EX UMBRA IN SOLEM

Running coupling corrections and nonlinear QCD evolution at small-x

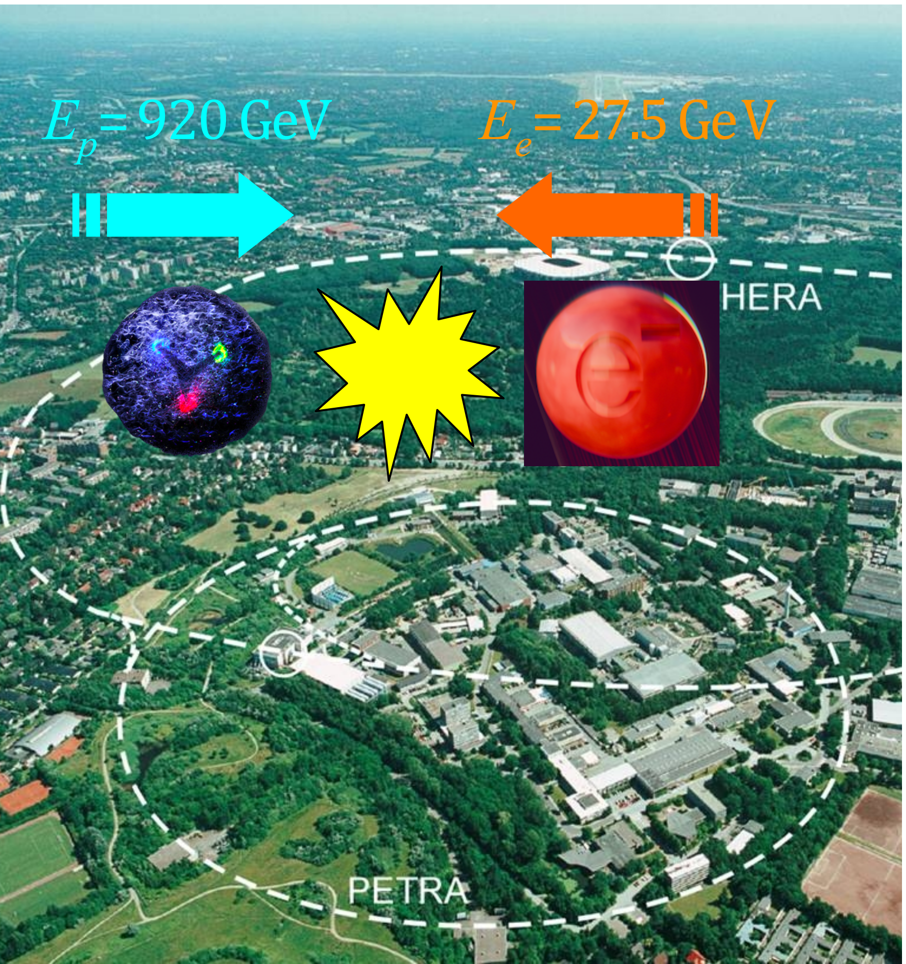
José Garrido
Universidad Técnica Federico Santa María

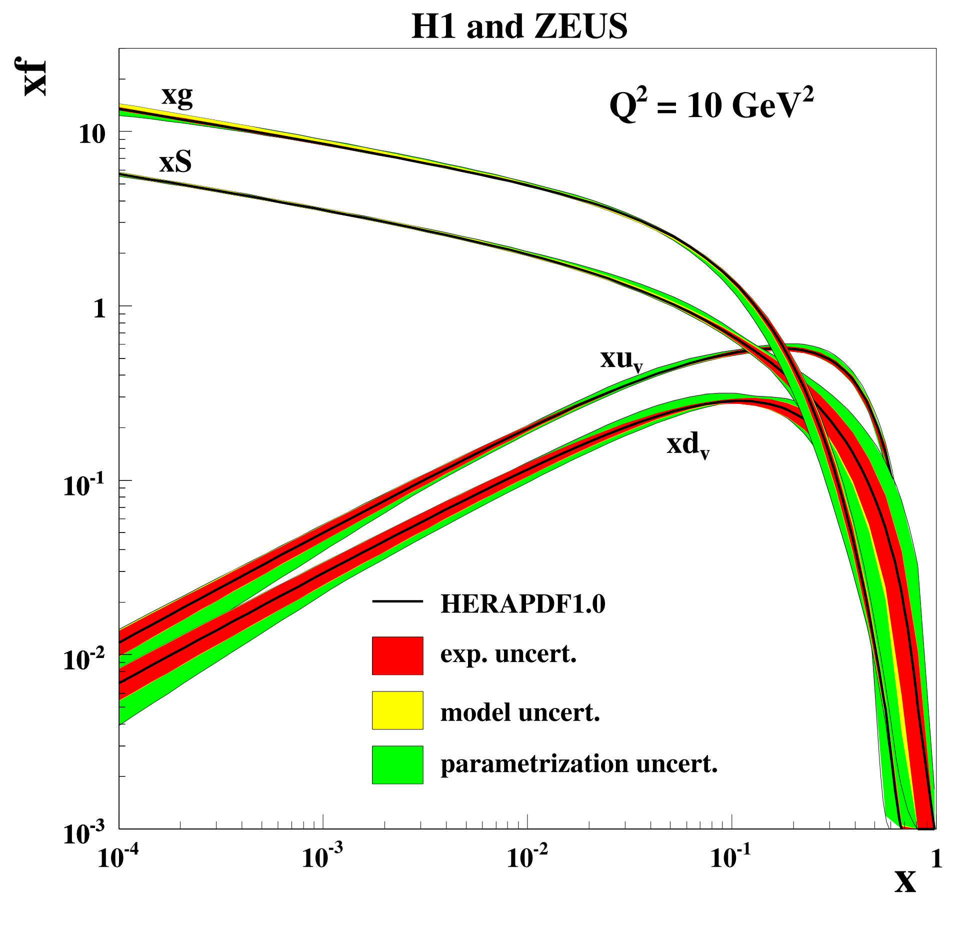
C. Contreras, J. Garrido, E. Levin: arXiv:2503.19771 [hep-ph]

6th WS on Nonperturbative Aspects of QCD, Valparaíso, Chile, December 1-5, 2025

Introduction

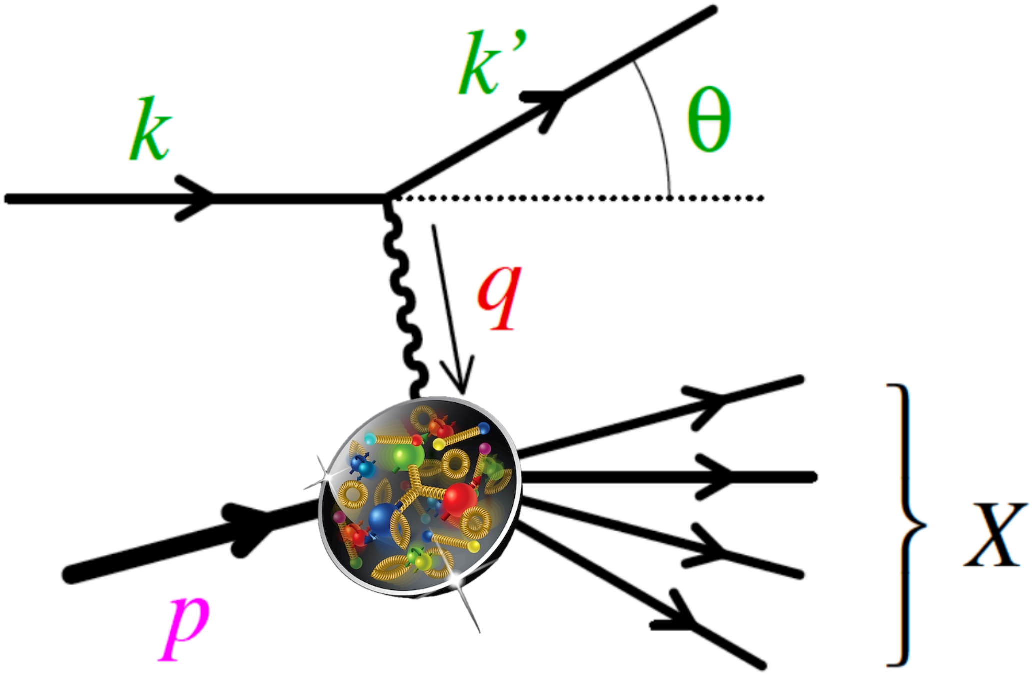
A key lesson from the HERA $e^{\pm}p$ DIS collider: "Gluons and sea quarks dominate the proton wave-function at high energies





Introduction

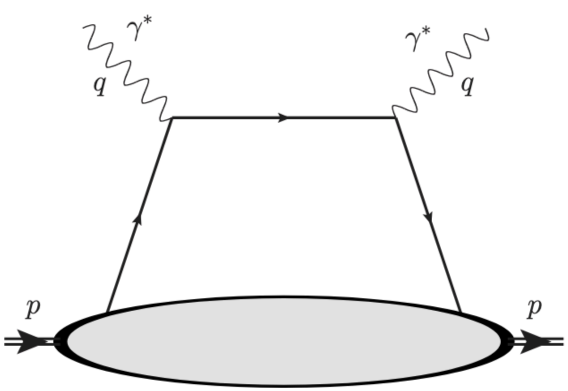
Deep Inelastic Scattering (DIS)

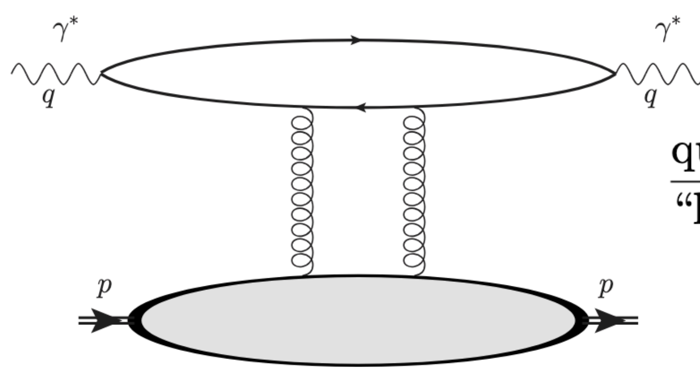


Kinematics:

$$
\begin{aligned}
s &= (P+q)^2 \\
x &= \frac{Q^2}{2P\cdot q} \\
Q^2 &= -q^2 = (k-k')^2 \\
y &= \frac{Q^2}{sx}
\end{aligned}
$$

$$\text{Small}-x \Leftrightarrow \text{High energy } s$$

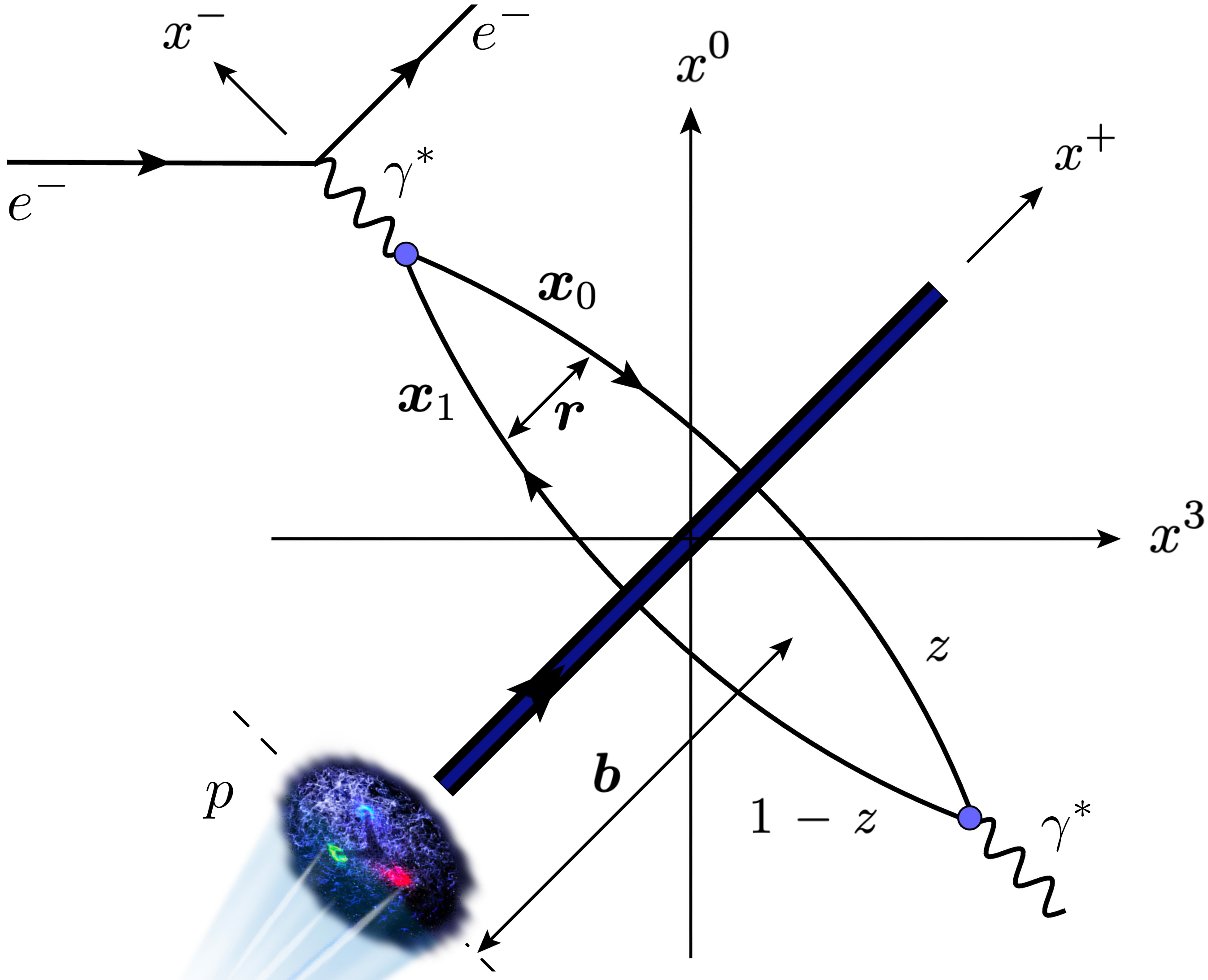




$$\frac{\text{quark loop diagram}}{\text{“handbag” diagram}} \propto \frac{\alpha_s}{x}.$$

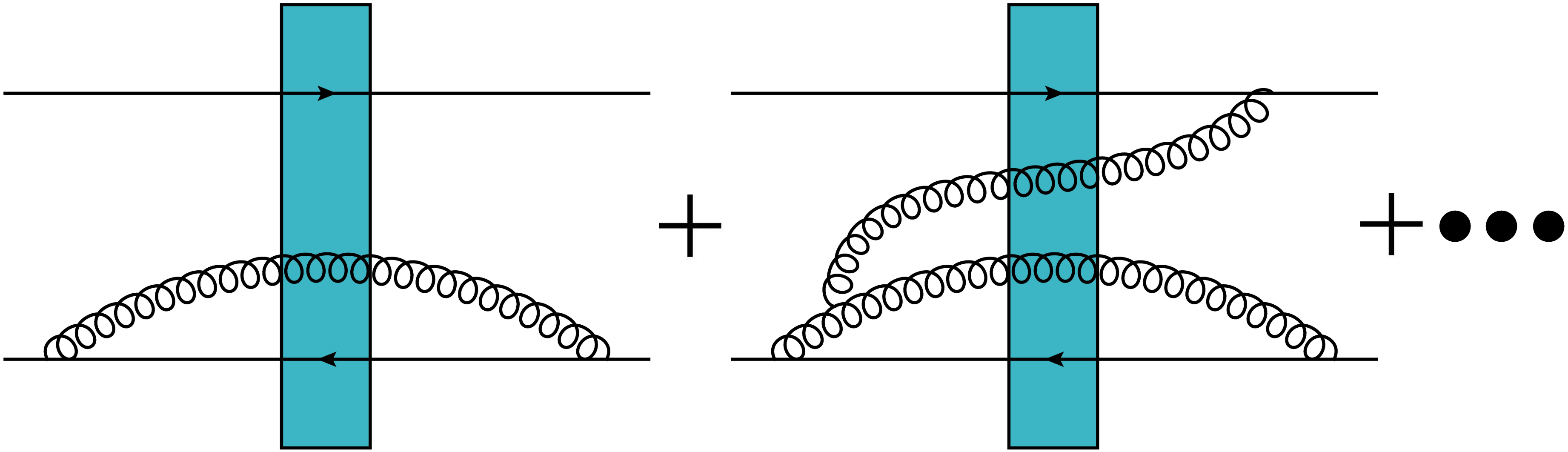
Introduction

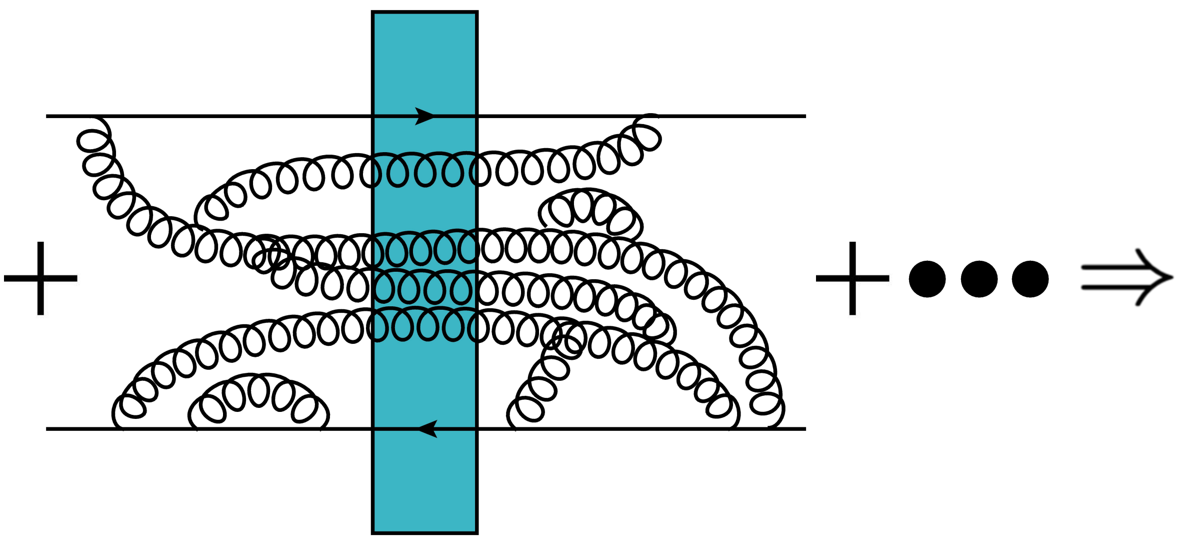
For x < 0.01, the 1/x enhancement overcomes the ~ α_s suppression factor, and the quark loop diagram dominates



Introduction

Quantum corrections:





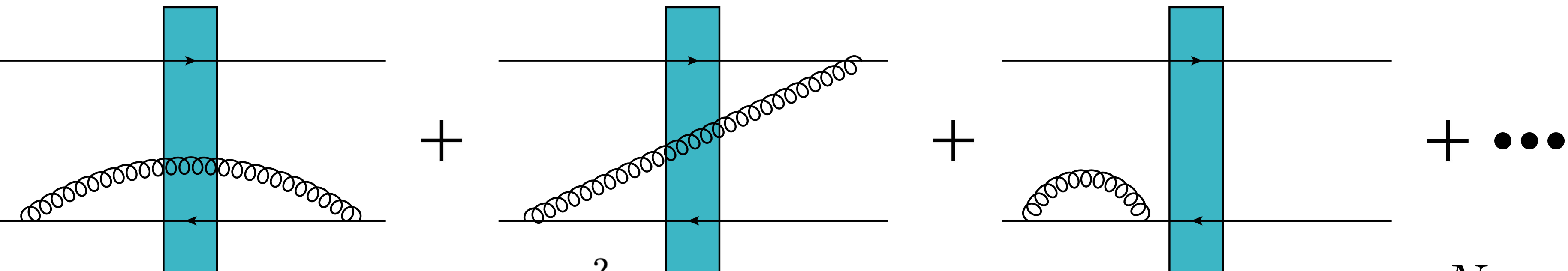
We cannot sum exactly. But we know how to sum for leading-log (LLA) or next-to-leading log (NLL).

Wee partons (x << 1) are intrinsically non-perturbative.

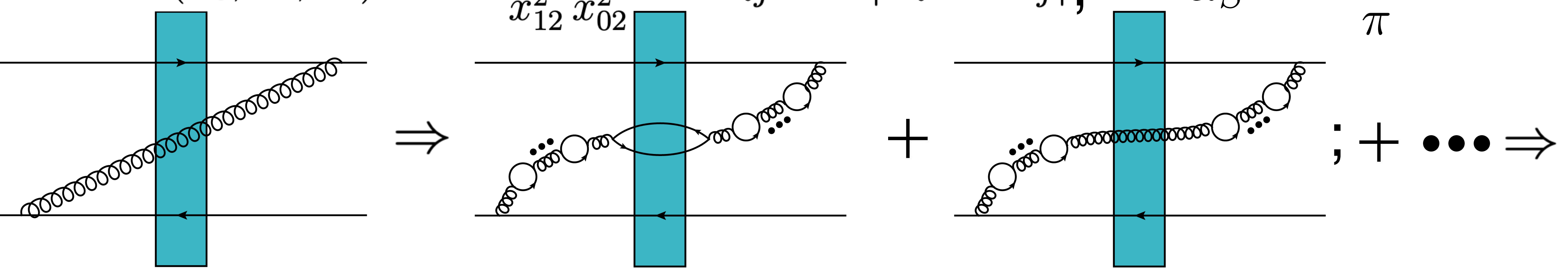
Introduction

The sum was first calculated by I. Balitsky '96 (effective lagrangian) and Y. Kovchegov '99 (large N_c QCD):

$$\frac{\partial N(\boldsymbol{x}_0, \boldsymbol{x}_1; Y)}{\partial Y} = \frac{1}{2\pi} \int d^2\boldsymbol{x}_2\, K(\boldsymbol{x}_0, \boldsymbol{x}_1; \boldsymbol{x}_2) \times \Big[N(\boldsymbol{x}_0, \boldsymbol{x}_2; Y) + N(\boldsymbol{x}_2, \boldsymbol{x}_1; Y) - N(\boldsymbol{x}_0, \boldsymbol{x}_1; Y) - N(\boldsymbol{x}_0, \boldsymbol{x}_2; Y) N(\boldsymbol{x}_2, \boldsymbol{x}_1; Y) \Big]$$

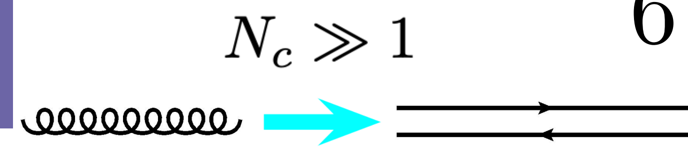


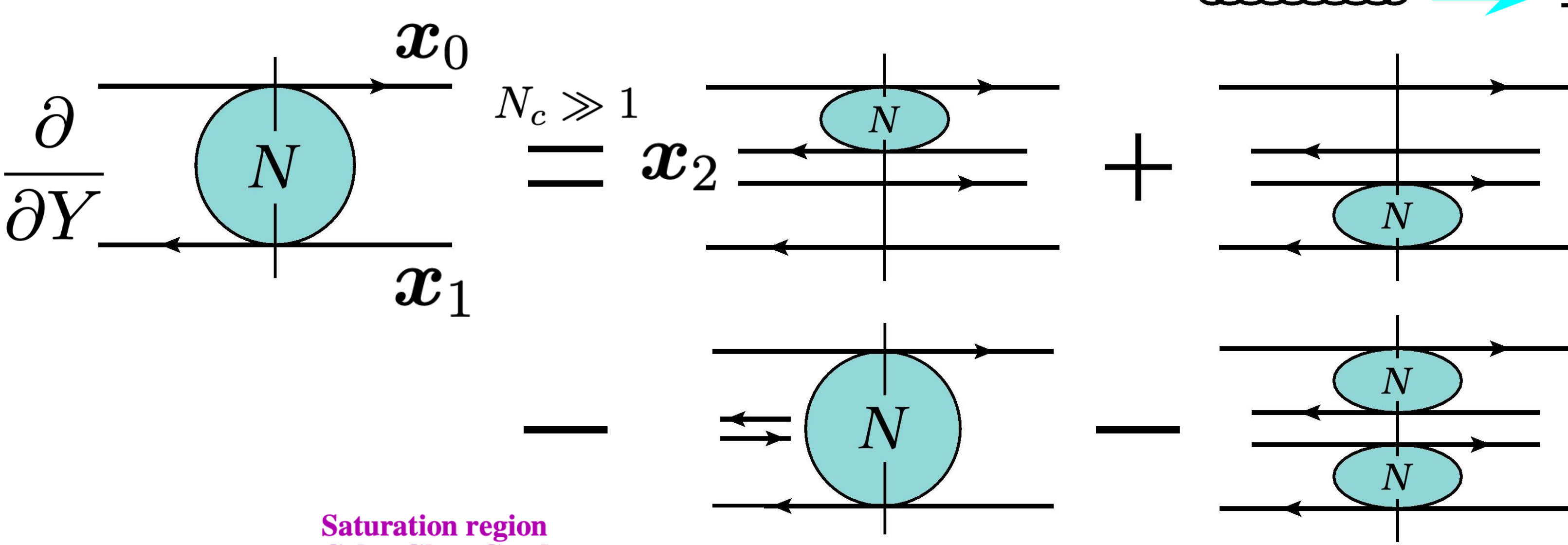
$$\Rightarrow K^{LO}(\boldsymbol{x}_0, \boldsymbol{x}_1; \boldsymbol{x}_2) = \bar{\alpha}_S \frac{x_{01}^2}{x_{12}^2\, x_{02}^2}; \quad x_{ij} = |\boldsymbol{x}_i - \boldsymbol{x}_j|; \quad \bar{\alpha}_S = \frac{\alpha_s N_c}{\pi}$$

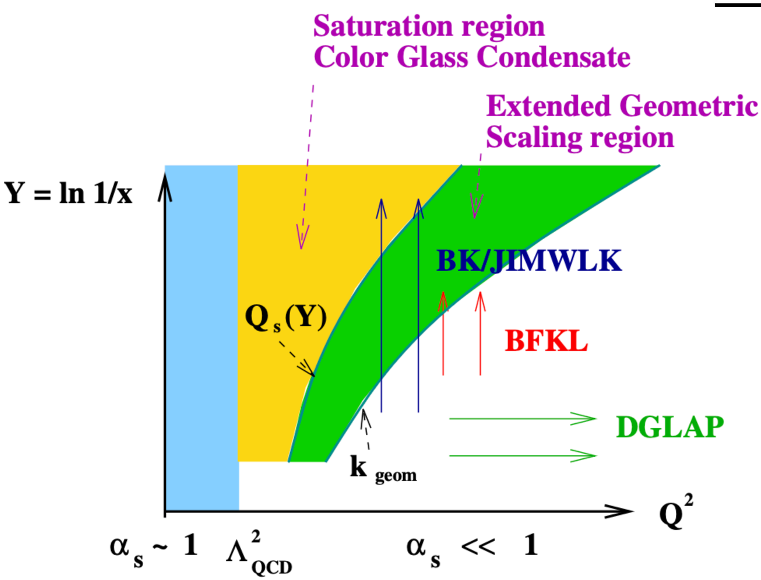


$$K^{\mathrm{Bal}}_{\mathrm{rcBK}}(\boldsymbol{x}_0, \boldsymbol{x}_1; \boldsymbol{x}_2) = \bar{\alpha}_S(1/x_{01}^2) \left[\frac{x_{01}^2}{x_{12}^2\, x_{02}^2} + \frac{1}{x_{12}^2} \left(\frac{\bar{\alpha}_S(1/x_{12}^2)}{\bar{\alpha}_S(1/x_{02}^2)} - 1 \right) + \frac{1}{x_{02}^2} \left(\frac{\bar{\alpha}_S(1/x_{02}^2)}{\bar{\alpha}_S(1/x_{12}^2)} - 1 \right) \right]$$

Introduction







The breakdown of the linear regime occurs when

$$N(r = 1/Q_s(Y), Y) \;=\; \text{const}$$

which gives

$$\kappa \;\equiv\; \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}$$

$$Q_s^2(Y) \;=\; Q_{s0}^2 \,\exp\left\{\bar{\alpha}_S\,\kappa\,Y \;-\; \frac{3}{2(1-\gamma_{cr})}\ln\bar{\alpha}_S\,Y + \mathcal{O}\left(\frac{1}{\sqrt{Y}}\right)\right\}; \quad \gamma_{cr} = 0.37$$

Introduction

Solution in the extended geometric region

$$\frac{\partial N(\boldsymbol{x}_0, \boldsymbol{x}_1; Y)}{\partial Y} = \frac{\bar{\alpha}_S}{2\pi} \int d^2 \boldsymbol{x}_2 \frac{(\boldsymbol{x}_0 - \boldsymbol{x}_1)^2}{(\boldsymbol{x}_0 - \boldsymbol{x}_2)^2 (\boldsymbol{x}_2 - \boldsymbol{x}_1)^2} \times \left[N(\boldsymbol{x}_0, \boldsymbol{x}_2; Y) + N(\boldsymbol{x}_2, \boldsymbol{x}_1; Y) - N(\boldsymbol{x}_0, \boldsymbol{x}_1; Y) \right]$$

$$\Rightarrow N(\boldsymbol{x}_0, \boldsymbol{x}_1; Y) = N_0 \left(r^2 Q_s^2(Y) \right)^{1-\gamma_{cr}}$$

Approximate solution in the saturation region: We simplify the kernel by taking into account only log contributions

$$\Rightarrow \frac{\partial N(\boldsymbol{x}_0, \boldsymbol{x}_1; Y)}{\partial Y} = (1 - N(\boldsymbol{x}_0, \boldsymbol{x}_1; Y)) \int\limits_{1/Q_s^2(Y)}^{r^2} \frac{d x_{02}^2}{x_{02}^2} N(\boldsymbol{x}_0, \boldsymbol{x}_2; Y)$$

with solution

$$N(\boldsymbol{x}_0, \boldsymbol{x}_1; Y) = 1 - e^{-\Omega(\boldsymbol{x}_0, \boldsymbol{x}_1; Y)}; \quad \Omega(\boldsymbol{x}_0, \boldsymbol{x}_1; Y) = \mathcal{U}^{-1}(\bar{\gamma} z);$$

$$\mathcal{U}(\Omega) = \int\limits_{\Omega_0}^{\Omega} \frac{d\Omega'}{\sqrt{\Omega_0^2 \left(1 - \frac{1}{\bar{\gamma}^2 \kappa}\right) + \frac{2}{\bar{\gamma}^2 \kappa} \left(-1 + \Omega' + e^{-\Omega'}\right)}}$$

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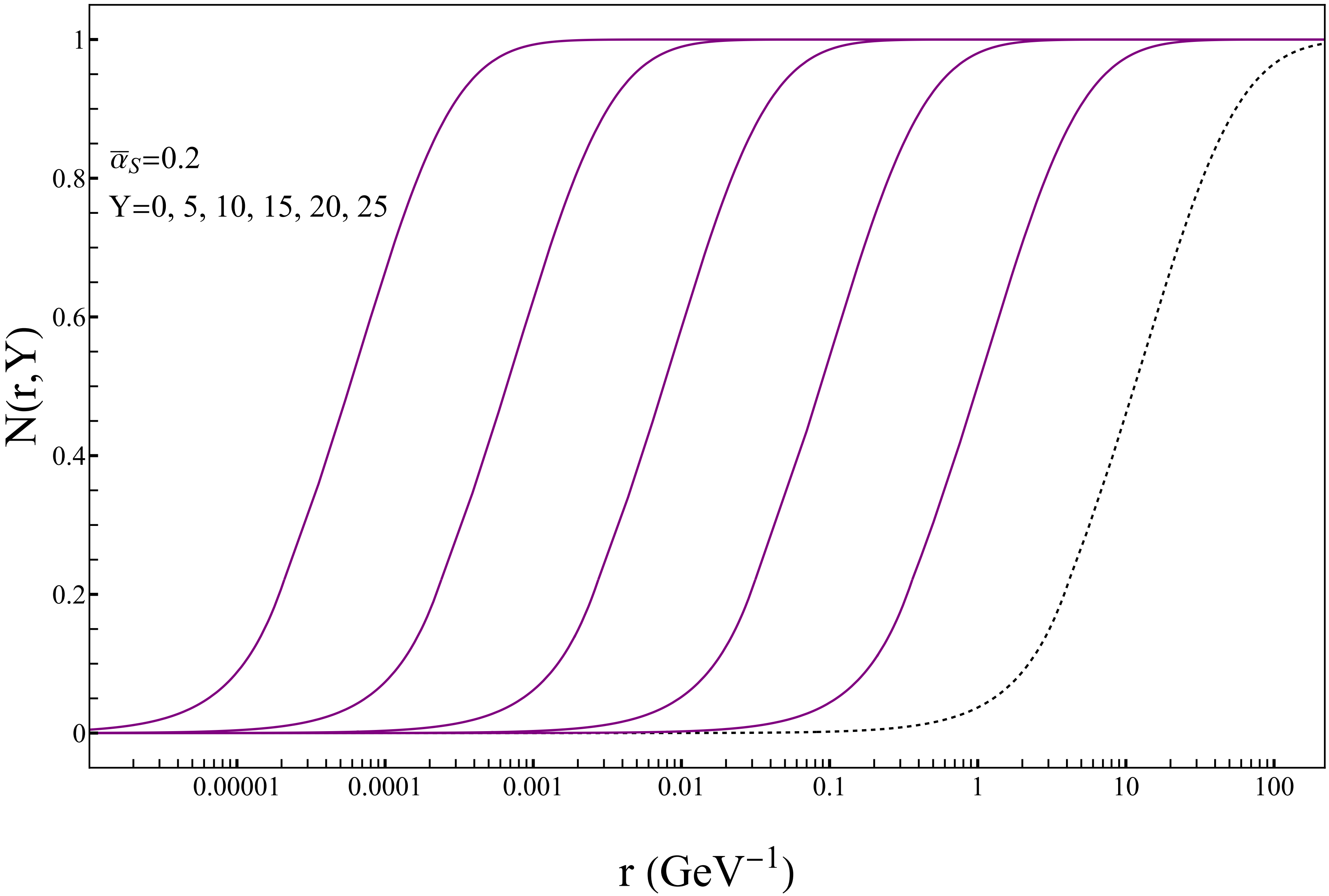
$$N(r,Y) \;=\; N_0\left(r^2 Q_s^2(Y)\right)^{\bar{\gamma}}\theta\left(1 \;-\; rQ_s(Y)\right) \;+\; \left(1 \;-\; e^{-\Omega(r,Y)}\right)\theta\left(rQ_s(Y)-1\right)$$



RC BK

Starting from

$$\frac{\partial N(\boldsymbol{x}_0,\boldsymbol{x}_1;Y)}{\partial Y} = \frac{1}{2\pi}\int d^2\boldsymbol{x}_2\, K^{\rm Bal}_{\rm rcBK}(\boldsymbol{x}_0,\boldsymbol{x}_1;\boldsymbol{x}_2)\times\Big[N(\boldsymbol{x}_0,\boldsymbol{x}_2;Y) + N(\boldsymbol{x}_2,\boldsymbol{x}_1;Y) - N(\boldsymbol{x}_0,\boldsymbol{x}_1;Y) - N(\boldsymbol{x}_0,\boldsymbol{x}_2;Y)N(\boldsymbol{x}_2,\boldsymbol{x}_1;Y)\Big]$$

In the vicinity of the saturation scale where

$$x_{01}^2 \approx x_{12}^2 \approx x_{02}^2 \approx 1/Q_s^2$$

we can consider $\bar{\alpha}_S(1/x_{ij}^2) \to \bar{\alpha}_S(1/x_{01}^2)$ so that

$$K^{\rm Bal}_{\rm rcBK}(\underline{x}_0,\underline{x}_1;\underline{x}_2) \approx \bar{\alpha}_S(1/x_{01}^2)\,\frac{x_{01}^2}{x_{02}^2\,x_{12}^2}$$

And taking only the log contributions, it reduces to

$$\frac{\partial N(\boldsymbol{x}_0,\boldsymbol{x}_1;Y)}{\partial Y} = \left(1 - N(\boldsymbol{x}_0,\boldsymbol{x}_1;Y)\right)\int\limits_{1/Q_s^2(Y)}^{r^2}\frac{dx_{02}^2}{x_{02}^2}\,\bar{\alpha}_S(1/x_{02}^2)\,N(\boldsymbol{x}_0,\boldsymbol{x}_2;Y)$$

RC BK

From the equations that determine the saturation momentum it is found

$$Q_s^2(Y) = \Lambda_{QCD}^2 e^{2.44\bar{\alpha}_S Y} \qquad Q_s^2(Y) = \Lambda_{QCD}^2 e^{\sqrt{39Y/3}}$$

(fixed) (running)

We are not able to find a solution with geometric scaling (GS) $z = \ln(r^2 Q_s^2(Y))$. What we found instead is

$$N(\boldsymbol{x}_0, \boldsymbol{x}_1; Y) = 1 - e^{-\Omega(\boldsymbol{x}_0, \boldsymbol{x}_1; Y)}; \; \Omega(\boldsymbol{x}_0, \boldsymbol{x}_1; Y) = \Omega_\zeta(\zeta) + \Omega'(Y, \zeta)$$

$$\Rightarrow \; \zeta \frac{d^2\Omega_\zeta(\zeta)}{d\zeta^2} + \frac{d\Omega_\zeta(\zeta)}{d\zeta} = 1 - e^{-\Omega_\zeta(\zeta)}$$

$$\Rightarrow \; \frac{\partial^2 \Omega'(\zeta, l - l_s; b)}{\partial Y \, \partial(l - l_s)} = e^{-\Omega_\zeta(\zeta)} \Omega'(\zeta, l - l_s; b) \; ; \; l - l_s = -\frac{4 N_c}{b_0} \ln\left(\frac{-\xi}{\xi_s}\right)$$

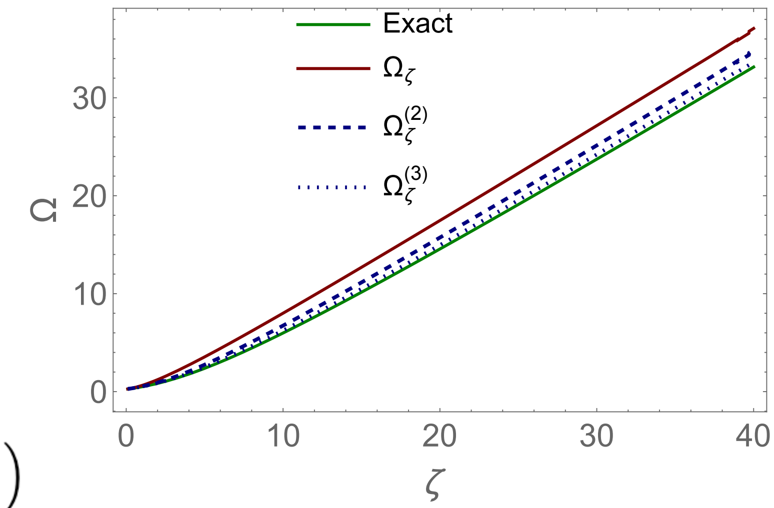
$$\zeta = -\frac{4N_c}{b_0} Y \ln\left(\bar{\alpha}_S(Q_s^2(Y))/\bar{\alpha}_S(1/r^2)\right)$$

RC BK

$$\zeta\,\frac{d^2\Omega^{(0)}\left(\zeta;b\right)}{d\zeta^2}\;+\;\frac{d\Omega^{(0)}\left(\zeta;b\right)}{d\zeta}\;=\;1$$

$$\zeta\,\frac{d^2\Omega^{(1)}\left(\zeta;b\right)}{d\zeta^2}\;+\;\frac{d\Omega^{(1)}\left(\zeta;b\right)}{d\zeta}\;=\;-\,e^{-\Omega^{(0)}\left(\zeta;b\right)}$$

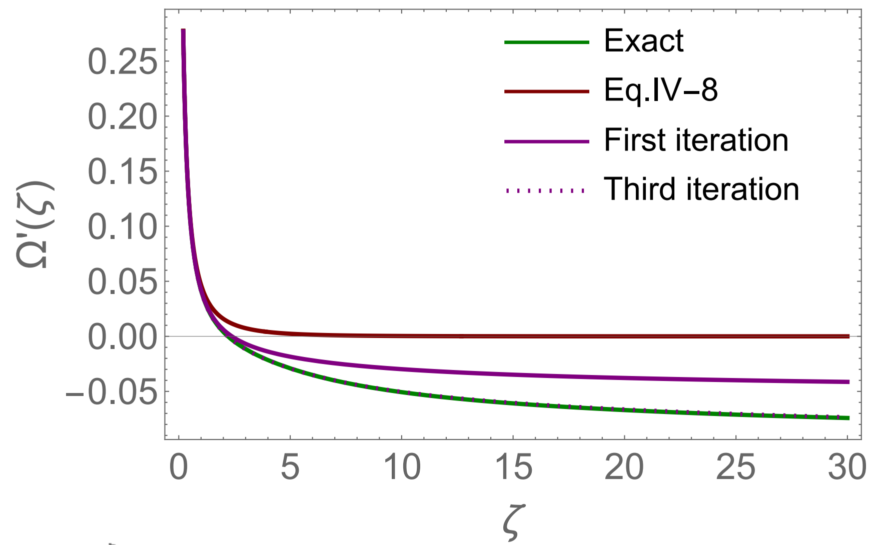
$$\frac{d}{d\,\zeta}\left(\zeta\frac{d}{d\zeta}\Omega^{(i+1)}\left(\zeta;b\right)\right)\;=\;\exp\left(-\sum_{l=0}^{i-1}\Omega^{(l)}\left(\zeta;b\right)\right)\left(1-e^{-\Omega^{(i)}\left(\zeta;b\right)}\right)$$



Iterative solution or numerical

$$\Omega'\left(\zeta,l-l_s;b\right)=\mathcal{Z}\left(\zeta\right)\;\mathcal{L}\left(l-l_s\right)$$

$$\frac{3}{2}\mathcal{Z}_\zeta\left(\zeta\right)\;+\;\zeta\,\mathcal{Z}''_{\zeta,\zeta}\left(\zeta\right)\;=\;e^{-\Omega_\zeta\left(\zeta\right)}\mathcal{Z}\left(\zeta\right)$$



$$\Delta^{(i)}\mathcal{Z}\left(\zeta\right)=\int_0^\zeta\frac{dt}{t^{3/2}}\int_0^t dt'\sqrt{t'}\,e^{-\Omega_\zeta\left(t'\right)}\Delta^{(i-1)}\mathcal{Z}\left(t'\right)=-\,2\int_0^\zeta dt\left(\sqrt{\frac{t}{\zeta}}-1\right)\,e^{-\Omega_\zeta\left(t\right)}\Delta^{(i-1)}\mathcal{Z}\left(t\right)$$

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To solve this last eq. we apply the homotopy perturbation method (HPM): $\mathscr{L}[u]+\mathscr{N}_{\mathscr{L}}[u]=0$

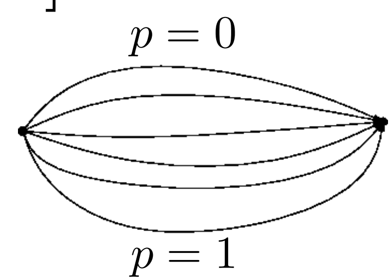
Homotopy: One can be "continuously deformed" into the other.

$\mathscr{L}$: Integro-differential operator

$\mathscr{N}_{\mathscr{L}}$: Non-linear part; arbitrary form

We construct a homotopy $u_p : \Omega \times [0,1] \to \mathbb{R}$ such that

$$\mathscr{H}(p,u) \;=\; \mathscr{L}[u_p] + p\,\mathscr{N}_{\mathscr{L}}[u_p] \;=\; 0$$



where $p\in[0,1]$ is an embedding parameter. The sol. is

$$u_p(Y,\underline{x}_{01},\underline{b}) \;=\; u_0(Y,\underline{x}_{01},\underline{b})\left(1+p\frac{u_1(Y,\underline{x}_{01},\underline{b})}{u_0(Y,\underline{x}_{01},\underline{b})}+p^2\frac{u_2(Y,\underline{x}_{01},\underline{b})}{u_0(Y,\underline{x}_{01},\underline{b})}+\cdots\right)$$

For $u=\lim_{p\to 1}u_p=u_0+u_1+u_2+\cdots$ it gives the solution to the nonlinear equation (J.H. He, 1999)

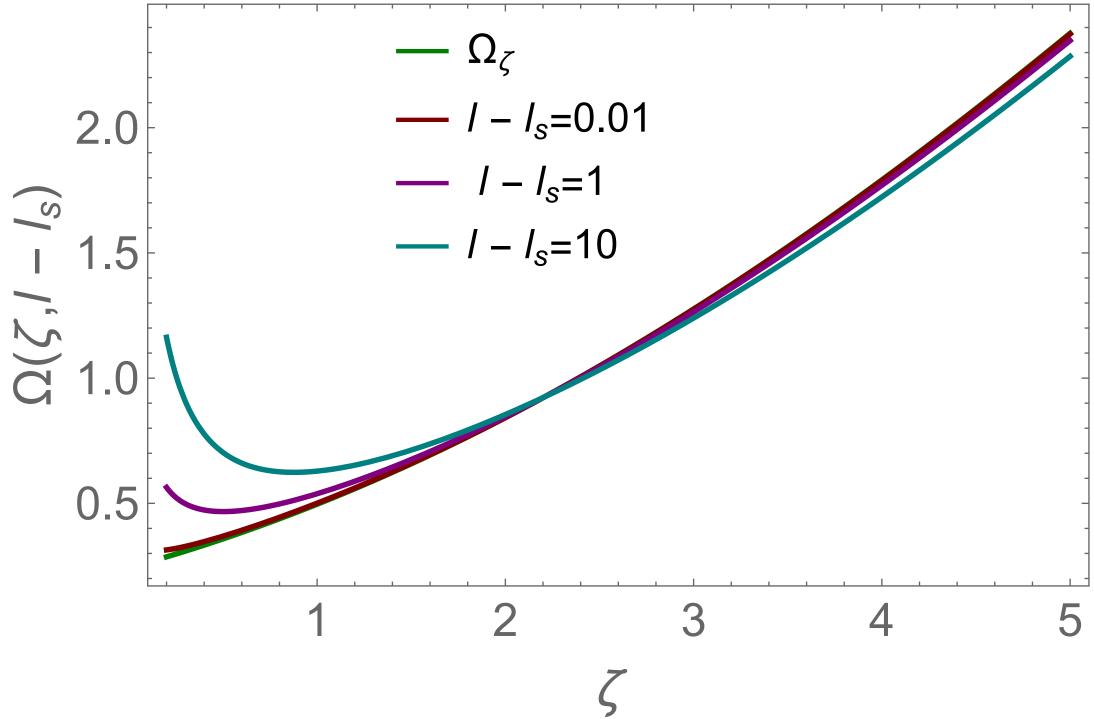
RC BK

Defining $$\mathscr{L}[\Omega^{(p)}] \;=\; \frac{\partial^2 \Omega^{(p)}(Y,r,b)}{\partial Y\,\partial l} \;-\;1\;+\;e^{-\Omega^{(p)}(Y,r,b)}$$

$$\mathcal{N}_{\mathscr{L}}[\Omega^{(p)}] = \int\limits_{r_1>r}^{1/\Lambda_{QCD}} \frac{d^2 r_1}{2\pi} K\left(r; r_1, r_2\right) \exp\left(-\,\Omega^{(p)}\left(r_1, Y; \boldsymbol{b} - \frac{1}{2}\boldsymbol{r}_2\right) - \Omega^{(p)}\left(r_2, Y; \boldsymbol{b} - \frac{1}{2}\boldsymbol{r}_1\right)\right)$$

For the zero iteration it gives

$$\Omega^{(0)}(\zeta, l - l_s) \;\equiv\; \Omega^{(3)}_{\zeta}(\zeta) \;+\; \Omega'(\zeta, l - l_s)$$

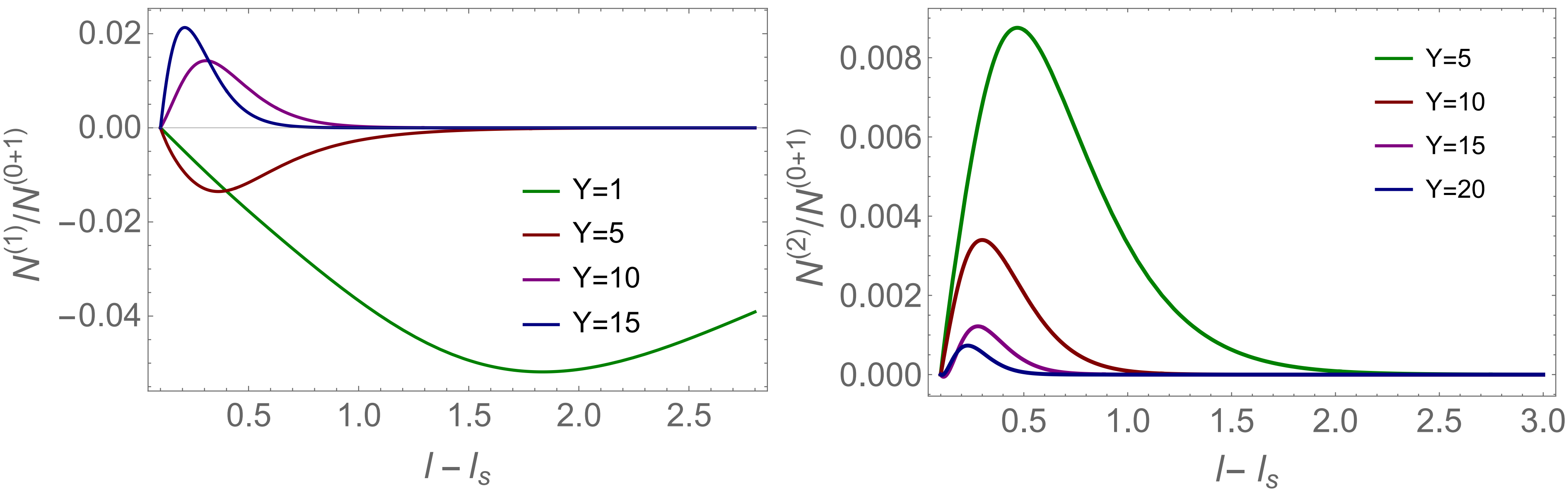


RC BK

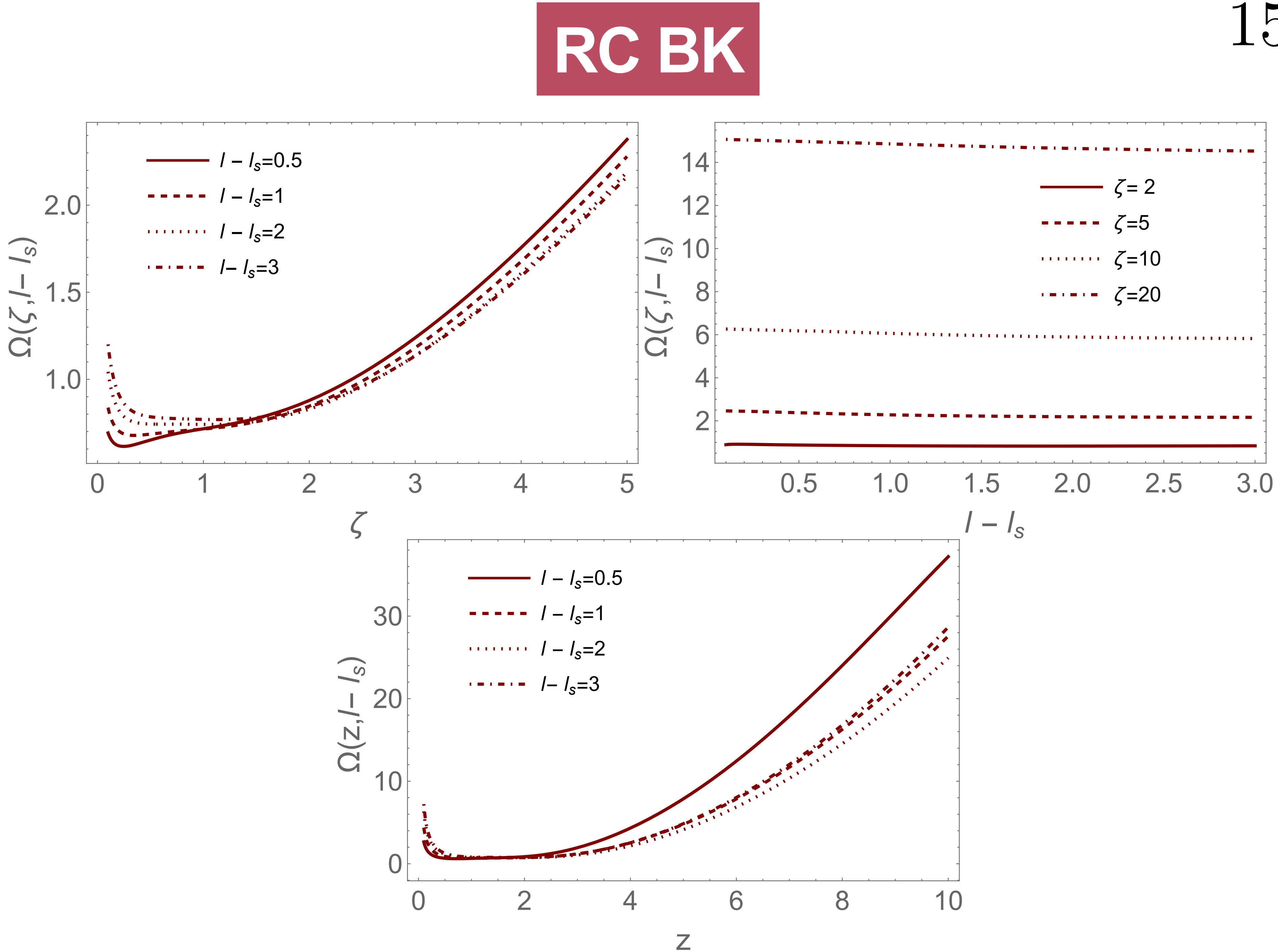
And for the corrections it gives

$$\frac{\partial^2\Omega^{(1)}\left(Y,l-l_s\right)}{\partial Y\,\partial l} = \left(1-e^{-\Omega^{(1)}(Y,l-l_s)}\right)\, e^{-\Omega^{(0)}(Y,l-l_s)} \;-\; \underbrace{\frac{\partial}{\partial l}\left(e^{\Omega^{(0)}(Y,l-l_s)}\mathscr{N}_{\mathscr{L}}[\Omega^{(0)}]\right)}_{DH^{(0)}(Y,l-l_s)}$$

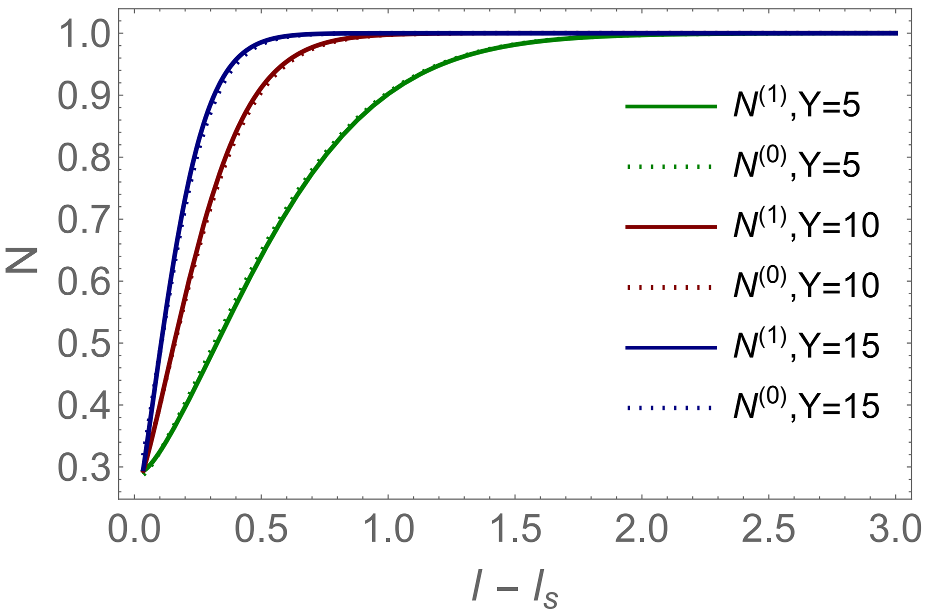
$$\frac{\partial^2\Omega^{(2)}\left(Y,l-l_s\right)}{\partial Y\,\partial (l-l_s)} = \Omega^{(2)}\left(Y,l-l_s\right)\, e^{-\Omega^{(0)}(Y,l-l_s)-\Omega^{(1)}(Y,l-l_s)} - \underbrace{\frac{\partial}{\partial\left(l-l_s\right)}\left(e^{\Omega^{(0)}(Y,l-l_s)+\Omega^{(1)}(Y,l-l_s)}\mathscr{N}_{\mathscr{L}}[\Omega^{(0)}+\Omega^{(1)}]\right)}_{DH^{(1)}(Y,l-l_s)}$$

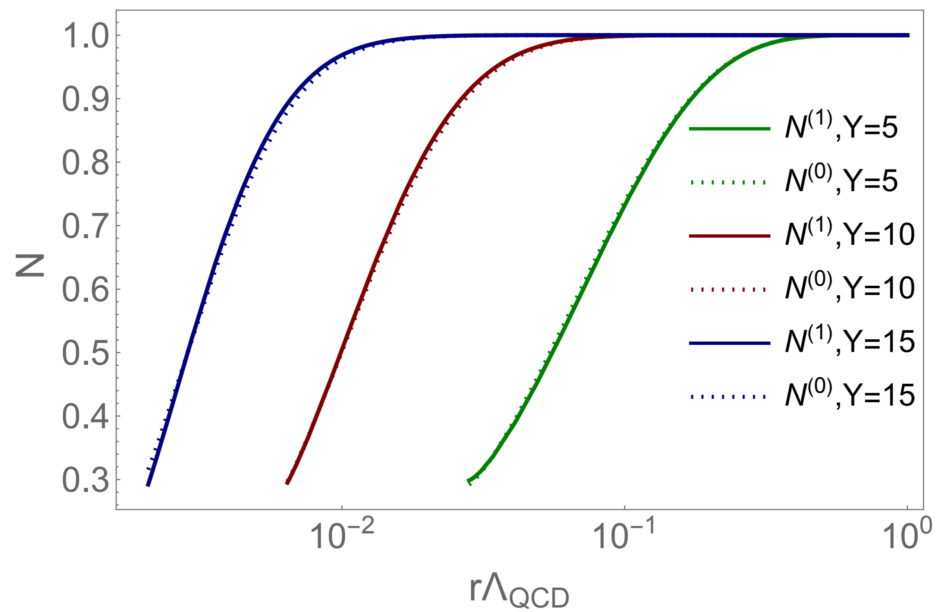


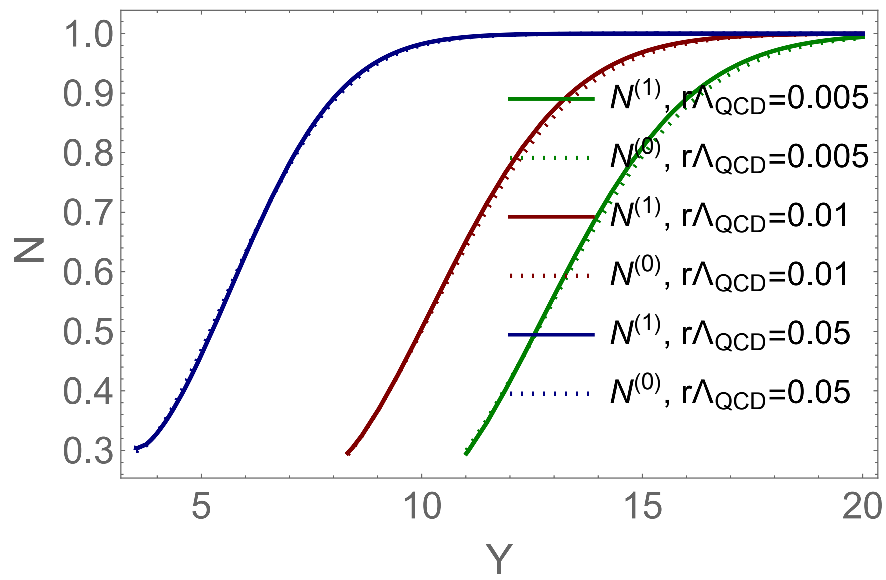
Not only studying the homotopy approach to nonlinear QCD, but also studying the scaling variable in this case


RC BK
$l - l_s$=0.5
$l - l_s$=1
$l - l_s$=2
$l - l_s$=3
$\Omega(\zeta, l - l_s)$
ζ
ζ= 2
ζ=5
ζ=10
ζ=20
$l - l_s$
$\Omega(z, l - l_s)$
z

RC BK


1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
N
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$l - l_s$
$N^{(1)}$,Y=5
$N^{(0)}$,Y=5
$N^{(1)}$,Y=10
$N^{(0)}$,Y=10
$N^{(1)}$,Y=15
$N^{(0)}$,Y=15

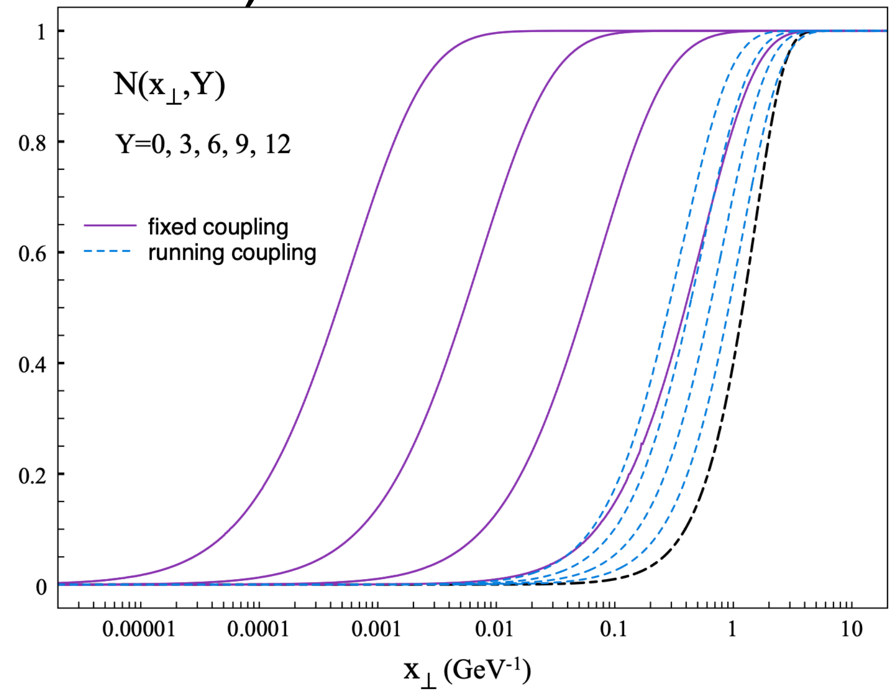

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
N
10^{-2}
10^{-1}
10^{0}
$r\Lambda_{QCD}$
$N^{(1)}$,Y=5
$N^{(0)}$,Y=5
$N^{(1)}$,Y=10
$N^{(0)}$,Y=10
$N^{(1)}$,Y=15
$N^{(0)}$,Y=15

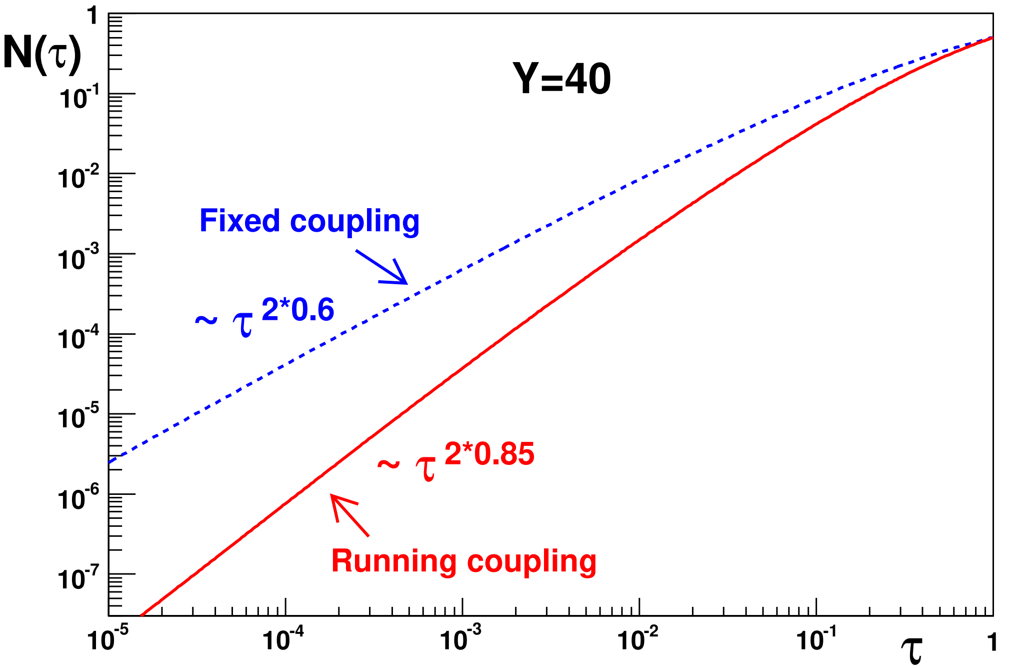

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
N
5
10
15
20
Y
$N^{(1)}$, $r\Lambda_{QCD}$=0.005
$N^{(0)}$, $r\Lambda_{QCD}$=0.005
$N^{(1)}$, $r\Lambda_{QCD}$=0.01
$N^{(0)}$, $r\Lambda_{QCD}$=0.01
$N^{(1)}$, $r\Lambda_{QCD}$=0.05
$N^{(0)}$, $r\Lambda_{QCD}$=0.05

Things to do

1.) Connect with BFKL solution for rQs(Y)<1 for a full picture, and compare with the fixed coupling case

2.) Check once again the geometric scaling behavior (since the numerical solution shows geometric scaling it seems)





(pictures from J. Albacete and Y. Kovchegov)

Thanks for listening!