldsymbol{\sigma} \cdot \mathbf{k}$$

Spectrum: $\epsilon_{\mathbf{k}} = v_F k$

Density of states

$$\rho(\epsilon) = \frac{N_f}{2\pi v_F^2} \{ |\epsilon| \Theta(1 - |\beta|) + \Lambda^2 \delta(\epsilon) \Theta(|\beta| - 1) \}$$

TB model with nonreciprocal hoppings: $|\beta| < 1$

v_H ~ average hopping amplitude

v_{NH} ~ hopping imbalance ($\delta t \sim t_{AB} - t_{BA}$)

Expt - ultracold atoms: NH Aharonov-Bohm chain
Liang et al., Phys. Rev. Lett. **129**, 070401 (2022).

NH Dirac operator: Symmetry protection

- Non-spatial discrete symmetries : (1) Time-reversal (T_+ & C_+),
of NH operator (H_{NH}) (2) Anti-unitary particle-hole (T_- & C_-)
(3) Unitary particle-hole (PH)
(4) Pseudo-Hermiticity (PSH)

D. Bernard & A. LeClair, A Classification of Non-Hermitian Random Matrices (Springer, 2002)
Kawabata, Shiozaki, Ueda, Sato, Phys. Rev. X 9, 041015 (2019).

- Symmetries operations:

$$\mathcal{T}_{\pm} H_{\text{NH}} \mathcal{T}_{\pm}^{-1} = \pm H_{\text{NH}} \quad \mathcal{C}_{\pm} H_{\text{NH}}^{\dagger} \mathcal{C}_{\pm}^{-1} = \pm H_{\text{NH}}$$

$$\text{PH}_{1,2} H_{\text{NH}} \text{PH}_{1,2}^{-1} = -H_{\text{NH}} \quad \text{PSH}_{1,2} H_{\text{NH}}^{\dagger} \text{PSH}_{1,2}^{-1} = H_{\text{NH}}$$

Mass terms	Symmetries										
	$\mathcal{T}_+$		$\mathcal{C}_+$		$\mathcal{T}_-$	$\mathcal{C}_-$		PH ₁	PH ₂	PSH ₁	PSH ₂
	$\mathcal{K}$	$iM_2M_3 \mathcal{K}$	$iM_1M_2 \mathcal{K}$	$iM_1M_3 \mathcal{K}$	$M_1 \mathcal{K}$	$M_2 \mathcal{K}$	$M_3 \mathcal{K}$	M_1	$i\Gamma_1\Gamma_2$	iM_1M_2	iM_1M_3
M_1	✓	✓	×	×	×	✓	✓	×	×	×	×
M_2	✓	×	×	✓	✓	×	✓	✓	×	×	✓
M_3	✓	×	✓	×	✓	✓	×	✓	×	✓	×
$i\Gamma_1\Gamma_2$	×	×	✓	✓	✓	×	×	×	×	✓	✓

V. J. & B. Roy, Comm Phys 2024

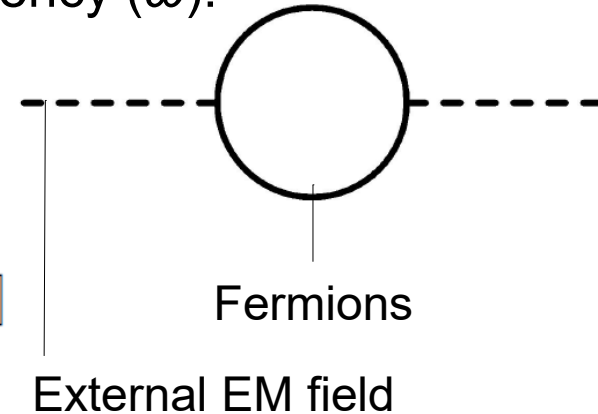
No constant mass term can be added without breaking at least one symmetry!!

Responses of NH Dirac materials

Optical conductivity (OC) @ zero temperature & finite frequency (ω):

Polarization tensor

$$\Pi_{jj}(i\omega) = -\text{Tr} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} [J_j G_F(i\omega + i\nu, \mathbf{k}) J_j G_F(i\nu, \mathbf{k})]$$



Current operator in j th direction: $J_j = (v_H + v_{NH} M) \Gamma_j$

$$\text{OC in } d=2: \quad \sigma(\omega) = \frac{\text{Im } \Pi(i\omega \rightarrow \omega + i\eta)}{\omega} \bigg|_{\eta \rightarrow 0} = N_f \frac{\pi}{4} \sigma_0$$

$$\text{OC in } d=3: \quad \sigma(\omega) = \frac{\text{Im } \Pi(i\omega \rightarrow \omega + i\eta)}{\omega} \bigg|_{\eta \rightarrow 0} = \frac{N_f \omega}{6v_F} \sigma_0$$

Same as in Hermitian systems with $v_F \rightarrow v_H$

N_f : Number of 4-component Dirac flavors & $\sigma_0 = e^2/h$

Interaction effects in NH Dirac semimetal

Local interaction: Dynamic mass generation

- Dirac fermions $\rightarrow$ Vanishing density of states: $\rho(E) \sim |E|$ in 2D or $|E|^2$ in 3D
- Local four-fermion interaction $\rightarrow$ Spontaneous symmetry breaking
(sublattice, chiral, TRS, spin-rotational,)

Dynamic generation of Dirac mass $\rightarrow$ Insulator (electric or thermal)

Ordered phase $\rightarrow$ Nontrivial expectation value of mass bilinear: $\langle \Psi^\dagger N \Psi \rangle \neq 0$

Hermitian matrix operator N : Must *anticommute* with the Dirac Hamiltonian

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- Scaling dimension of local four-fermions interaction: $g_N (\Psi^\dagger N \Psi)^2$

$[g_N] = -1$ in 2D & -2 in 3D $\rightarrow$ (a) Weak interactions: Irrelevant

I.F. Herbut, V.J., B. Roy, PRB 2009
I.F. Herbut, V.J., O. Vafek, PRB 2009

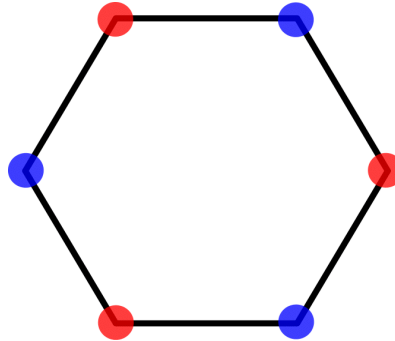
(b) Broken symmetry phase @ strong coupling via quantum phase transition

QMC: Herbut, Assaad, PRX 2013;
Otsuka, Yonuki, Sorella, PRX 2016

Examples of Dirac masses

- Charge density wave (CDW):
Breaks sublattice symmetry

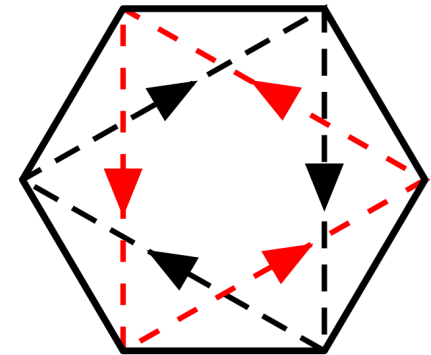
Semenoff PRL **53**, 2449 (1983)



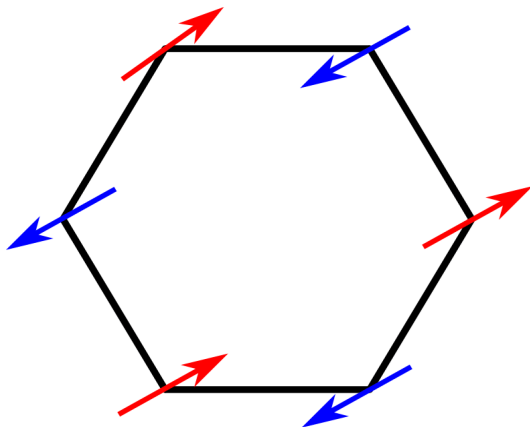
Each mass: Represented by
unique mass matrix (N)

- Quantum anomalous Hall insulator (QAHI):
Breaks sublattice & time reversal symmetries

Haldane PRL **61**, 2015 (1988)



- Antiferromagnet (AFM): Breaks sublattice
& spin rotational symmetries



Other examples of masses: translational
symmetry breaking valence bond order, s-wave and
f-wave pairings, Kekule pair-density-wave etc.

S. Ryu, C. Mudry, C.-Y Hou & C. Chamon, PRB **80**, 205319 (2010)
A. L. Szabo & B. Roy, PRB **103**, 205319 (2019)

Mass: Fragmentation in NH Dirac systems

- Lorentz invariant construction of NH Dirac operator : [V. J. & B. Roy, Comm Phys 2024](#)

Selects a mass matrix M from the set of all N s



Causes fragmentation among all N s, depending on algebra between M and N



Two classes of mass orders in NH Dirac systems

- Commuting class mass (CCM): $[N, M] = 0$
- Anti-commuting class mass (ACM): $\{N, M\} = 0$

Effective field theory: Gross-Neveu-Yukawa theory

Scalar bosons: NH Yukawa theory

- Dynamic mass generation → Condensation of bosonic order-parameter field:
Composite of fermionic fields



V. J. & B. Roy, Comm Phys 2024

Yukawa interaction between bosonic & fermionic degrees of freedom

$$S_Y = g \int d\tau \int d^d \mathbf{x} \sum_{j=1}^n \Phi_j(\tau, \mathbf{x}) \Psi^\dagger(\tau, \mathbf{x}) N_j \Psi(\tau, \mathbf{x})$$

Marginal in $d=3+1$

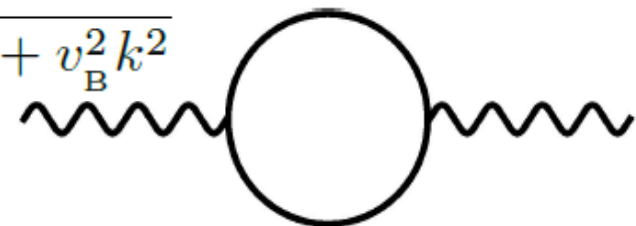
- Fermions: Two velocity parameters (v_H , v_{NH}) & effective Fermi velocity (v_F)
- Bosons: Unique bosonic velocity (v_B) → Bosonic Green's function:

$$G_B(i\omega, \mathbf{k}) = \frac{1}{\omega^2 + v_B^2 k^2}$$



Fermionic self-energy

Lorentz symmetry?



Bosonic self-energy

CCM: Emergent NH Lorentz symmetry

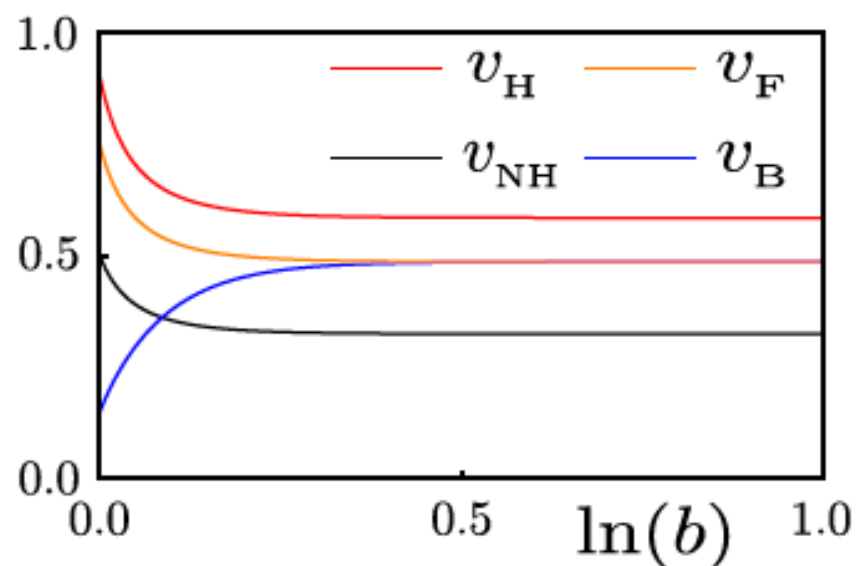
- RG Flow equations: Hermitian & anti-Hermitian velocity components of Dirac fermions

$$\beta_{v_H} = -\frac{4g^2n}{3v_B(v_F + v_B)^2} \left(1 - \frac{v_B}{v_F}\right) v_H \quad \& \quad \beta_{v_{NH}} = -\frac{4g^2n}{3v_B(v_F + v_B)^2} \left(1 - \frac{v_B}{v_F}\right) v_{NH}$$

- RG Flow equation: Bosonic velocity $\beta_{v_B} = -N_f \frac{g^2n}{2v_F^3} \left(1 - \frac{v_F^2}{v_B^2}\right) v_B$

RG fixed point

The system acquires a unique terminal velocity for fermionic & bosonic degrees of freedom
 $\rightarrow v_F = v_B$ with both v_H & v_{NH} being non-zero!!



Emergent Yukawa-Lorentz symmetry (Non-Hermiticity: retained)

ACM: Emergent Hermitian Lorentz symmetry

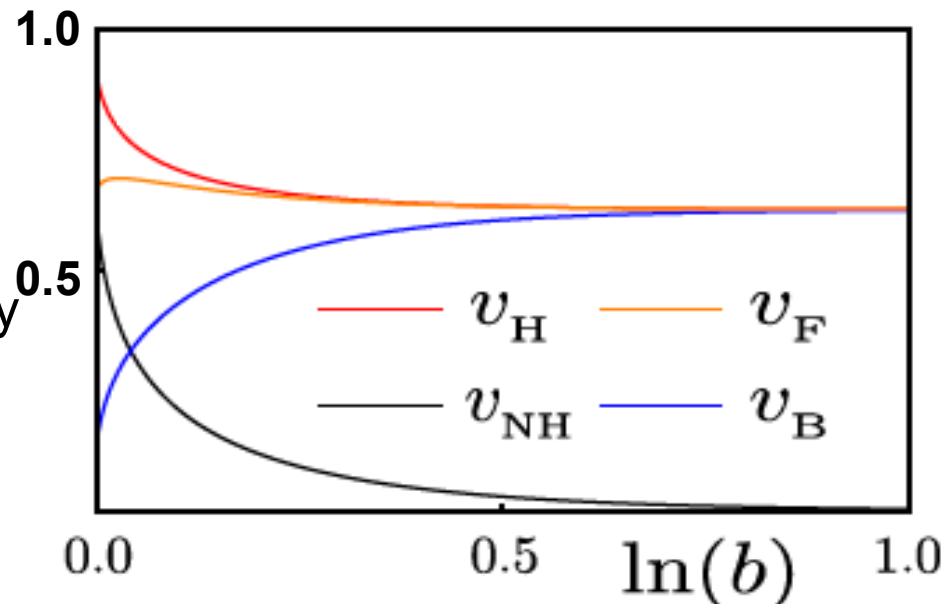
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- RG Flow equation: Bosonic velocity $\beta_{v_B} = -N_f \frac{g^2n}{2v_F^3} \left(\frac{v_H^2}{v_F^2} - \frac{v_H^2}{v_B^2} - \frac{2v_{NH}^2}{3v_B^2} \right) v_B$

RG fixed point

The system acquires a unique terminal velocity for fermionic & bosonic degrees of freedom
 $\rightarrow v_F = v_H = v_B$ with $v_{NH} = 0$!!



Emergent Hermitian Lorentz symmetry (Hermiticity: restored)

Tilted NH Dirac Semimetal

NH tilted strongly-coupled DSM:

S. Pino-Alarcon & V. J., PRB **111**, 195126 (2025)

Hermitian tilted strongly-coupled DSM:

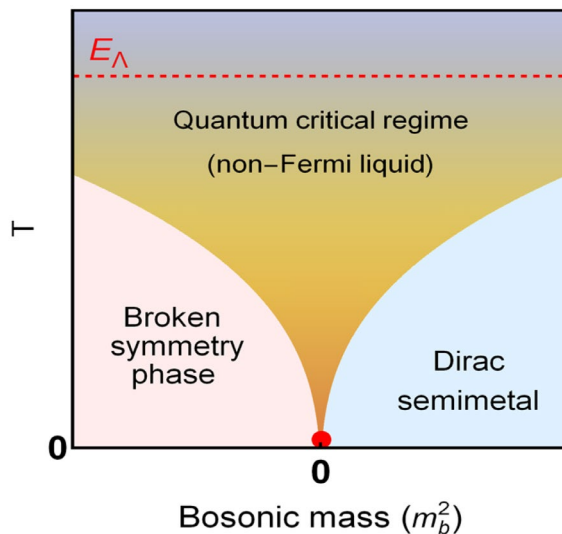
H. Rostami & V. J., PRR **2**, 013069 (2020).

P. Reiser & V. J., JHEP **2024**, 1 (2024).

$$H_{\text{NH}} = (v_{\text{H}} + v_{\text{NH}}M)h_0 + \alpha v_{\text{H}}T k_x$$

$$\{M, H_{\text{D}}\} = 0 \quad v_{\text{F}} = \sqrt{v_{\text{H}}^2 - v_{\text{NH}}^2}$$

Tilt matrices - Symmetric & Asymmetric $T_{\text{S}} = \sigma_1 \otimes \tau_0$ and $T_{\text{AS}} = \sigma_0 \otimes \tau_0$



Lorentz symmetry close to the QCP:
Velocities are not necessarily equal

The Density of States

S. Pino-Alarcon & V. J., PRB 2025

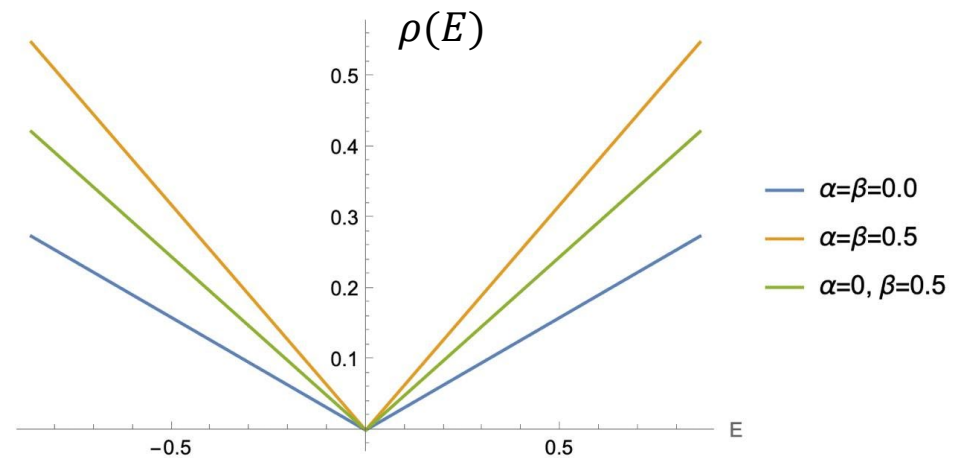
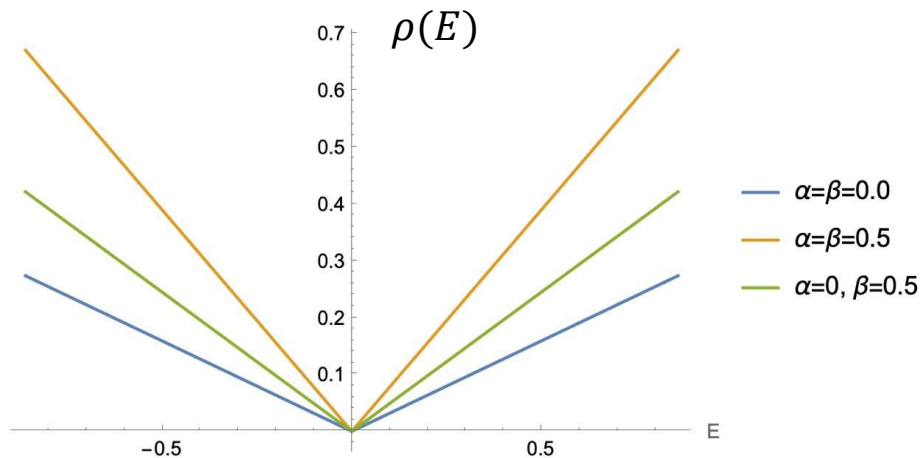
$$\rho(E) = -\frac{1}{\pi} \lim_{\eta \rightarrow 0^+} \text{Im} \{ \text{Tr} G_F(i\omega \rightarrow \omega + i\eta, \mathbf{k}) \} \quad \beta = v_{\text{NH}}/v_{\text{H}}$$

Asymmetric tilt

$$\rho_{\text{AS}}(E) = \frac{\sqrt{1 - \beta^2} |E|}{\pi(1 - \alpha^2 - \beta^2)^{3/2} v_{\text{H}}^2}$$

Symmetric tilt

$$\rho_{\text{S}}(E) = \frac{|E|}{\pi(1 - \alpha^2 - \beta^2) v_{\text{H}}^2}$$



$$\alpha^2 + \beta^2 < 1$$



$$\rho(E \rightarrow 0) \rightarrow 0$$

Weak interactions
irrelevant

Zero-frequency optical conductivity: Tilt probes non-Hermiticity

S. Pino-Alarcon, J.P. Esparza & V. J., in preparation.

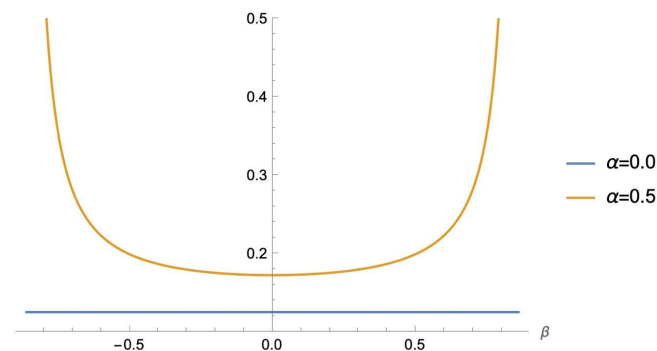
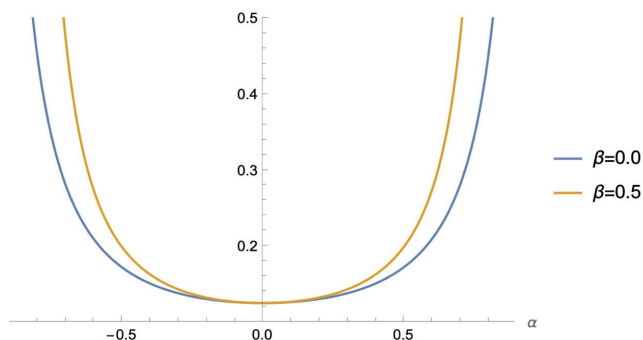
Kubo formula

$$\sigma(\omega) = \frac{\text{Im } \Pi(i\omega \rightarrow \omega + i\eta)}{\omega} \bigg|_{\eta \rightarrow 0}$$

$$\Pi_{lm}(i\omega) = -\text{Tr} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int \frac{d^2\mathbf{k}}{(2\pi)^2} [J_l G_F(i\omega + i\nu, \mathbf{k}) J_m G_F(i\nu, \mathbf{k})]$$

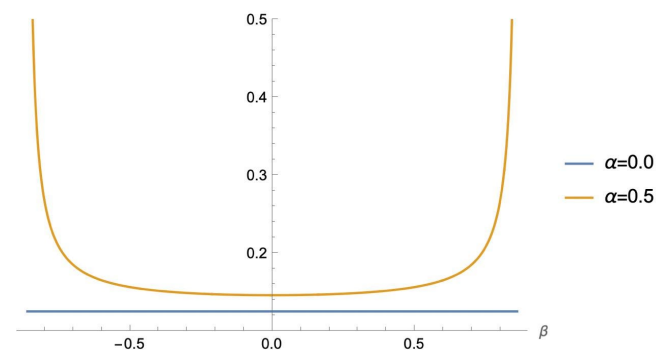
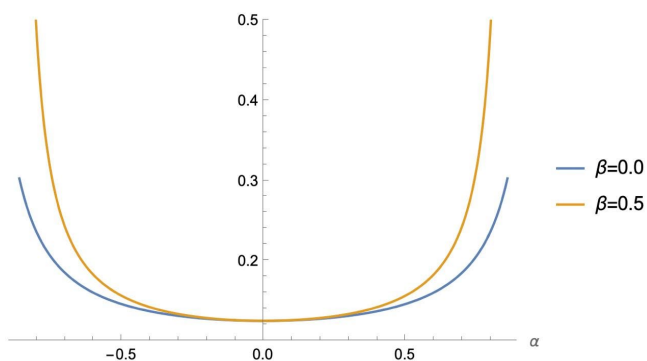
Symmetric tilt

$$\sigma_S(\omega) = \frac{N_f}{8} \left(1 + \frac{\alpha^2}{2(1 - \alpha^2 - \beta^2)} \right)$$



Asymmetric tilt

$$\sigma_{AS}(\omega) = \frac{N_f}{32} \left(\frac{\alpha^2 (-2\alpha^2 + \beta^2 - 1) (\alpha^2 + 2\beta^2 - 2)}{(1 - \beta^2)^{3/2} (-\alpha^2 - \beta^2 + 1)^{3/2}} + \frac{2(\alpha^2 - 2\beta^2 + 2)}{1 - \beta^2} \right)$$



Strongly-coupled tilted NH Dirac Semimetal

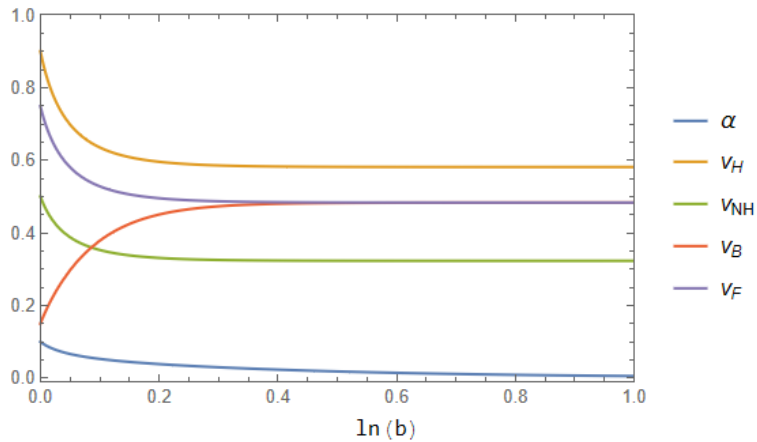
S. Pino-Alarcon & V. J., PRB 2025

ϵ – expansion close to $d = 3$ spatial dimensions near CDW QCP

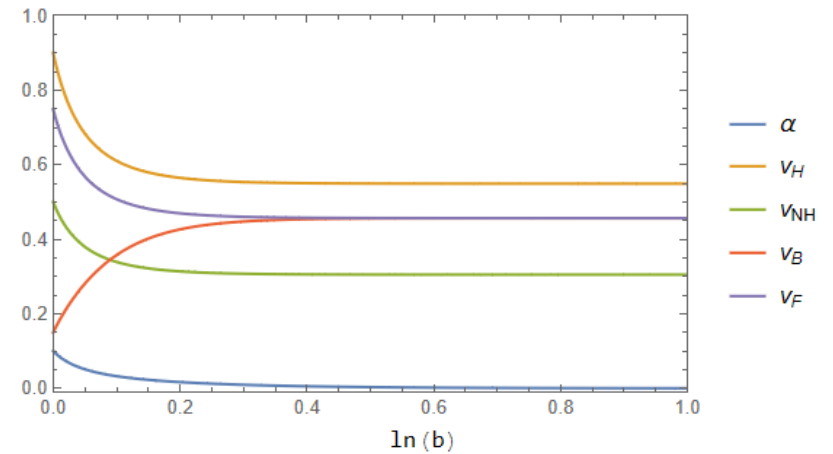
Asymmetric tilt

Symmetric tilt

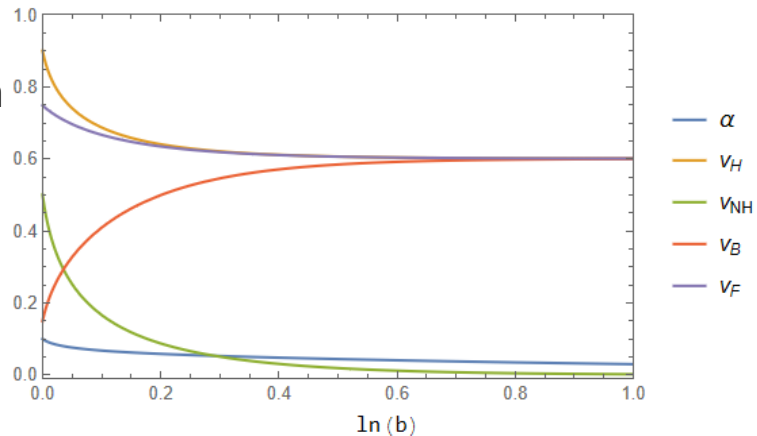
CCM:
NH
Lorentz



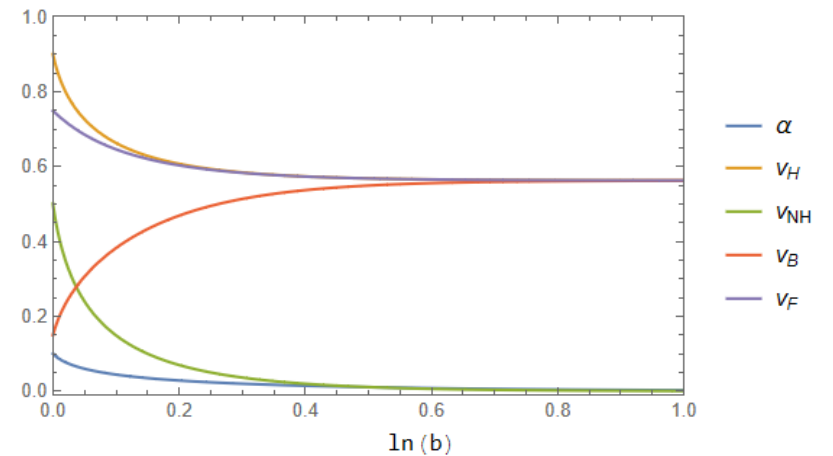
CCM



ACM:
Hermitian
Lorentz



ACM



In all cases: Restoration of the Lorentz symmetry – CCM: Yukawa-Lorentz symmetry

Conclusions

- General principle of construction for NH Dirac Hamiltonians
- Strongly coupled NH Dirac fermions: Emergent Lorentz symmetry in NH or open quantum systems (system-to-environment coupling)
- Tilted NH Dirac semimetals: Tilt can probe non-Hermiticity (OC)
- Lorentz symmetry restores at strong coupling – tilt irrelevant at QCP: CCM – non-Hermiticity retained