Sixth Workshop on Nonperturbative Aspects of QCD

Universidad Técnica Federico Santa María 03.12.2025

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Universidad San Sebastián, Santiago, Chile

Transcendental Structures and Nonperturbative Resummation in Supersymmetric Two-Point Functions

Motivation

- **What?**
  - This talk is rooted in **perturbation theory** in *Quantum Field Theory*
  - The idea is to explore its structure and **uncover hidden properties**
- **Why?**
  - What kind of **numbers** and **functions** appear in perturbative predictions for physics observables?
  - Can we leverage this knowledge to **push computational boundaries** in QFTs?
- **How?**
  - Let us work under the paradigm

    Symmetry = Simplicity

    and see if we get any mileage

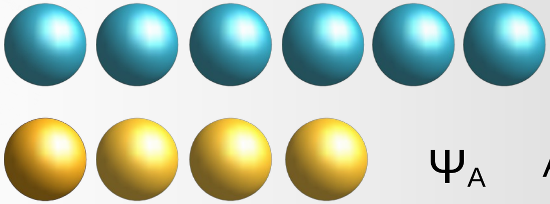
Symmetries on steroids

N = *4* SYM in one slide

10 d Yang-Mills *SU(**N**)* → four dimensions

$$\mathcal{L} = \mathrm{tr}\left(-\frac{1}{2}F_{MN}F^{MN} + i\overline{\Psi}\Gamma^N\mathcal{D}_N\Psi\right)$$

+



Φ_{AB}

Ψ_A A=1...4

adjoint matter: gluon +
six scalars + four fermions

extended supersymmetry: **unique** interactions

$$\lambda = \frac{g^2 N}{16\pi^2}$$

only one coupling ***g*** and one parameter ***N***.

faint *resemblance* with *massless QCD*?

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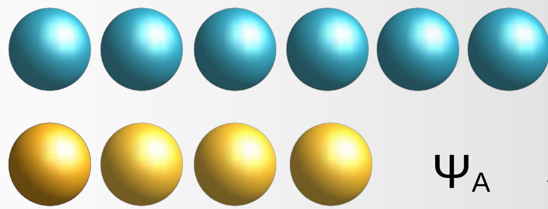
\+  Φ_{AB} Ψ_A A=1...4

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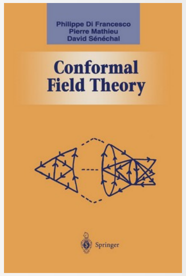

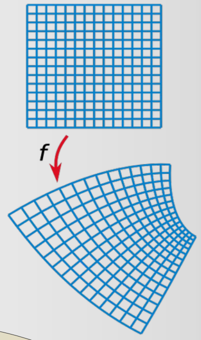

<u>Rich structure</u>:

N* = *4 is UV finite
conformal: PSU(2,2|4)

N = 4 SYM holography

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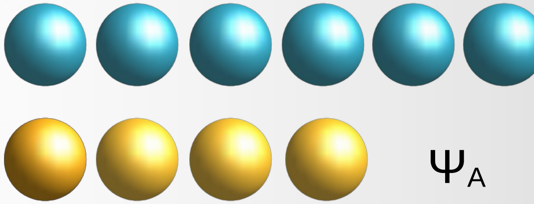

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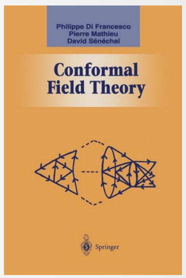

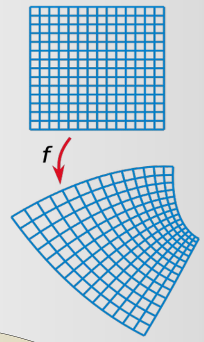

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N = 4 is UV finite
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strong coupling dual
IIB strings *AdS₅ x S⁵*

strong coupling **holography**

$\mathcal{N}$ = 4 SYM exact results

10 d Yang-Mills $SU(N)$ → four dimensions

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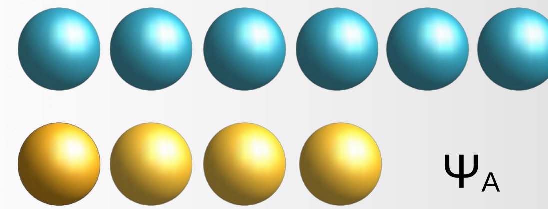

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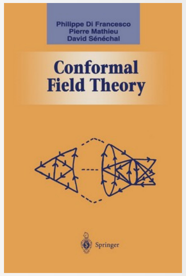

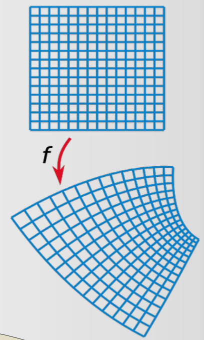

<u>Rich structure</u>:

***N* = 4** is UV finite
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strong coupling dual
IIB strings *AdS*$_5$ *x S*^{5}

integrability

supersymmetry:
localization

strong coupling **holography**

exact results!!!

N = 4 SYM solvability

10 d Yang-Mills *SU(**N**)* → four dimensions

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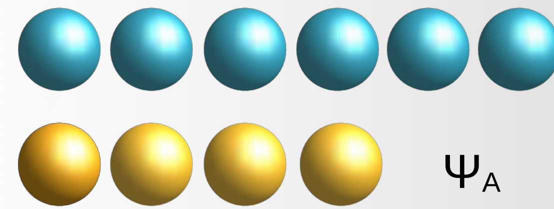

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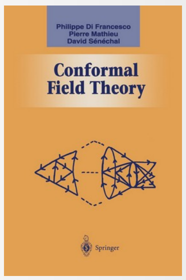
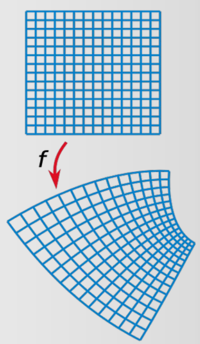




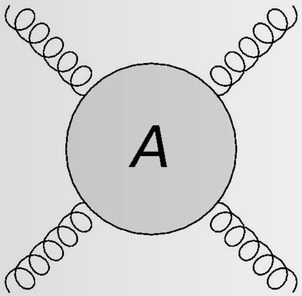
<u>Why it is interesting</u>:

solve a QFT

- **inspire** novel techniques
- many data of objects whose perturbative expansion is **similar** to massless *QCD* counterpart

An example: scattering amplitudes

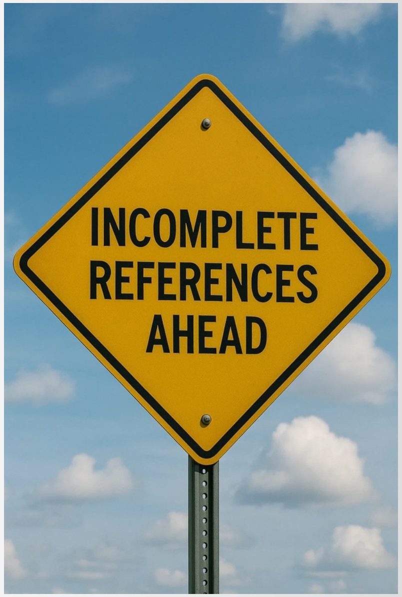
N = *4* scattering amplitudes



on-shell methods

Bern et al 1994-

emergent properties of
scattering amplitudes
in *N* = *4* SYM



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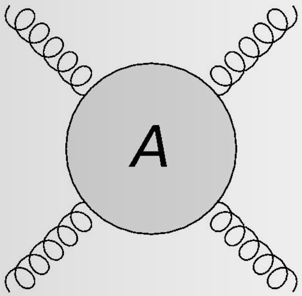

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BDS
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uniform
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progress in
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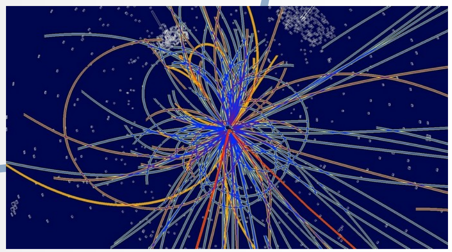
Henn 2013

bootstrap methods

Dixon et al 2011-

inspired
advances
in ***pheno***



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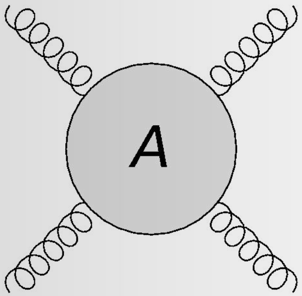

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strong coupling interpretation

Alday et al 2007

uniform **transcendentality**

duality with Wilson loops

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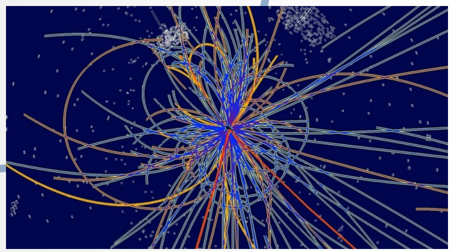
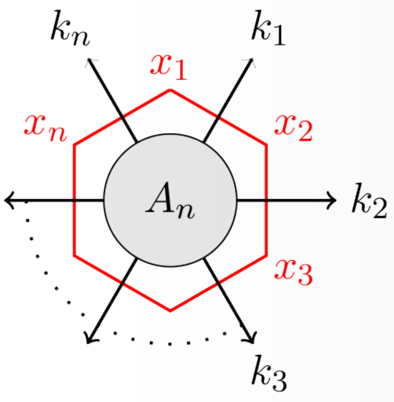


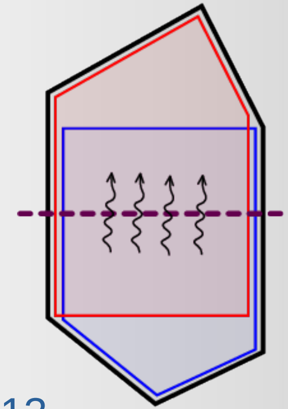
inspired advances in ***pheno***

dual conformal / Yangian **symmetry**



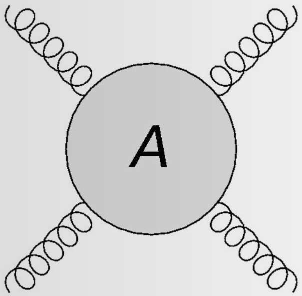
evidence for integrability

integrability prescription



Basso et al 2013

N = *4* scattering amplitudes



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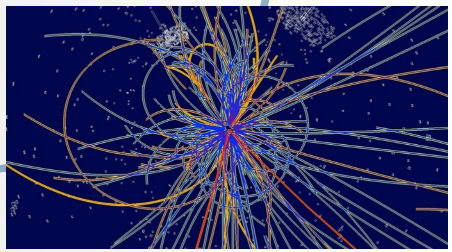

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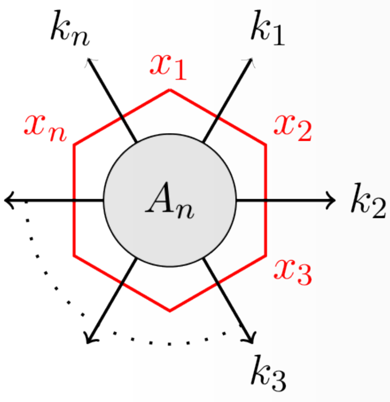

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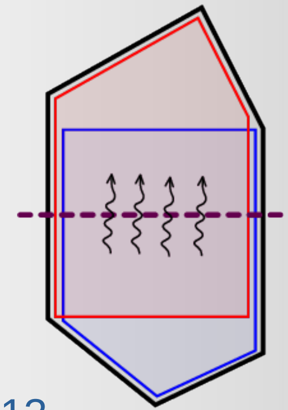
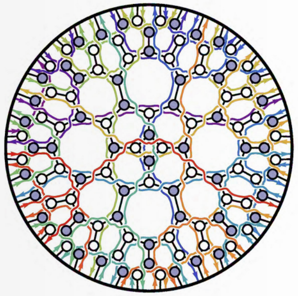


Basso et al 2013

reformulation of **QFT**

Arkani-Hamed et al 2013



Focus on this: transcendentality

uniform
transcendentality

- Meaning? It is a mathematical property exhibited by the **perturbative series** of certain observables calculated with **dimensional regularization**
- One associates a **degree of transcendentality** to certain transcendental **functions** and **numbers**
- Uniform transcendentality occurs when such a degree is found to be **fixed** for all terms at each perturbative order

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- One associates a **degree of transcendentality** to certain transcendental **functions** and **numbers**
- Uniform transcendentality occurs when such a degree is found to be **fixed** for all terms at each perturbative order
- It is a highly remarkable occurrence with **little explanation**
- It has been leveraged for perturbative calculations of scattering amplitudes, both in ***N* = *4*** and for **phenomenological applications**

bootstrap methods
Dixon et al 2011-

canonical form
Henn 2013

The power of transcendentality

Goncharov polylogarithms

In 2008 a bunch of heroes computed a very complicated six-point scattering amplitude at two loops in *N = 4 SYM*

Del Duca et al 2008

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\begin{aligned}
&R^{(2)}_{6,WL}(u_1,u_2,u_3) = \qquad\qquad\qquad\qquad\qquad\qquad \text{(H.1)}\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\\
&\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\\
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&\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+\\
&G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\\
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&\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+\\
&\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)+\\
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\end{aligned}
$$

In terms of **transcendental** iterated integrals called **Goncharov polylogarithms**

$$G(a_1,\ldots,a_n;x)=\int_0^x \frac{dt}{t-a_1}G(a_2,\ldots,a_n;t)$$

Transcendental weight
= # iterated integrals

Goncharov polylogarithms

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Del Duca et al 2008

$$R^{(2)}_{6,WL}(u_1,u_2,u_3) = \tag{H.1}$$

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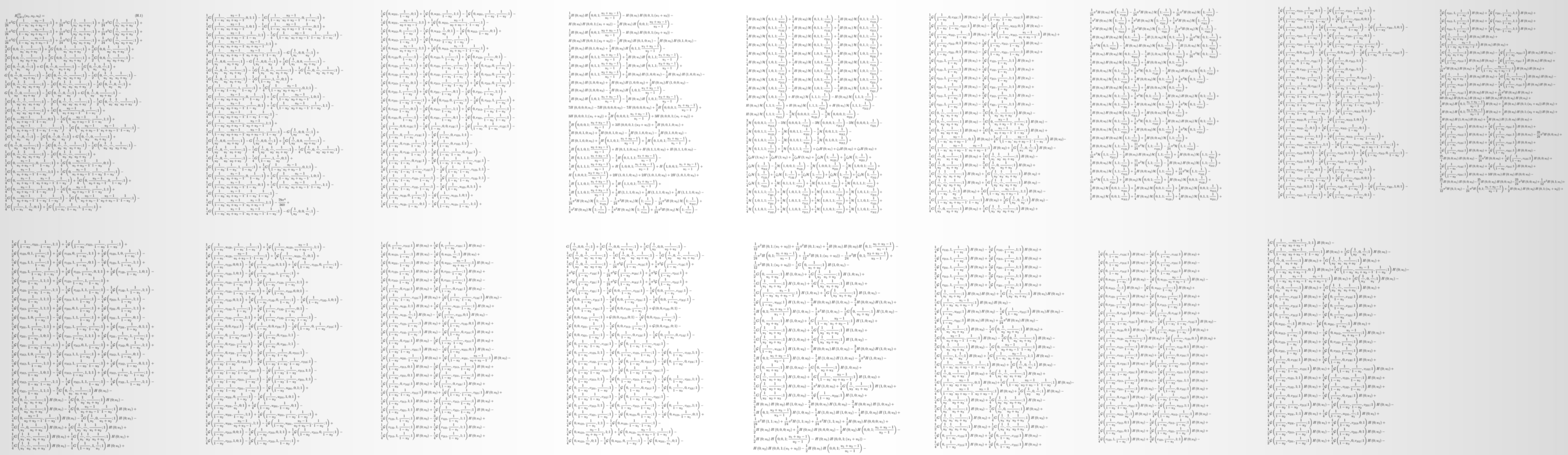
$$\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+$$
$$\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+$$
$$G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_1};1\right)+$$
$$G\left(0,\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_3};1\right)+G\left(0,\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_1};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)+$$
$$G\left(0,\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)-$$
$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3},0;1\right)+\frac{1}{2}G\left(\frac{1}{1-u_1},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+$$

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1,0;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},1;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_2};1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1},0;1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{1-u_2},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1,0;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},1;1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_1};1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_2},0;1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{1-u_3},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1,0;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{79\pi^4}{360}+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)-G\left(\frac{1}{u_3},0,0,\frac{1}{u_1};1\right)-$$

Goncharov polylogarithms

In 2008 a bunch of heroes computed a very complicated six-point scattering amplitude at two loops in *N = 4 SYM*

Del Duca et al 2008



All terms exhibit the **same transcendentality weight**!

17 pages like that

Goncharov polylogarithms

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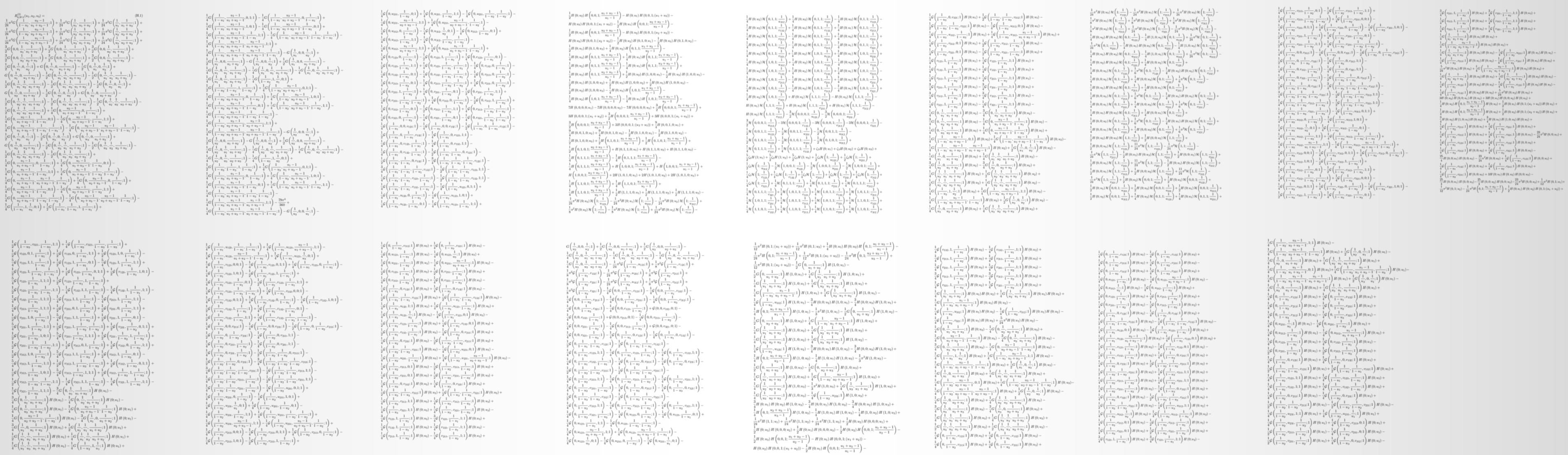
Goncharov et al 2010



Thanks to properties of **transcendental functions** it was possible to reduce the result to

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right)$$

$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

New uniformly transcendental magnitudes

MB 2306.06239

My calculation: the simplest ever

For some reason I have been computing **2pt** functions of some **supersymmetric** operators in *N = 4 SYM* of the form

$$O_0 = \mathrm{Tr}\,(XX)$$

← complex scalars

Their **conformal dimension 2** is **protected** by SUSY and so are their **2pt** and **3pt** functions in 4 dimensions

tree level exact

protected 2pt functions

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complex scalars

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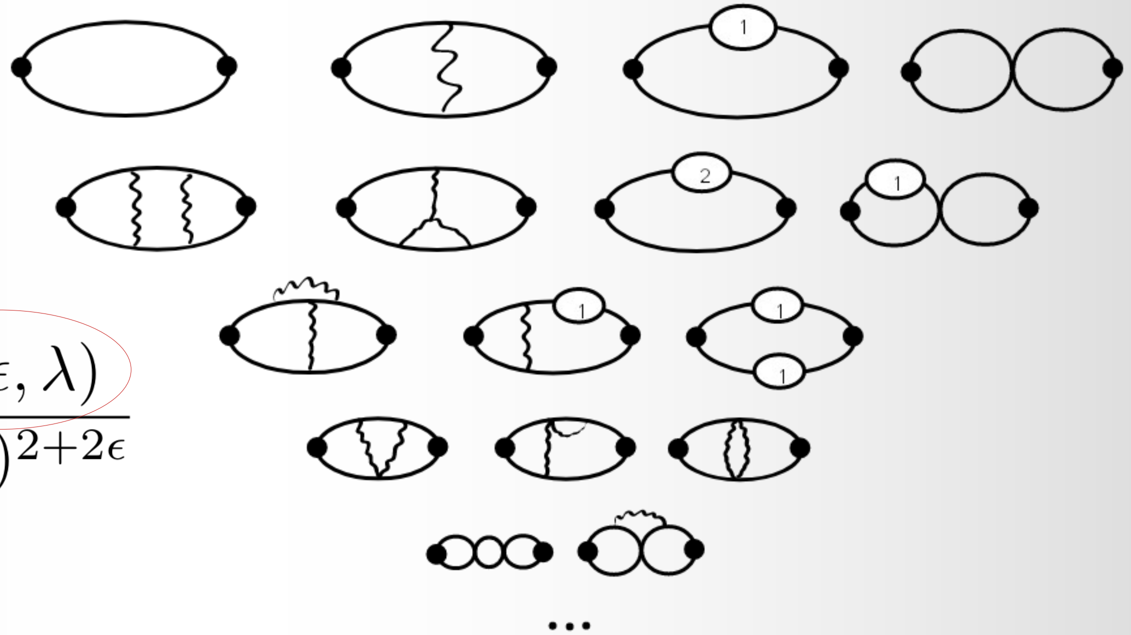
tree level exact

Yet, working in ***d = 4-2ε*** there can be *O(ε)* **quantum corrections**:

$$\langle \mathcal{O}_0(x_1)\mathcal{O}_0(x_2))\rangle = \frac{N(\epsilon,\lambda)}{(x_{12}^2)^{2+2\epsilon}}$$

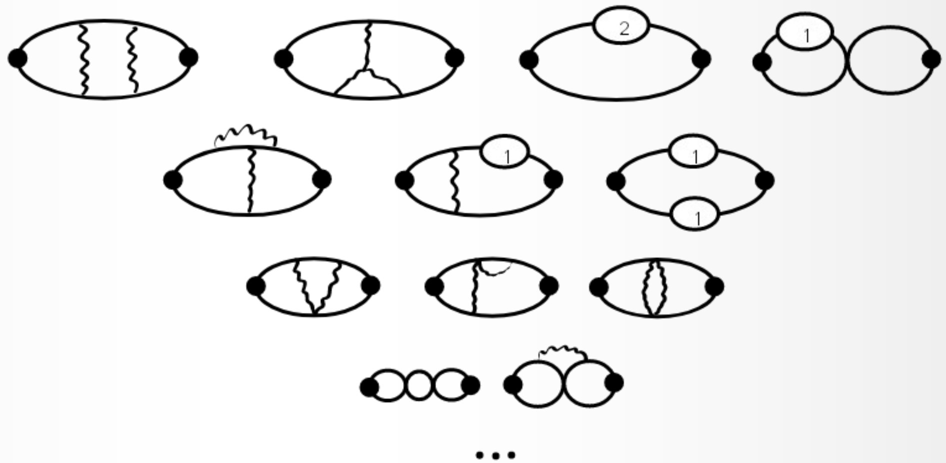
to the 2pt function

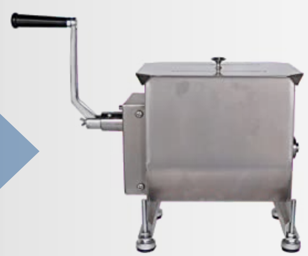


Credits

For 2pt-function:

$$\langle O_0(p) O_0(-p) \rangle^{(2)} =$$





:

Feynman diagram generation
QGRAF

Nogueira 1993

Feynman diagram evaluation

FORM

Vermaseren 1991

Color package

van Ritbergen et al 1999

Integrals:

- Master integrals reduction
- master integral substitution

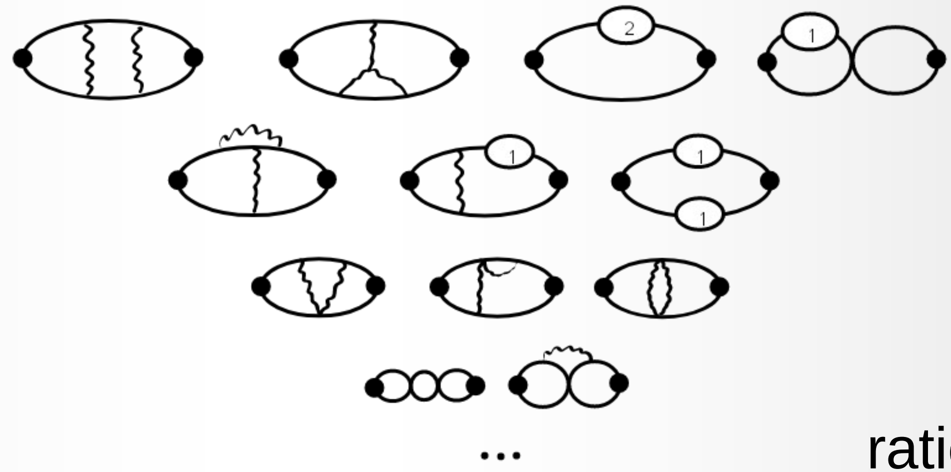
Forcer

Ruijl et al 2017

master integrals reduction example

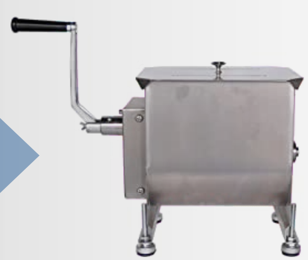
For 2pt-function:

$$\langle O_0(p) O_0(-p)\rangle^{(2)} =$$

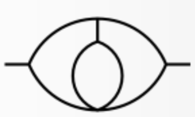


master integrals

rational coefficients in *d*



$= \frac{4(1-2\epsilon)^2}{\epsilon^2}$  $- \frac{24\left(6\epsilon^2-5\epsilon+1\right)}{\epsilon^2}$  $+ \frac{16\left(8\epsilon^2-6\epsilon+1\right)}{\epsilon^2}$ 

$+ \frac{16(3\epsilon-1)(4\epsilon-1)}{\epsilon^2}$  $- \frac{12(2\epsilon-1)}{\epsilon}$  $+$ 

# of protected 2pt functions

For some reason I have been computing **2pt** functions of some **supersymmetric** operators in *N = 4 SYM* of the form

$$O_0 = \mathrm{Tr}\,(XX)$$

complex scalars

Their **conformal dimension 2** is **protected** by SUSY and so are their **2pt** and **3pt** functions in 4 dimensions

tree level exact

Yet, working in ***d = 4-2ε***
there can be *O(ε)*
quantum corrections:

$$\langle \mathcal{O}_0(x_1)\mathcal{O}_0(x_2)) \rangle = \frac{N(\epsilon,\lambda)}{(x_{12}^2)^{2+2\epsilon}}$$

to the 2pt function

What numbers will appear?

Zeta values and transcendentality

Riemann's Zeta: $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

Example: $$\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = 1.202056903159594285399...$$

Zeta values and transcendentality

Riemann's Zeta: $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

*Multiple zeta values (**MZV**):* $$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

depth: $d = k$ weight: $w = \Sigma s_i$ ← ***transcendental weight***

Example: $\zeta(5,3) = 0.037707672528307 23\ldots$ → $d = 3$; $w = 9$

Zeta values and transcendentality

Riemann's Zeta:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

Multiple zeta values (***MZV***)*:*

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

depth: $d = k$ weight: $w = \Sigma s_i$ ← ***transcendental weight***

Why *MZVs*?

- **Ubiquitous** numbers appearing evaluating Feynman integrals
- *Conjecturally*, they are all **transcendental** numbers
- Linearly ***independent*** over rationals if their weights are different: vector space is direct sum of subspaces or **graded** *Q*-algebra
- *w* defines ***transcendental weight*** for *MZVs*

For instance: QCD beta function

At lowest order:

$$\beta_0 = 11 - \frac{2}{3} n_f$$

n_f is the number of "*quarks*"

possesses zeta values

At lowest order:

n_f is the number of "*quarks*"

$$\beta_0 = 11 - \frac{2}{3} n_f$$

At higher order ζs pop up!

$$\beta_1 = 102 - \frac{38}{3} n_f \qquad \beta_2 = \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2$$

$$\beta_3 = \frac{149753}{6} + 3564\,\zeta_3 + n_f \left(-\frac{1078361}{162} - \frac{6508}{27}\zeta_3\right) + n_f^2 \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) + \frac{1093}{729} n_f^3$$

$$\beta_4 = n_f \left(\frac{4157}{6} + 128\,\zeta_3\right) + n_f^2 \left(-\frac{7462}{9} - 992\,\zeta_3 + 2720\,\zeta_5\right) + n_f^3 \left(-\frac{21758}{81} + \frac{16000}{27}\zeta_3 - \frac{416}{3}\zeta_4 - \frac{1280}{3}\zeta_5\right) + n_f^4 \left(\frac{856}{243} + \frac{128}{27}\zeta_3\right)$$

Quantum corrections

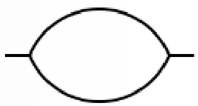
Back to my calculation

Quantum corrections

Back to my calculation

Factoring tree level result

$$\langle O_0(p) O_0(-p) \rangle = 2(N^2 - 1)$$



$$\left(1 + n^{(1)}\lambda + n^{(2)}\lambda^2 + n^{(3)}\lambda^3 + \mathcal{O}(\lambda^4)\right)$$

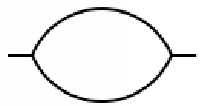
Quantum corrections are exposed, which are **order ε**

$$\lambda = \frac{g^2 N}{16\pi^2}$$

Quantum corrections

Back to my calculation

Factoring tree level result

$$\langle O_0(p) O_0(-p)\rangle = 2(N^2-1)$$  $$\left(1 + n^{(1)}\lambda + n^{(2)}\lambda^2 + n^{(3)}\lambda^3 + \mathcal{O}(\lambda^4)\right)$$

Quantum corrections are exposed, which are **order ε** $\lambda = \frac{g^2 N}{16\pi^2}$

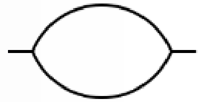
@NLO: in **DimRed scheme** $\int d^{4-2\epsilon} l$ Siegel 1979

$$\begin{aligned}
n^{(1)} = &-12\zeta_3\epsilon - 18\zeta_4\epsilon^2 + (6\zeta_2\zeta_3 - 84\zeta_5)\,\epsilon^3 + \left(64\zeta_3^2 - \tfrac{657}{4}\zeta_6\right)\epsilon^4 + \left(\tfrac{741}{4}\zeta_3\zeta_4 + 42\zeta_2\zeta_5\right.\\
&\left.-588\zeta_7\right)\epsilon^5 + \left(-32\zeta_2\zeta_3^2 + \tfrac{3872}{5}\zeta_5\zeta_3 - \tfrac{18285}{16}\zeta_8\right)\epsilon^6 + \left(-\tfrac{566}{3}\zeta_3^3 + \tfrac{46647}{32}\zeta_6\zeta_3 + \tfrac{22287}{20}\zeta_4\zeta_5\right.\\
&\left.+294\zeta_2\zeta_7 - 4260\zeta_9\right)\epsilon^7 + \left(-813\zeta_4\zeta_3^2 - \tfrac{1936}{5}\zeta_2\zeta_5\zeta_3 + \tfrac{35824}{7}\zeta_7\zeta_3 + \tfrac{11424}{5}\zeta_5^2 - \tfrac{4843593}{640}\zeta_{10}\right)\epsilon^8\\
&+\left(\tfrac{283}{3}\zeta_2\zeta_3^3 - \tfrac{49702}{15}\zeta_5\zeta_3^2 + \tfrac{2493779}{256}\zeta_8\zeta_3 + \tfrac{1366341}{160}\zeta_5\zeta_6 + \tfrac{205683}{28}\zeta_4\zeta_7 + 2130\zeta_2\zeta_9\right.\\
&\left.-31836\zeta_{11}\right)\epsilon^9 + \left(\tfrac{10784}{27}\zeta_3^4 - \tfrac{49421}{8}\zeta_6\zeta_3^2 - \tfrac{47524}{5}\zeta_4\zeta_5\zeta_3 - \tfrac{17912}{7}\zeta_2\zeta_7\zeta_3 + \tfrac{108544}{3}\zeta_9\zeta_3 - \tfrac{5712}{5}\zeta_2\zeta_5^2\right.\\
&\left.+\tfrac{149328}{5}\zeta_5\zeta_7 - \tfrac{71432982333}{1415168}\zeta_{12}\right)\epsilon^{10} + \mathcal{O}\left(\epsilon^{11}\right)
\end{aligned}$$

Quantum corrections

Back to my calculation

Factoring tree level result

$$\langle O_0(p) O_0(-p) \rangle = 2(N^2-1)$$  $$\left(1 + n^{(1)}\lambda + n^{(2)}\lambda^2 + n^{(3)}\lambda^3 + \mathcal{O}(\lambda^4)\right)$$

Quantum corrections are exposed, which are **order ε**

$$\lambda = \frac{g^2 N}{16\pi^2}$$

@NLO: **w=2** w=3-1 w=4-2 w=5-3 w=6-4 w=7-5 w(**ε**)=-1

$$n^{(1)} = -12\zeta_3\epsilon - 18\zeta_4\epsilon^2 + (6\zeta_2\zeta_3 - 84\zeta_5)\,\epsilon^3 + \left(64\zeta_3^2 - \frac{657}{4}\zeta_6\right)\epsilon^4 + \left(\frac{741}{4}\zeta_3\zeta_4 + 42\zeta_2\zeta_5 - 588\zeta_7\right)\epsilon^5 + \left(-32\zeta_2\zeta_3^2 + \frac{3872}{5}\zeta_5\zeta_3 - \frac{18285}{16}\zeta_8\right)\epsilon^6 + \left(-\frac{566}{3}\zeta_3^3 + \frac{46647}{32}\zeta_6\zeta_3 + \frac{22287}{20}\zeta_4\zeta_5 + 294\zeta_2\zeta_7 - 4260\zeta_9\right)\epsilon^7 + \left(-813\zeta_4\zeta_3^2 - \frac{1936}{5}\zeta_2\zeta_5\zeta_3 + \frac{35824}{7}\zeta_7\zeta_3 + \frac{11424}{5}\zeta_5^2 - \frac{4843593}{640}\zeta_{10}\right)\epsilon^8 + \left(\frac{283}{3}\zeta_2\zeta_3^3 - \frac{49702}{15}\zeta_5\zeta_3^2 + \frac{2493779}{256}\zeta_8\zeta_3 + \frac{1366341}{160}\zeta_5\zeta_6 + \frac{205683}{28}\zeta_4\zeta_7 + 2130\zeta_2\zeta_9 - 31836\zeta_{11}\right)\epsilon^9 + \left(\frac{10784}{27}\zeta_3^4 - \frac{49421}{8}\zeta_6\zeta_3^2 - \frac{47524}{5}\zeta_4\zeta_5\zeta_3 - \frac{17912}{7}\zeta_2\zeta_7\zeta_3 + \frac{108544}{3}\zeta_9\zeta_3 - \frac{5712}{5}\zeta_2\zeta_5^2 + \frac{149328}{5}\zeta_5\zeta_7 - \frac{71432982333}{1415168}\zeta_{12}\right)\epsilon^{10} + \mathcal{O}\left(\epsilon^{11}\right)$$

Heuristic uniform transcendentality

@NNLO:

$$n^{(2)} = 100\zeta_5\epsilon + \left(244\zeta_3^2 + 250\zeta_6\right)\epsilon^2 + \left(732\zeta_3\zeta_4 - 100\zeta_2\zeta_5 + 1718\zeta_7\right)\epsilon^3 + \left(\tfrac{7288}{3}\zeta_5\zeta_3 - \frac{1296}{5}\zeta_{5,3} - 244\zeta_2\zeta_3^2 + \frac{179647}{30}\zeta_8\right)\epsilon^4 + \left(-\frac{10712}{3}\zeta_3^3 + \frac{10717}{3}\zeta_6\zeta_3 + 5763\zeta_4\zeta_5 - 1718\zeta_2\zeta_7 + \frac{221140}{9}\zeta_9\right)\epsilon^5 + \left(\frac{1296}{5}\zeta_2\zeta_{5,3} - \frac{16014}{7}\zeta_{7,3} - 15641\zeta_4\zeta_3^2 - \frac{7288}{3}\zeta_2\zeta_5\zeta_3 + \frac{23420}{3}\zeta_7\zeta_3 + \frac{140021431}{1400}\zeta_{10} - \frac{4278}{7}\zeta_5^2\right)\epsilon^6 + \left(9504\zeta_3\zeta_{5,3} - \frac{58592}{5}\zeta_{5,3,3} + \frac{10712}{3}\zeta_2\zeta_3^3 - \frac{3124952}{45}\zeta_5\zeta_3^2 - \frac{836833}{18}\zeta_8\zeta_3 + \frac{164813}{4}\zeta_5\zeta_6 + \frac{276033}{10}\zeta_4\zeta_7 + \frac{36679873}{30}\zeta_{11} - \frac{4967092}{9}\zeta_2\zeta_9\right)\epsilon^7 + \mathcal{O}(\epsilon^8)$$

@NNNLO:

$$n^{(3)} = -980\zeta_7\epsilon + \left(-5560\zeta_3\zeta_5 - 3430\zeta_8\right)\epsilon^2 + \left(-\frac{17560}{3}\zeta_3^3 - 13900\zeta_6\zeta_3 - 8340\zeta_4\zeta_5 + 1470\zeta_2\zeta_7 - \frac{292220}{9}\zeta_9\right)\epsilon^3 + \left(\frac{18330}{7}\zeta_{7,3} - 26340\zeta_4\zeta_3^2 + 8340\zeta_2\zeta_5\zeta_3 - 76170\zeta_7\zeta_3 - \frac{391050}{7}\zeta_5^2 - \frac{4537861}{28}\zeta_{10}\right)\epsilon^4 + \left(64368\zeta_3\zeta_{5,3} - 77152\zeta_{5,3,3} + 8780\zeta_2\zeta_3^3 + 52040\zeta_5\zeta_3^2 - 384846\zeta_8\zeta_3 + \frac{40041999}{8}\zeta_{11} - \frac{179855}{2}\zeta_5\zeta_6 - \frac{10269410}{3}\zeta_2\zeta_9 - \frac{1616769}{4}\zeta_4\zeta_7\right)\epsilon^5 + \left(70184\zeta_4\zeta_{5,3} + \frac{1223545}{7}\zeta_2\zeta_{7,3} - \frac{178720}{3}\zeta_{6,4,1,1} - \frac{1569410}{9}\zeta_{9,3} + \frac{1620880}{9}\zeta_3^4 + \frac{218205}{2}\zeta_6\zeta_3^2 - \frac{2257405}{6}\zeta_4\zeta_5\zeta_3 + \frac{2129965}{3}\zeta_2\zeta_7\zeta_3 - \frac{39914030}{27}\zeta_9\zeta_3 + \frac{6138365}{21}\zeta_2\zeta_5^2 - \frac{1883141}{3}\zeta_5\zeta_7 - \frac{400117736081}{66336}\zeta_{12}\right)\epsilon^6 + \mathcal{O}(\epsilon^7)$$

Heuristic uniform transcendentality

@NNLO: **w=4** w=5-1 w=6-2 w=7-3 w=8-4

$$
\begin{aligned}
n^{(2)} = {} & 100\zeta_5\epsilon + \left(244\zeta_3^2 + 250\zeta_6\right)\epsilon^2 + \left(732\zeta_3\zeta_4 - 100\zeta_2\zeta_5 + 1718\zeta_7\right)\epsilon^3 + \Big(\tfrac{7288}{3}\zeta_5\zeta_3 \\
& - \tfrac{1296}{5}\zeta_{5,3} - 244\zeta_2\zeta_3^2 + \tfrac{179647}{30}\zeta_8\Big)\epsilon^4 + \Big(-\tfrac{10712}{3}\zeta_3^3 + \tfrac{10717}{3}\zeta_6\zeta_3 + 5763\zeta_4\zeta_5 - 1718\zeta_2\zeta_7 \\
& + \tfrac{221140}{9}\zeta_9\Big)\epsilon^5 + \Big(\tfrac{1296}{5}\zeta_2\zeta_{5,3} - \tfrac{16014}{7}\zeta_{7,3} - 15641\zeta_4\zeta_3^2 - \tfrac{7288}{3}\zeta_2\zeta_5\zeta_3 + \tfrac{23420}{3}\zeta_7\zeta_3 \\
& + \tfrac{140021431}{1400}\zeta_{10} - \tfrac{4278}{7}\zeta_5^2\Big)\epsilon^6 + \Big(9504\zeta_3\zeta_{5,3} - \tfrac{58592}{5}\zeta_{5,3,3} + \tfrac{10712}{3}\zeta_2\zeta_3^3 - \tfrac{3124952}{45}\zeta_5\zeta_3^2 \\
& - \tfrac{836833}{18}\zeta_8\zeta_3 + \tfrac{164813}{4}\zeta_5\zeta_6 + \tfrac{276033}{10}\zeta_4\zeta_7 + \tfrac{36679873}{30}\zeta_{11} - \tfrac{4967092}{9}\zeta_2\zeta_9\Big)\epsilon^7 + \mathcal{O}(\epsilon^8)
\end{aligned}
$$

w($\boldsymbol{\varepsilon}$)=-1

@NNNLO: **w=6** w=7-1 w=8-2 w=9-3 w=10-4

$$
\begin{aligned}
n^{(3)} = {} & -980\zeta_7\epsilon + \left(-5560\zeta_3\zeta_5 - 3430\zeta_8\right)\epsilon^2 + \Big(-\tfrac{17560}{3}\zeta_3^3 - 13900\zeta_6\zeta_3 - 8340\zeta_4\zeta_5 \\
& + 1470\zeta_2\zeta_7 - \tfrac{292220}{9}\zeta_9\Big)\epsilon^3 + \Big(\tfrac{18330}{7}\zeta_{7,3} - 26340\zeta_4\zeta_3^2 + 8340\zeta_2\zeta_5\zeta_3 - 76170\zeta_7\zeta_3 - \tfrac{391050}{7}\zeta_5^2 \\
& - \tfrac{4537861}{28}\zeta_{10}\Big)\epsilon^4 + \Big(64368\zeta_3\zeta_{5,3} - 77152\zeta_{5,3,3} + 8780\zeta_2\zeta_3^3 + 52040\zeta_5\zeta_3^2 - 384846\zeta_8\zeta_3 \\
& + \tfrac{40041999}{8}\zeta_{11} - \tfrac{179855}{2}\zeta_5\zeta_6 - \tfrac{10269410}{3}\zeta_2\zeta_9 - \tfrac{1616769}{4}\zeta_4\zeta_7\Big)\epsilon^5 + \Big(70184\zeta_4\zeta_{5,3} + \tfrac{1223545}{7}\zeta_2\zeta_{7,3} \\
& - \tfrac{178720}{3}\zeta_{6,4,1,1} - \tfrac{1569410}{9}\zeta_{9,3} + \tfrac{1620880}{9}\zeta_3^4 + \tfrac{218205}{2}\zeta_6\zeta_3^2 - \tfrac{2257405}{6}\zeta_4\zeta_5\zeta_3 + \tfrac{2129965}{3}\zeta_2\zeta_7\zeta_3 \\
& - \tfrac{39914030}{27}\zeta_9\zeta_3 + \tfrac{6138365}{21}\zeta_2\zeta_5^2 - \tfrac{1883141}{3}\zeta_5\zeta_7 - \tfrac{400117736081}{66336}\zeta_{12}\Big)\epsilon^6 + \mathcal{O}(\epsilon^7)
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Uniform transcendentality conjecture

Observation: up to NNNLO quantum corrections exhibit **uniform transcendentality**

Conjecture ***1***: the perturbative series of 2pt functions of $N = 4$ *SYM* dimension-2 protected operators in dimensional regularization has **uniform transcendental weight *2L*** @ loop *L* to ***all orders***

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***Non-MZV* numbers** are expected to appear in propagator integrals @8 loops. If they also appear in the two-point function, some extension in defining *transcendentality* beyond *MZV*s is required. But they might not …

Panzer et al 2017

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Panzer et al 2017

Can we infer a pattern for the *leading terms* highlighted in ?

Ask the OEIS

Reconstruct sequence for red circled *O(ε)* terms: **ask OEIS**

The OEIS is supported by the many generous donors to the OEIS Foundation.

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A000888 a(n) = (2*n)!^2 / ((n+1)!*n!^3). 17

1, 2, 12, 100, 980, 10584, 121968, 1472328, 18404100, 236390440, 3103161776, 41469525552, 562496897872, 7726605740000, 107289439704000, 1503840313184400, 21252802073091300, 302539888334593800, 4334635827016110000, 62464383654579522000, 904841214653480504400 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS a(n) is the number of walks of 2n unit steps North, East, South, or West, starting at the origin, bounded above by y=x, below by y=-x and terminating on the ray y = x >= 0. Example: a(1) counts EN, EW; a(2) counts ESNN, ESNW, ENSN, ENSW, ENEN, ENEW, EENN, EENW, EEWN, EEWW, EWEN, EWEW. - David Callan, Oct 11 2005

Bijective proof: given such a NESW walk, construct a pair (P_1, P_2) of lattice paths of upsteps U=(1,1) and downsteps D=(1,-1) as follows. To get P_1, replace each E and S with U and each W and N with D. To get P_2, replace each N and E with U and each S and W with D. For

A second conjecture

*Conjecture **2***: according to the OEIS oracle, the *O(ε)* reads to all orders

$$\frac{\langle O_0(p)O_0(-p)\rangle^{(l)}}{\langle O_0(p)O_0(-p)\rangle^{(0)}} = (-1)^l \frac{(2l+2)!^2}{(l+2)(l+1)!^4} \zeta_{2l+1}\, \epsilon + \mathcal{O}\left(\epsilon^2\right)$$

Here is my all loop resummation!

Re-summing the series

The perturbative series for the *O(ε)* result

$$\frac{\langle O_0(p)O_0(-p)\rangle^{(l)}}{\langle O_0(p)O_0(-p)\rangle^{(0)}} = (-1)^l \frac{(2l+2)!^2}{(l+2)(l+1)!^4} \zeta_{2l+1}\,\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

can be **re-summed** using integral form of zeta and Σ ↔ ∫

$$\frac{\langle O_0(p)O_0(-p)\rangle}{\langle O_0(p)O_0(-p)\rangle^{(0)}} = 1 + \int_0^\infty \frac{2}{e^x - 1}\left({}_2F_3\left(\begin{matrix}\frac{3}{2} & \frac{3}{2} & \\ \frac{1}{2} & 2 & 3\end{matrix}\middle| -4x^2\lambda\right) - 1\right) dx\,\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

planar limit understood

Re-summing the series

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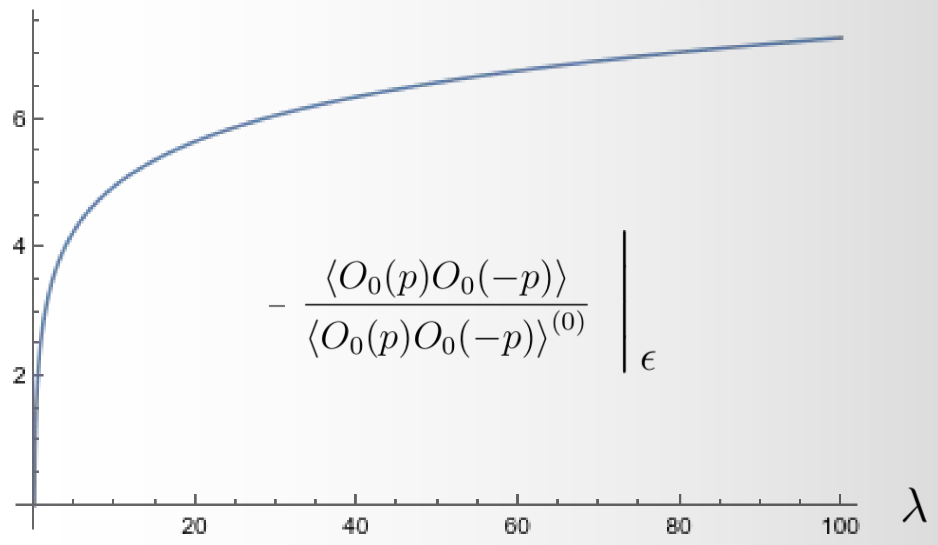
Some colleagues managed to check the conjecture via a technique called supersymmetric localization !!!

Pestun 2005, many others...

Re-summing the series

interpolating function from weak to strong coupling



can be **re-summed**

$$\frac{\langle O_0(p)O_0(-p)\rangle}{\langle O_0(p)O_0(-p)\rangle^{(0)}} = 1 + \int_0^\infty \frac{2}{e^x-1}\left({}_2F_3\left(\begin{matrix}\frac{3}{2} & \frac{3}{2} & \\ \frac{1}{2} & 2 & 3\end{matrix}\middle| -4x^2\lambda\right)-1\right)dx\,\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

This **invites a holographic calculation** of the correlation function at strong coupling using AdS/CFT (***in d = 4-2ε, though***)

?

Applications

MB in progress

First, get a link on the OEIS

A000888 a(n) = (2*n)!^2 / ((n+1)!*n!^3). +40 18

1, 2, 12, 100, 980, 10584, 121968, 1472328, 18404100, 236390440, 3103161776, 41469525552, 562496897872, 7726605740000, 107289439704000, 1503840313184400, 21252802073091300, 302539888334593800, 4334635827016110000, 62464383654579522000, 904841214653480504400

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If A is a random matrix in USp(4) (4 X 4 complex matrices that are unitary and symplectic), then a(n) = E[(tr(A^3))^{2n}]. - Andrew V. Sutherland, Apr 01 2008

Number of walks within N^2 (the first quadrant of Z^2) starting at (0,0), ending on the vertical axis and consisting of 2 n steps taken from {(-1,-1), (-1,1), (1,-1), (1,1)}. - Manuel Kauers, Nov 18 2008

a(n) is equal to the n-th moment of the following positive function defined on x in (0,16), in Maple notation: (EllipticK(sqrt(1-x/16)) - EllipticE(sqrt(1-x/16)))/(Pi^2*sqrt(x)). This is the solution of the Hausdorff moment problem and thus it is unique. - Karol A. Penson, Feb 11 2011

The partial sums of a(n)/A013709(n) absolutely converge to 1/Pi. - Ralf Steiner, Jan 21 2016

REFERENCES Eldon R. Hansen, A Table of Series and Products, Prentice-Hall, Englewood Cliffs, NJ, 1975, p. 93.

Thomas M. MacRobert, Functions of a Complex Variable, 4th ed., Macmillan & Co., London, 1958, p. 177.

LINKS Vincenzo Librandi, Table of n, a(n) for n = 0..100

Marco S. Bianchi, Protected and uniformly transcendental, arXiv:2306.06239 [hep-th], 2023.

Practical application?

- The UT property "***predicts***" that certain combinations of master integrals are of uniform transcendental weight
- This could be **useful** for **reconstructing** their **analytic** expansion from numerics
- Though at the level I am working (4 loops) MIs have been already expanded up to transcendentality 12 and **most** can be made UT by some overall rescalings

Lee, Smirnov, Smirnov 2011

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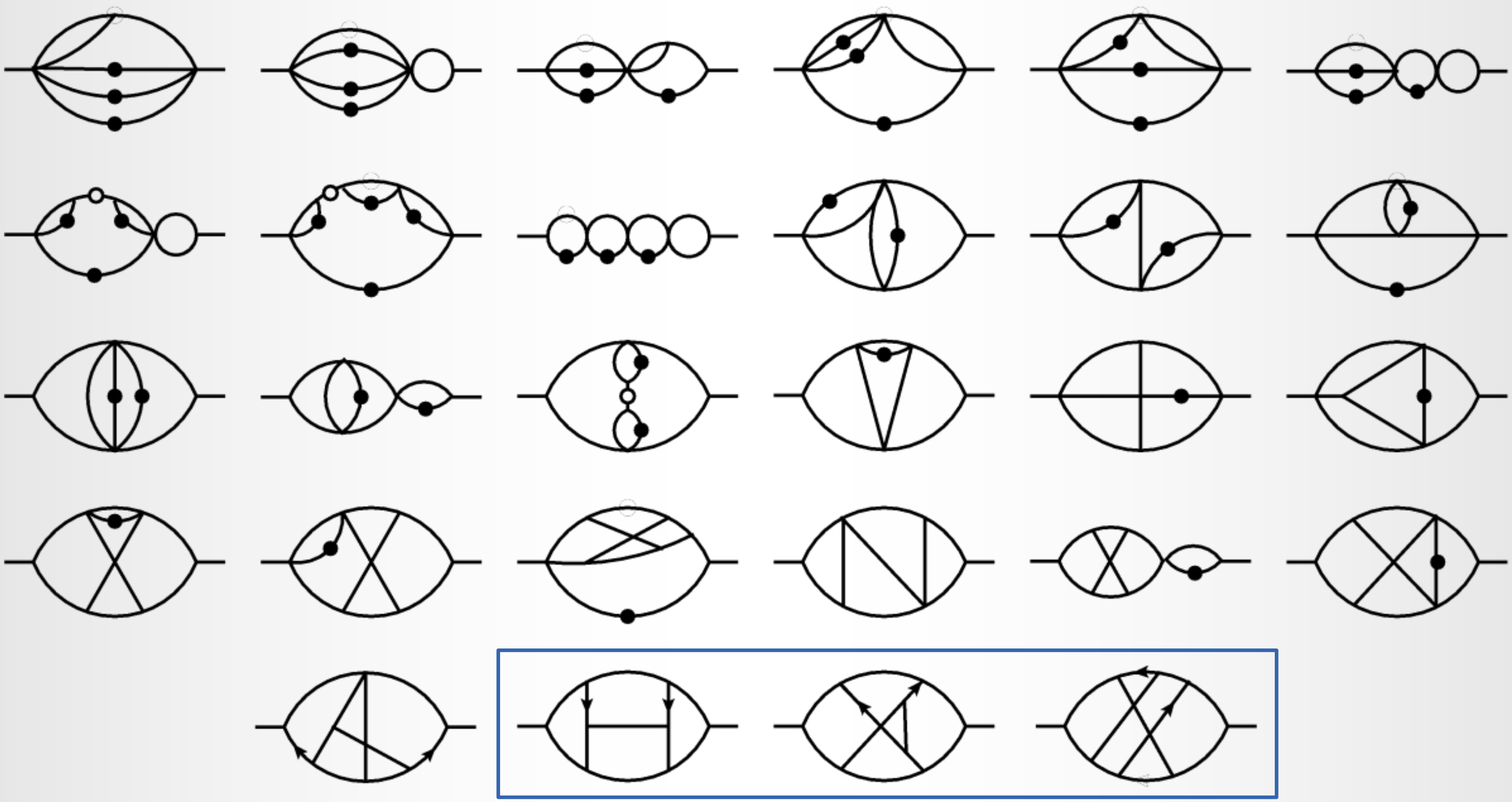
Lee, Smirnov, Smirnov 2011

- Our result suggests that we should be able to do better and find a **full basis of uniformly transcendental integrals**!

Uniformly transcendental basis

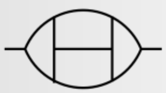
- Let's try to construct a basis of uniformly transcendental master integrals at four loops

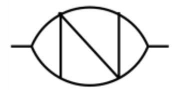
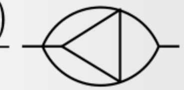
MB in progress



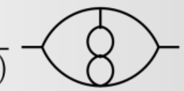
Hidden simplicity

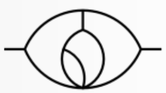
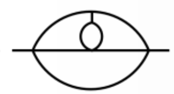
- Let's appreciate the UT basis in action
- The four-loop result in a non UT basis looks like

$\frac{(3\epsilon+1)^2}{\epsilon(5\epsilon+1)}$  $-\frac{2(3\epsilon+1)}{5\epsilon+1}$  $-\frac{2(3\epsilon+1)(4\epsilon+1)}{\epsilon(5\epsilon+1)}$  $-\frac{16(6\epsilon-1)(6\epsilon+1)}{(4\epsilon+1)(5\epsilon+1)}$ 

$+\frac{20(5\epsilon-1)\left(19\epsilon^2+13\epsilon+2\right)}{3\epsilon(3\epsilon+1)(5\epsilon+1)}$  $-\frac{8\epsilon}{(4\epsilon+1)(5\epsilon+1)}$  $-\frac{16(4\epsilon-1)(5\epsilon-1)\left(104\epsilon^2+69\epsilon+10\right)}{9\epsilon^2(4\epsilon+1)(5\epsilon+1)}$ 

$-\frac{4(6\epsilon+1)(2\epsilon-1)}{\epsilon(4\epsilon+1)}$  $+\frac{4(2\epsilon-1)}{\epsilon}$  $+\frac{4(2\epsilon-1)}{\epsilon}$  $+\frac{32\left(363\epsilon^2+205\epsilon+26\right)(2\epsilon-1)^2}{9\epsilon^2(3\epsilon+1)(5\epsilon+1)}$ 

$-\frac{8\left(300\epsilon^2+142\epsilon+15\right)(2\epsilon-1)^2}{3\epsilon^2(4\epsilon+1)(5\epsilon+1)}$  $+\frac{8(3\epsilon-1)^2\left(329\epsilon^3+223\epsilon^2+49\epsilon+4\right)(2\epsilon-1)}{3\epsilon^3(3\epsilon+1)(4\epsilon+1)(5\epsilon+1)}$  $+\frac{16(2\epsilon-1)^2}{(3\epsilon+1)(5\epsilon+1)}$ 

$-\frac{12(3\epsilon-1)\left(128\epsilon^2+65\epsilon+8\right)(2\epsilon-1)}{\epsilon^2(4\epsilon+1)(5\epsilon+1)}$  $+\frac{60(5\epsilon-1)(2\epsilon-1)}{\epsilon(3\epsilon+1)}$  $+\frac{16(4\epsilon-1)\left(324\epsilon^2+157\epsilon+18\right)(2\epsilon-1)^2}{3\epsilon^3(4\epsilon+1)(5\epsilon+1)}$ 

$+\frac{4(4\epsilon-1)(5\epsilon-1)\left(3592\epsilon^3+2938\epsilon^2+761\epsilon+63\right)(2\epsilon-1)}{3\epsilon^3(3\epsilon+1)(4\epsilon+1)(5\epsilon+1)}$  $-\frac{4(3\epsilon-1)(5\epsilon-1)\left(3288\epsilon^3+2629\epsilon^2+579\epsilon+38\right)(2\epsilon-1)}{9\epsilon^3(3\epsilon+1)(4\epsilon+1)(5\epsilon+1)}$ 

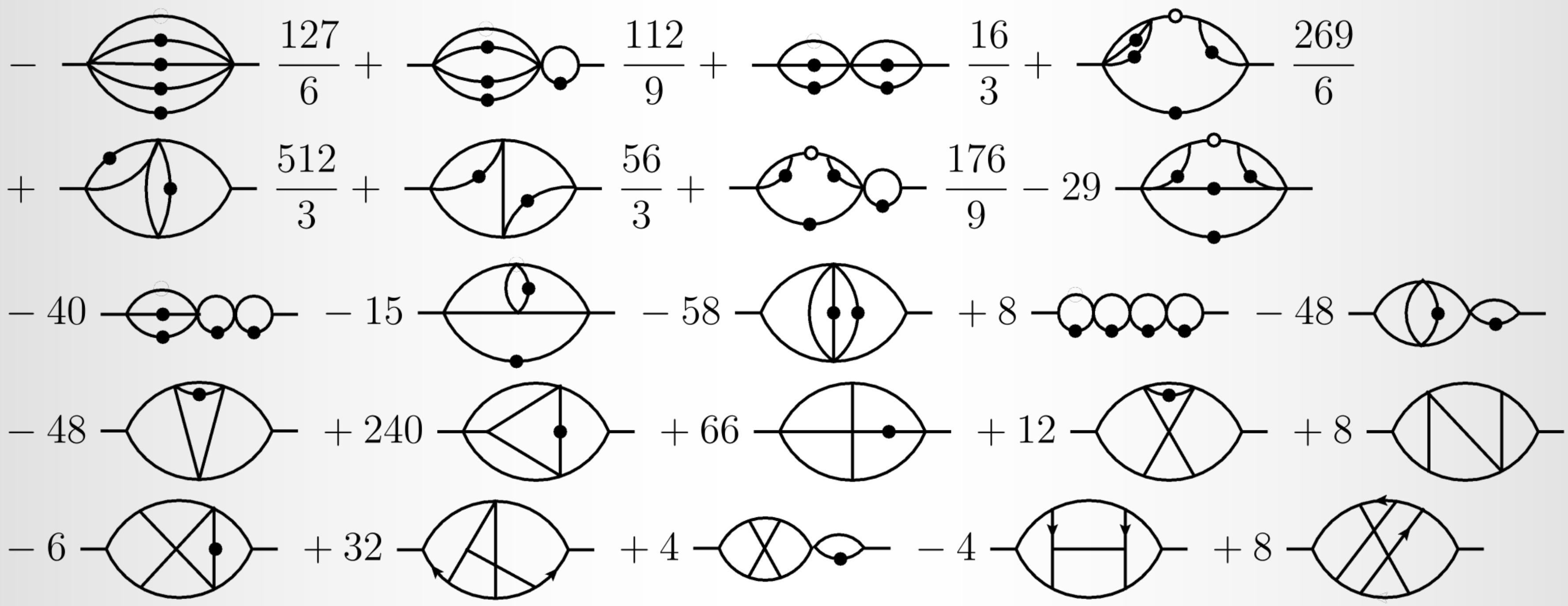
$-\frac{20(5\epsilon-1)(2\epsilon-1)^2}{\epsilon^3}$  $-\frac{4(3\epsilon-1)(4\epsilon-1)(5\epsilon-1)\left(14484\epsilon^3+10379\epsilon^2+2433\epsilon+190\right)}{9\epsilon^3(3\epsilon+1)(4\epsilon+1)(5\epsilon+1)}$ 

$-\frac{48(4\epsilon+1)(2\epsilon-1)}{\epsilon(5\epsilon+1)}$  $+\frac{16(3\epsilon-1)(4\epsilon-1)\left(1257\epsilon^3+833\epsilon^2+165\epsilon+10\right)(2\epsilon-1)}{9\epsilon^3(3\epsilon+1)(4\epsilon+1)(5\epsilon+1)}$ 

$-\frac{48(2\epsilon-1)^2}{\epsilon^2}$  $-\frac{4(3\epsilon-1)\left(408\epsilon^2+187\epsilon+21\right)(2\epsilon-1)^2}{\epsilon^3(4\epsilon+1)(5\epsilon+1)}$  $+\frac{8(2\epsilon-1)^3}{\epsilon^3}$ 

Hidden simplicity

- Let's appreciate the UT basis in action
- In a UT basis it simplifies to



- With **only rational coefficients**!
- Hallmark of hidden simplicity of *N=4* SYM

Conclusions

Conclusions

- **Transcendentality** has been playing a prominent role in perturbative calculations in QFTs
- **Uniform transcendentality** appears in several magnitudes of $N = 4$ SYM
- **No deep explication** is available for such a property
- Yet it has been **leveraged** for phenomenological applications
- We have uncovered this property in some **new magnitudes**: **two-point functions** of (lowest dimension) protected operators
- This transcendental structure helped guessing an **all-loop re-summation**
- We exposed simplicity by constructing a **basis** of master integrals with manifest **uniform transcendentality**

Thank you!