Bloch's Theorem and The Lattice Gluon Propagator

Tereza Mendes

in collaboration with Attilio Cucchieri

Instituto de Física de São Carlos

Universidade de São Paulo

Abstract

Exploiting the similarity between Bloch's theorem for electrons in crystalline solids and the problem of Landau gauge-fixing in Yang-Mills theory on a "replicated" lattice allows one to obtain essentially infinite-volume results from numerical simulations performed on regular-size lattices

We review our study of the method applied to the gluon propagator and propose a novel interpretation, which might improve our understanding of color confinement

In particular, we show how to map numerical simulations performed on the "replicated" lattice onto the original (smaller), lattice, or "unit cell". Special emphasis is given to the rôle played by boundary conditions

Abstract

Exploiting the similarity between Bloch's theorem for electrons in crystalline solids and the problem of Landau gauge-fixing in Yang-Mills theory on a "replicated" lattice allows one to obtain essentially infinite-volume results from numerical simulations performed on regular-size lattices

We review our study of the method applied to the gluon propagator and propose a novel interpretation, which might improve our understanding of color confinement

In particular, we show how to map numerical simulations performed on the "replicated" lattice onto the original (smaller), lattice, or "unit cell". Special emphasis is given to the rôle played by boundary conditions

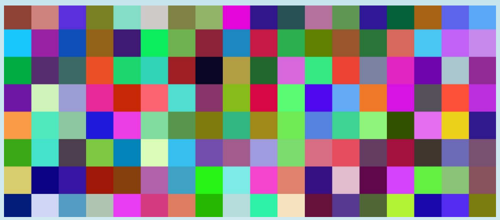
In short: we discuss a way to "stretch" lattice sizes considerably, by taking advantage of Bloch's theorem, from condensed-matter physics

General Aspects

Lattice gauge theories provide a nonperturbative way to investigate non-Abelian gauge theories, such as QCD

General Aspects

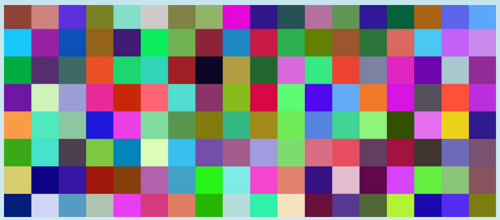
Lattice gauge theories provide a nonperturbative way to investigate non-Abelian gauge theories, such as QCD



Lattice formulation considers a discretized version of the theory; physical quantities are obtained in the limit of zero lattice spacing

General Aspects

Lattice gauge theories provide a nonperturbative way to investigate non-Abelian gauge theories, such as QCD

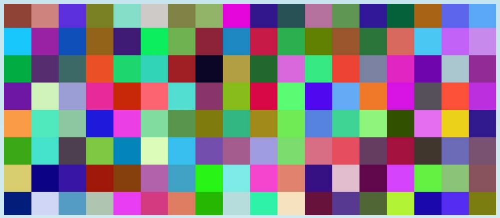


Lattice formulation considers a discretized version of the theory; physical quantities are obtained in the limit of zero lattice spacing

Statistical mechanics tools, such as numerical (Monte Carlo) simulation ⇒ New approach to QFT, direct access to (representative) gauge-field configurations

General Aspects

Lattice gauge theories provide a nonperturbative way to investigate non-Abelian gauge theories, such as QCD



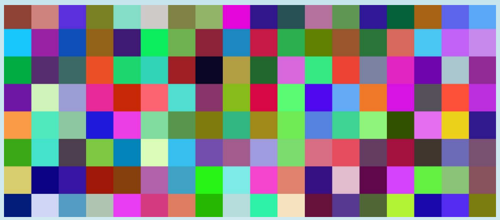
Lattice formulation considers a discretized version of the theory; physical quantities are obtained in the limit of zero lattice spacing

Statistical mechanics tools, such as numerical (Monte Carlo) simulation $\Rightarrow$ New approach to QFT, direct access to (representative) gauge-field configurations

First principles study of low-energy QCD properties (including confinement, chiral-symmetry breaking, dynamical mass generation).

General Aspects

Lattice gauge theories provide a nonperturbative way to investigate non-Abelian gauge theories, such as QCD



Lattice formulation considers a discretized version of the theory; physical quantities are obtained in the limit of zero lattice spacing

Statistical mechanics tools, such as numerical (Monte Carlo) simulation $\Rightarrow$ New approach to QFT, direct access to (representative) gauge-field configurations

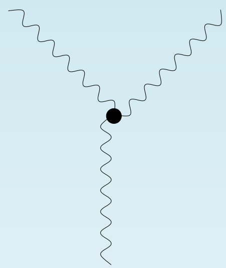
First principles study of low-energy QCD properties (including confinement, chiral-symmetry breaking, dynamical mass generation). In this case, one of the challenges: Infrared limit requires large lattice volumes

Origin of Confinement in QCD

Note: contribution $F^a_{\mu\nu} \sim g_0 \, f^{abc} \, A^b_\mu \, A^c_\nu$ means that in addition to quadratic terms (propagators) and the usual vertex

$\mathcal{L}_{\bar{\psi}\psi A} = g_0 \, \bar{\psi} \, \gamma^\mu A_\mu \, \psi$ (quark-quark-gluon vertex)

Lagrangian contains terms with 3 and 4 gauge fields



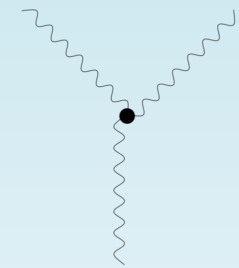
$\mathcal{L}_{AAA} = g_0 \, f^{abc} \, A^\mu_a \, A^\nu_b \, \partial_\mu A^c_\nu \Rightarrow$ three-gluon vertex

Origin of Confinement in QCD

Note: contribution $F^a_{\mu\nu} \sim g_0\, f^{abc}\, A^b_\mu\, A^c_\nu$ means that in addition to quadratic terms (propagators) and the usual vertex

$\mathcal{L}_{\bar{\psi}\psi A} = g_0\, \bar{\psi}\, \gamma^\mu A_\mu\, \psi$ (quark-quark-gluon vertex)

Lagrangian contains terms with 3 and 4 gauge fields



$\mathcal{L}_{AAA} = g_0\, f^{abc}\, A^\mu_a\, A^\nu_b\, \partial_\mu A^c_\nu \Rightarrow$ three-gluon vertex

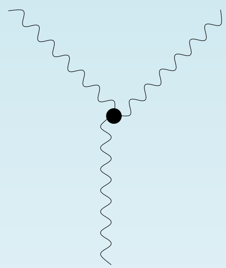
$\Rightarrow$ gluons interact (have color charge), determining the peculiar properties and the nonperturbative nature of low-energy QCD (nonlinear effects)

Origin of Confinement in QCD

Note: contribution $F^a_{\mu\nu} \sim g_0 \, f^{abc} \, A^b_\mu \, A^c_\nu$ means that in addition to quadratic terms (propagators) and the usual vertex

$\mathcal{L}_{\bar{\psi}\psi A} = g_0 \, \bar{\psi} \, \gamma^\mu A_\mu \, \psi$ (quark-quark-gluon vertex)

Lagrangian contains terms with 3 and 4 gauge fields



$\mathcal{L}_{AAA} = g_0 \, f^{abc} \, A^\mu_a \, A^\nu_b \, \partial_\mu A^c_\nu \Rightarrow$ three-gluon vertex

$\Rightarrow$ gluons interact (have color charge), determining the peculiar properties and the nonperturbative nature of low-energy QCD (nonlinear effects)

$\Rightarrow$ Running coupling $\alpha_s(p)$ instead of $\alpha \approx 1/137$

Confinement vs. Aymptotic Freedom

- At high energies: deep inelastic scattering of electrons reveals proton made of partons: pointlike and free. In this limit $\alpha_s(p) \ll 1$ (asymptotic freedom) and QCD is perturbative

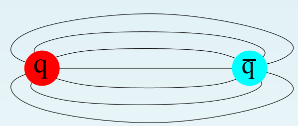
$$\alpha_s(p) = \frac{4\pi}{\beta_0 \log\left(p^2/\Lambda^2\right)} \left[1 - \frac{2\beta_1}{\beta_0^2} \frac{\log\left(\log\left(p^2/\Lambda^2\right)\right)}{\log\left(p^2/\Lambda^2\right)} + \ldots\right]$$

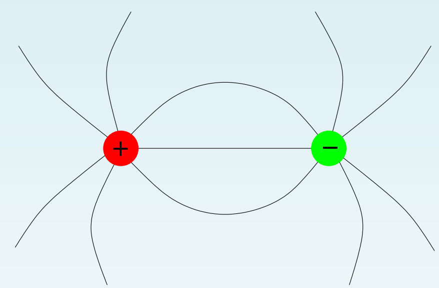
Confinement vs. Aymptotic Freedom

- At high energies: deep inelastic scattering of electrons reveals proton made of partons: pointlike and free. In this limit $\alpha_s(p) \ll 1$ (asymptotic freedom) and QCD is perturbative

$$\alpha_s(p) = \frac{4\pi}{\beta_0 \log\left(p^2/\Lambda^2\right)} \left[1 - \frac{2\beta_1}{\beta_0^2} \frac{\log\left(\log\left(p^2/\Lambda^2\right)\right)}{\log\left(p^2/\Lambda^2\right)} + \ldots\right]$$

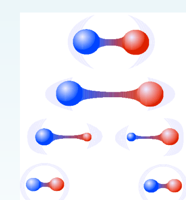
- At low energies: interaction gets stronger, $\alpha_s \approx 1$ and confinement occurs. Color field may form flux tubes





linear increase of inter-quark potential $\rightarrow$ string tension

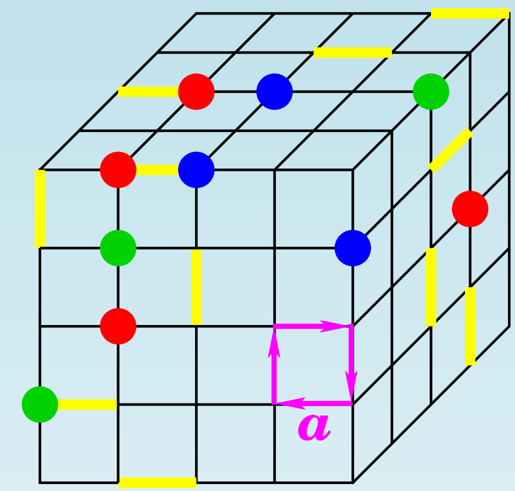
At large distances $\rightarrow$ string breaks



Nonperturbative QCD ⇒ Lattice

Three ingredients

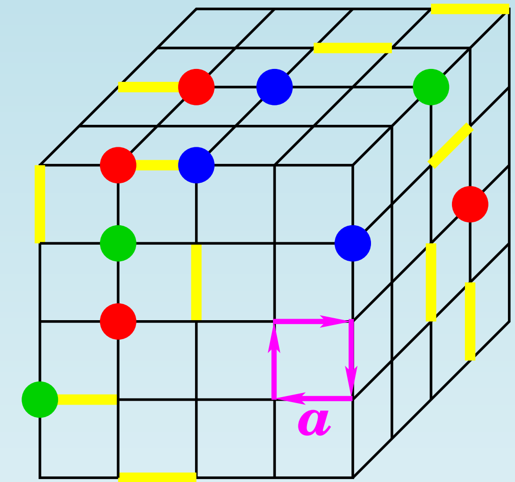
1. Quantization by path integrals ⇒ sum over configurations with "weights" $e^{i\,S/\hbar}$
2. Euclidean formulation (analytic continuation to imaginary time) ⇒ weight becomes $e^{-S/\hbar}$
3. Discrete space-time ⇒ UV cut at momenta $p \lesssim 1/a$ ⇒ regularization



Nonperturbative QCD $\Rightarrow$ Lattice

Three ingredients

1. Quantization by path integrals $\Rightarrow$ sum over configurations with "weights" $e^{i\,S/\hbar}$
2. Euclidean formulation (analytic continuation to imaginary time) $\Rightarrow$ weight becomes $e^{-S/\hbar}$
3. Discrete space-time $\Rightarrow$ UV cut at momenta $p \lesssim 1/a$ $\Rightarrow$ regularization

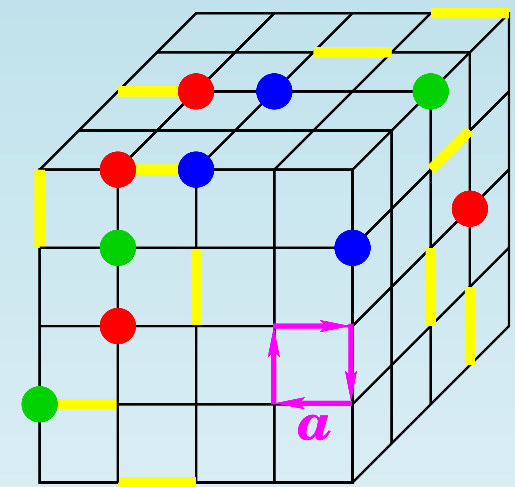


Also: finite-size lattices $\Rightarrow$ IR cut for small momenta $p \approx 1/L$

Nonperturbative QCD ⇒ Lattice

Three ingredients

1. Quantization by path integrals ⇒ sum over configurations with "weights" $e^{i\,S/\hbar}$
2. Euclidean formulation (analytic continuation to imaginary time) ⇒ weight becomes $e^{-S/\hbar}$
3. Discrete space-time ⇒ UV cut at momenta $p \lesssim 1/a$ ⇒ regularization



Also: finite-size lattices ⇒ IR cut for small momenta $p \approx 1/L$

The Wilson action

- is written for the gauge links $U_{x,\mu} \;\equiv\; e^{i g_0\, a A^b_\mu(x) T_b}$
- reduces to the usual action for $a \to 0$
- is gauge-invariant

The Lattice Action

The Wilson action (1974)

$$S \;=\; -\frac{\beta}{3}\sum_{\Box}\text{ReTr}\,U_{\Box}\,, \quad U_{x,\mu}\;\equiv\; e^{i g_0 a A^b_\mu(x) T_b}\,, \quad \beta \;=\; 6/{g_0}^2$$

- written in terms of oriented plaquettes formed by the link variables $U_{x,\mu}$, which are group elements
- under gauge transformations: $U_{x,\mu} \to g(x)\,U_{x,\mu}\,g^\dagger(x+\mu)$, where $g \in SU(3)$ $\Rightarrow$ closed loops are gauge-invariant quantities
- integration volume is finite: no need for gauge-fixing

At small β (i.e. strong coupling) we can perform an expansion analogous to the high-temperature expansion in statistical mechanics. At lowest order, the only surviving terms are represented by diagrams with “double” or “partner” links, i.e. the same link should appear in both orientations, since $\int dU\, U_{x,\mu} = 0$

Confinement and Area Law

Considering a rectangular loop with sides R and T (the Wilson loop) as our observable, the leading contribution to the observable's expectation value is obtained by "tiling" its inside with plaquettes, yielding the area law

$$\langle W(R,T)\rangle \;\sim\; \beta^{RT}$$

But this observable is related to the interquark potential for a static quark-antiquark pair

$$\langle W(R,T)\rangle \;=\; e^{-V(R)T}$$

We thus have $V(R) \sim \sigma R$, demonstrating confinement at strong coupling (small β)!

Problem: the physical limit is at large β...

(Numerical) Lattice QCD

Classical Statistical-Mechanics model with the partition function

$$Z = \int \mathcal{D}U \, e^{-S_g} \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; e^{-\int d^4x \, \overline{\psi}(x) \, K \, \psi(x)} = \int \mathcal{D}U \, e^{-S_g} \, \det K(U)$$

(Numerical) Lattice QCD

Classical Statistical-Mechanics model with the partition function

$$Z = \int \mathcal{D}U \, e^{-S_g} \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; e^{-\int d^4x \, \overline{\psi}(x) \, K \, \psi(x)} = \int \mathcal{D}U \, e^{-S_g} \, \det K(U)$$

Evaluate expectation values

$$\langle \mathcal{O} \rangle = \int \mathcal{D}U \, \mathcal{O}(U) \, P(U) = \frac{1}{N} \sum_i \mathcal{O}(U_i)$$

with the weight

$$P(U) = \frac{e^{-S_g(U)} \, \det K(U)}{Z}$$

(Numerical) Lattice QCD

Classical Statistical-Mechanics model with the partition function

$$Z = \int \mathcal{D}U \, e^{-S_g} \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; e^{-\int d^4x \, \overline{\psi}(x) \, K \, \psi(x)} = \int \mathcal{D}U \, e^{-S_g} \, \det K(U)$$

Evaluate expectation values

$$\langle \mathcal{O} \rangle = \int \mathcal{D}U \, \mathcal{O}(U) \, P(U) = \frac{1}{N} \sum_i \mathcal{O}(U_i)$$

with the weight

$$P(U) = \frac{e^{-S_g(U)} \, \det K(U)}{Z}$$

Very complicated (high-dimensional) integral to compute!

(Numerical) Lattice QCD

Classical Statistical-Mechanics model with the partition function

$$Z = \int \mathcal{D}U \, e^{-S_g} \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; e^{-\int d^4x \, \overline{\psi}(x) \, K \, \psi(x)} = \int \mathcal{D}U \, e^{-S_g} \, \det K(U)$$

Evaluate expectation values

$$\langle \mathcal{O} \rangle = \int \mathcal{D}U \, \mathcal{O}(U) \, P(U) = \frac{1}{N} \sum_i \mathcal{O}(U_i)$$

with the weight

$$P(U) = \frac{e^{-S_g(U)} \, \det K(U)}{Z}$$

Very complicated (high-dimensional) integral to compute!

$\Rightarrow$ Monte Carlo simulation: sample representative configurations, then compute O and take average

Limits of Lattice Simulations

Physics is obtained after 3 limits:

1) The Thermodynamic Limit ($V = N^d \to \infty$): need $N \to \infty$ to keep physical lengths $L = aN$ fixed. Need $N > \xi_{\rm latt}$, while $\xi_{\rm latt}(a)$ diverges!

Limits of Lattice Simulations

Physics is obtained after 3 limits:

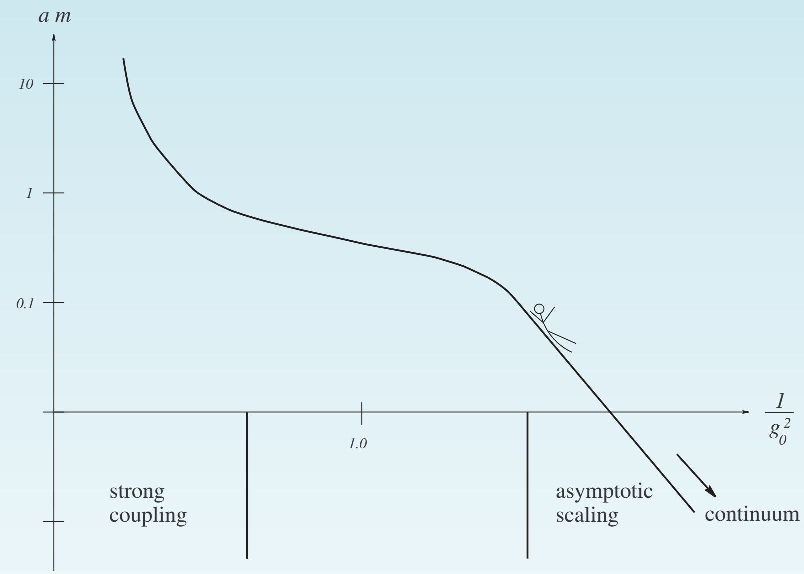
1) The Thermodynamic Limit ($V = N^d \to \infty$): need $N \to \infty$ to keep physical lengths $L = aN$ fixed. Need $N > \xi_{\rm latt}$, while $\xi_{\rm latt}(a)$ diverges!

2) The Continuum Limit ($a \to 0$): correlation length $\leftrightarrow$ mass^{-1}

from renormalization group:

$\log(\xi_{\rm latt}) = \log(1/ma) \sim 1/g_0^2 \sim \beta$

thus continuum limit given by $g_0 \to 0$, $\beta \to \infty$ and $\xi_{\rm latt} \sim e^{\beta}$ (asymptotic scaling), i.e. $\xi = 1/m \sim a\, e^{\beta} \Rightarrow$ eliminate e^{β} computing mass ratios (scaling law) or fix a using an experimental input (renormalization)



Limits of Lattice Simulations

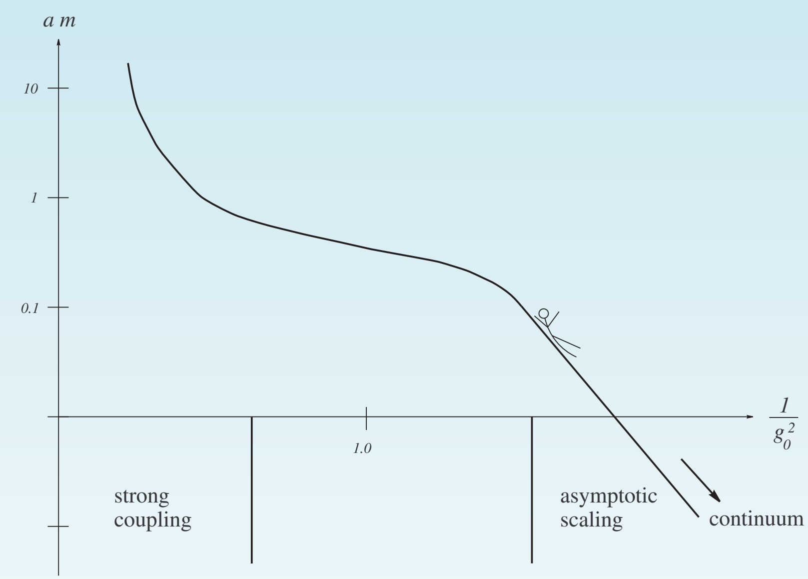
Physics is obtained after 3 limits:

1) The Thermodynamic Limit ($V = N^d \to \infty$): need $N \to \infty$ to keep physical lengths $L = aN$ fixed. Need $N > \xi_{\rm latt}$, while $\xi_{\rm latt}(a)$ diverges!

2) The Continuum Limit ($a \to 0$): correlation length $\leftrightarrow$ mass^{-1} from renormalization group:

$\log(\xi_{\rm latt}) = \log(1/ma) \sim 1/g_0^2 \sim \beta$

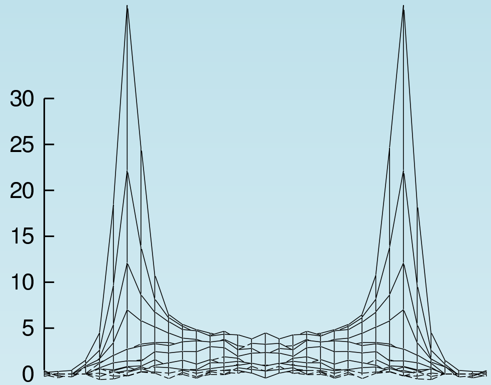
thus continuum limit given by $g_0 \to 0$, $\beta \to \infty$ and $\xi_{\rm latt} \sim e^\beta$ (asymptotic scaling), i.e. $\xi = 1/m \sim a\,e^\beta \Rightarrow$ eliminate e^β computing mass ratios (scaling law) or fix a using an experimental input (renormalization)

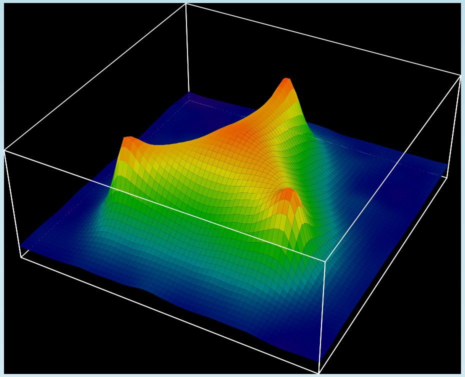


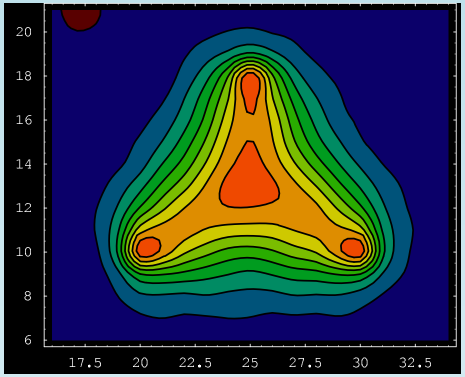
3) The Chiral Limit (small m_q): fit results to chiral perturbation theory predictions and extrapolate to physical masses

Manifestations of Confinement

May observe formation of flux tubes

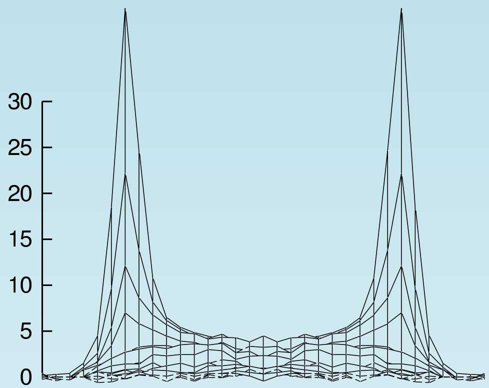


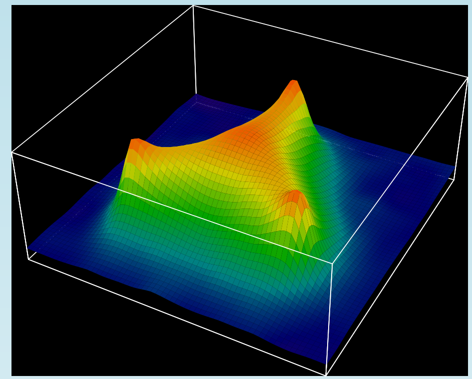


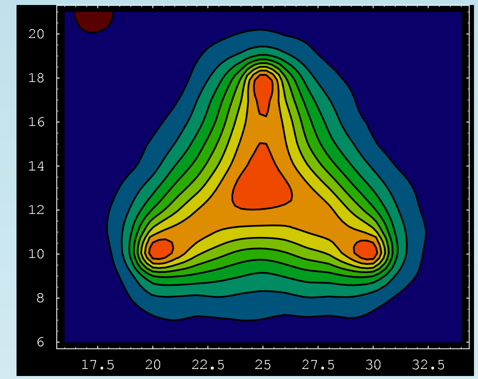


Manifestations of Confinement

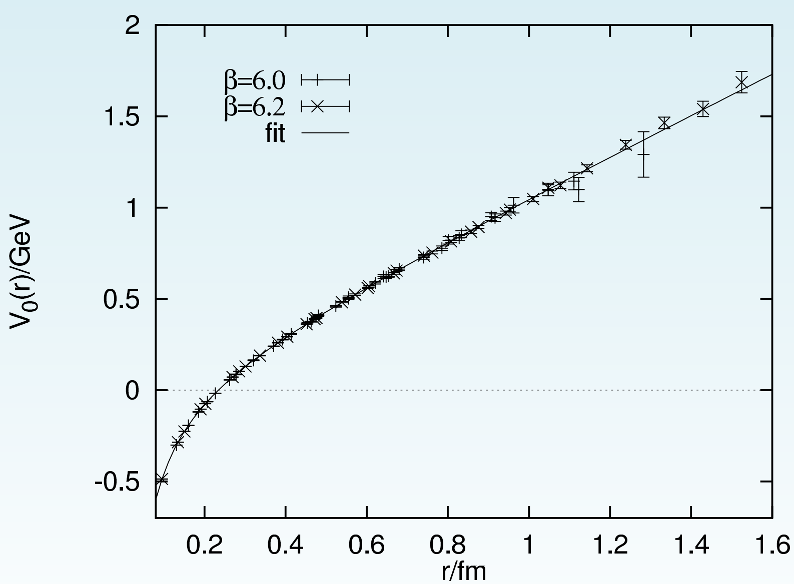
May observe formation of flux tubes

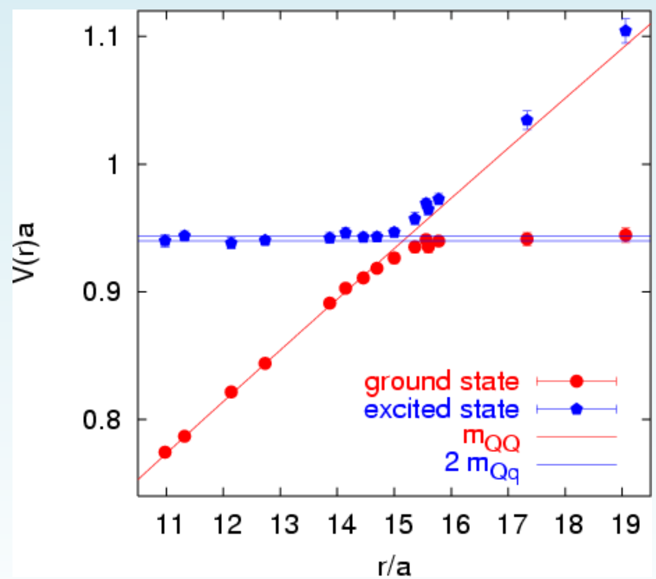






Linear Growth of potential between quarks, string breaking





Understanding Confinement

- How does linearly rising potential (seen in lattice QCD) come about?

Understanding Confinement

- How does linearly rising potential (seen in lattice QCD) come about?
- Models of confinement include: dual superconductivity (electric flux tube connecting magnetic monopoles), condensation of center vortices, but also merons, calorons

Understanding Confinement

- How does linearly rising potential (seen in lattice QCD) come about?
- Models of confinement include: dual superconductivity (electric flux tube connecting magnetic monopoles), condensation of center vortices, but also merons, calorons
- Proposal by Mandelstam (1979) linking linear potential to infrared behavior of gluon propagator as $1/p^4$

$$V(r) \sim \int \frac{d^3p}{p^4} e^{ip\cdot r} \sim r$$

Understanding Confinement

- How does linearly rising potential (seen in lattice QCD) come about?
- Models of confinement include: dual superconductivity (electric flux tube connecting magnetic monopoles), condensation of center vortices, but also merons, calorons
- Proposal by Mandelstam (1979) linking linear potential to infrared behavior of gluon propagator as $1/p^4$

$$V(r) \sim \int \frac{d^3p}{p^4} e^{ip\cdot r} \sim r$$

- Gribov-Zwanziger confinement scenario based on suppressed gluon propagator and enhanced ghost propagator in the infrared

GZ Scenario: Confinement by Ghost

Formulated for Landau gauge, predicts gluon propagator

$$D^{ab}_{\mu\nu}(p) = \sum_x e^{-2i\pi k\cdot x} \langle A^a_\mu(x)\, A^b_\nu(0)\rangle = \delta^{ab} \left(g_{\mu\nu} - \frac{p_\mu\, p_\nu}{p^2} \right) D(p^2)$$

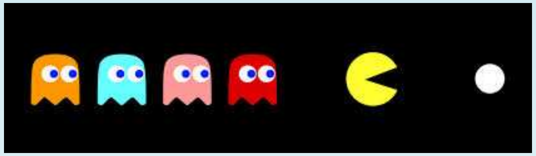
suppressed in the IR limit $\Rightarrow$ gluon confinement

GZ Scenario: Confinement by Ghost

Formulated for Landau gauge, predicts gluon propagator

$$D^{ab}_{\mu\nu}(p) = \sum_x e^{-2i\pi k \cdot x} \langle A^a_\mu(x)\, A^b_\nu(0) \rangle = \delta^{ab} \left(g_{\mu\nu} - \frac{p_\mu\, p_\nu}{p^2} \right) D(p^2)$$

suppressed in the IR limit ⇒ gluon confinement



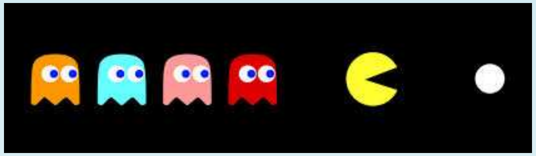
Long range effects are felt in the ghost propagator $G(p)$:

GZ Scenario: Confinement by Ghost

Formulated for Landau gauge, predicts gluon propagator

$$D^{ab}_{\mu\nu}(p) = \sum_x e^{-2i\pi k\cdot x} \langle A^a_\mu(x)\, A^b_\nu(0)\rangle = \delta^{ab} \left(g_{\mu\nu} - \frac{p_\mu\, p_\nu}{p^2} \right) D(p^2)$$

suppressed in the IR limit ⇒ gluon confinement



Long range effects are felt in the ghost propagator $G(p)$:

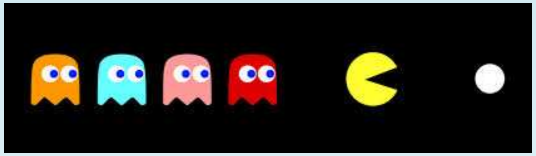
- Infinite volume favors configurations on the first Gribov horizon, where minimum nonzero eigenvalue λ_{min} of Faddeev-Popov operator $\mathcal{M}$ goes to zero

GZ Scenario: Confinement by Ghost

Formulated for Landau gauge, predicts gluon propagator

$$D^{ab}_{\mu\nu}(p) = \sum_x e^{-2i\pi k\cdot x} \langle A^a_\mu(x)\, A^b_\nu(0)\rangle = \delta^{ab} \left(g_{\mu\nu} - \frac{p_\mu\, p_\nu}{p^2} \right) D(p^2)$$

suppressed in the IR limit $\Rightarrow$ gluon confinement

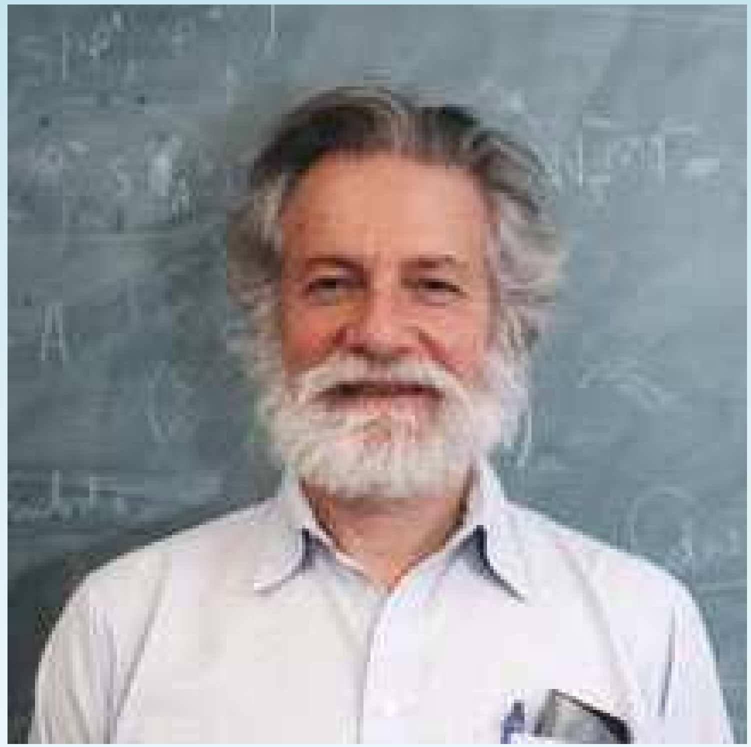


Long range effects are felt in the ghost propagator $G(p)$:

- Infinite volume favors configurations on the first Gribov horizon, where minimum nonzero eigenvalue λ_{min} of Faddeev-Popov operator $\mathcal{M}$ goes to zero
- In turn, $G(p)$ should be IR enhanced, introducing long-range effects, which are related to the color-confinement mechanism

Dan Zwanziger (1935–2024)

Wikipedia Daniel Zwanziger ($*$20. Mai 1935 in New York City; † 2024) war ein US-amerikanischer theoretischer Physiker. Er befasste sich mit Quantenfeldtheorie, mathematischer Physik und Elementarteilchenphysik



[Zwanziger *Local and renormizable action from the Gribov horizon*, Nucl. Phys. B, 1989]

[Vandersickel, Zwanziger *The Gribov problem and QCD dynamics*, Phys. Rep., 2012]

Gauge-Related Lattice Features

- Gauge action written in terms of oriented plaquettes formed by the link variables $U_{x,\mu}$, which are group elements

Gauge-Related Lattice Features

- Gauge action written in terms of oriented plaquettes formed by the link variables $U_{x,\mu}$, which are group elements
- under gauge transformations: $U_{x,\mu} \to g(x)\, U_{x,\mu}\, g^{\dagger}(x+\mu)$, where $g \in SU(3)$ $\Rightarrow$ closed loops are gauge-invariant quantities

Gauge-Related Lattice Features

- Gauge action written in terms of oriented plaquettes formed by the link variables $U_{x,\mu}$, which are group elements
- under gauge transformations: $U_{x,\mu} \to g(x)\, U_{x,\mu}\, g^\dagger(x+\mu)$, where $g \in SU(3) \Rightarrow$ closed loops are gauge-invariant quantities
- integration volume is finite: no need for gauge-fixing

Gauge-Related Lattice Features

- Gauge action written in terms of oriented plaquettes formed by the link variables $U_{x,\mu}$, which are group elements
- under gauge transformations: $U_{x,\mu} \to g(x)\, U_{x,\mu}\, g^{\dagger}(x+\mu)$, where $g \in SU(3)$ $\Rightarrow$ closed loops are gauge-invariant quantities
- integration volume is finite: no need for gauge-fixing
- when gauge fixing, procedure is incorporated in the simulation, no need to consider Faddeev-Popov matrix

Gauge-Related Lattice Features

- Gauge action written in terms of oriented plaquettes formed by the link variables $U_{x,\mu}$, which are group elements
- under gauge transformations: $U_{x,\mu} \to g(x)\, U_{x,\mu}\, g^{\dagger}(x+\mu)$, where $g \in SU(3) \Rightarrow$ closed loops are gauge-invariant quantities
- integration volume is finite: no need for gauge-fixing
- when gauge fixing, procedure is incorporated in the simulation, no need to consider Faddeev-Popov matrix
- get FP matrix without considering ghost fields explicitly

Gauge-Related Lattice Features

- Gauge action written in terms of oriented plaquettes formed by the link variables $U_{x,\mu}$, which are group elements
- under gauge transformations: $U_{x,\mu} \to g(x)\, U_{x,\mu}\, g^{\dagger}(x+\mu)$, where $g \in SU(3)$ $\Rightarrow$ closed loops are gauge-invariant quantities
- integration volume is finite: no need for gauge-fixing
- when gauge fixing, procedure is incorporated in the simulation, no need to consider Faddeev-Popov matrix
- get FP matrix without considering ghost fields explicitly
- Lattice momenta given by $\hat{p}_\mu = 2\,\sin\left(\pi\, n_\mu/N\right)$ with $n_\mu = 0, 1, \ldots, N/2$ $\Leftrightarrow$ $p_{min} \sim 2\pi/(a\,N) = 2\pi/L$, $p_{max} = d/a$ in physical units

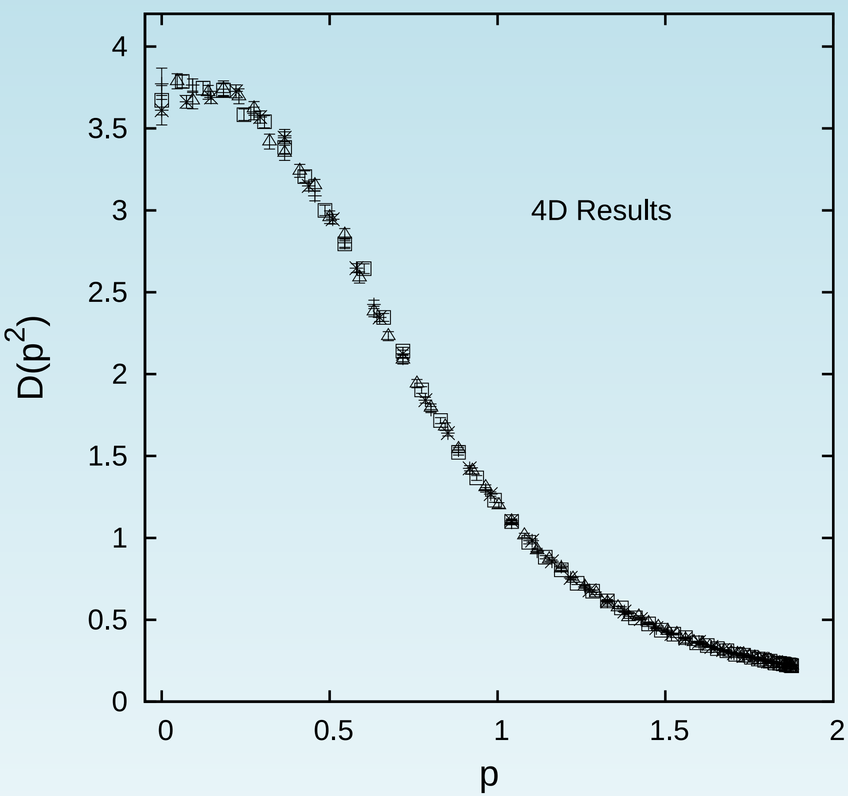
3-Step Code

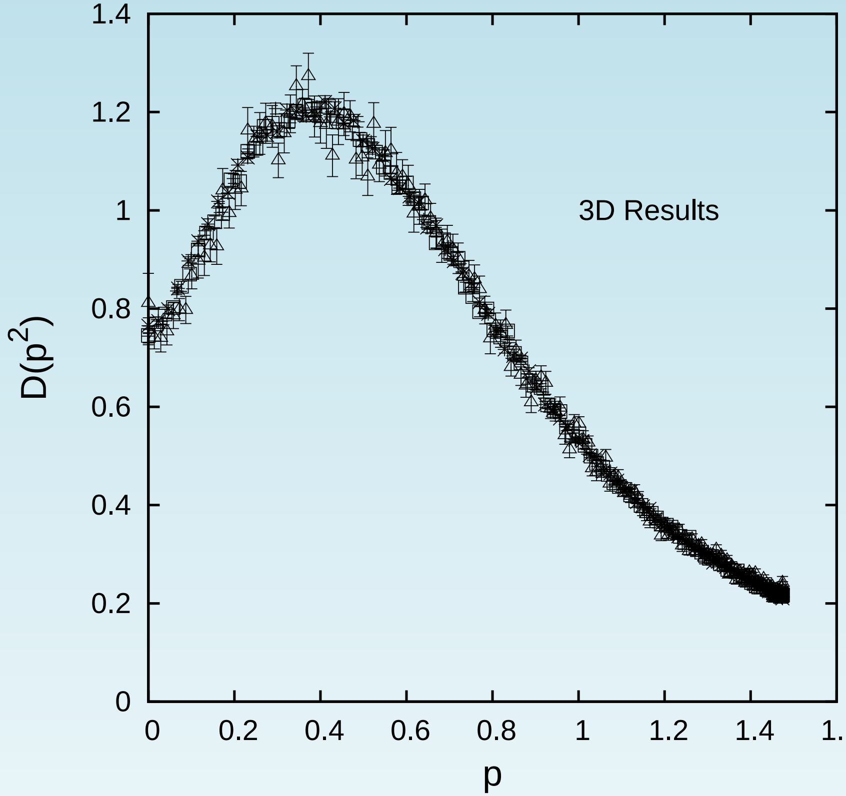
```
main() {
/* set parameters: beta, number of configurations NC,
                   number of thermalization sweeps NT */
     read_parameters();
/* {U} is the link configuration */
     set_initial_configuration(U);
/* cycle over NC configurations */
     for (int c=0; c < NC; c++) {
          thermalize(U,beta,NT);
          gauge_fix(U,g);
          evaluate_propagators(U[g]);
     }
}
```

Algorithms: Heat-Bath and Micro-canonical (thermalization), overrelaxation and simulated annealing (gauge fixing), conjugate gradient and Fourier transform (propagators, etc.).

Gluon Propagator at “Infinite” Volume

Attilio Cucchieri & T.M. (PRL, 2008)

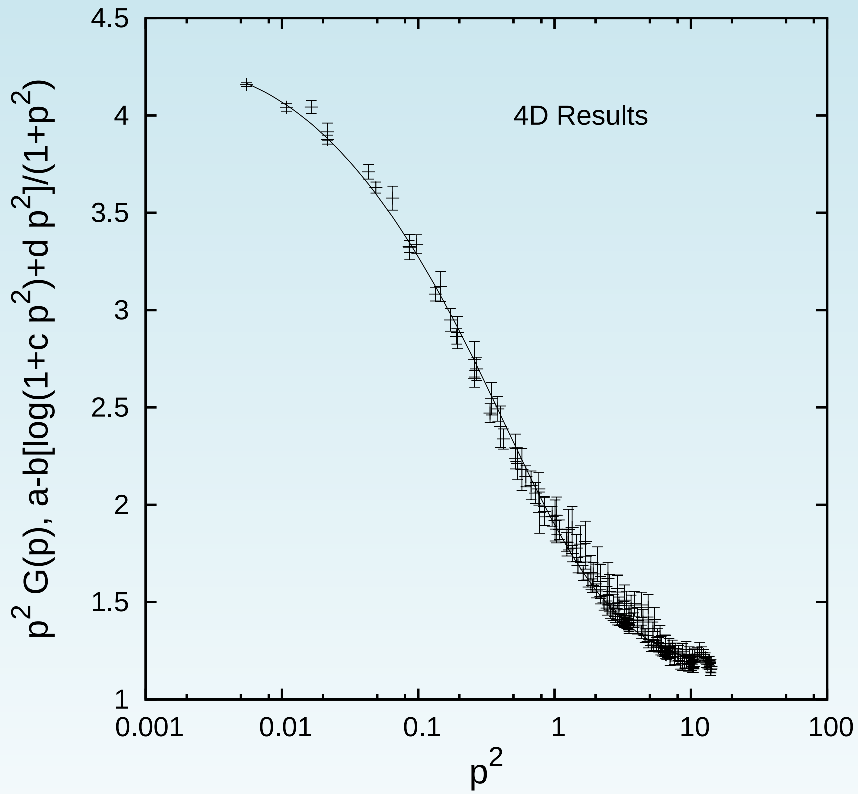




Gluon propagator $D(k)$ as a function of the lattice momenta k (both in physical units) for the pure-$SU(2)$ case in $d = 4$ (left), considering volumes of up to 128^4 (lattice extent ~ 27 fm) and $d = 3$ (right), considering volumes of up to 320^3 (lattice extent ~ 85 fm)

Ghost Propagator Results

Fit of the ghost dressing function $p^2 G(p^2)$ as a function of p^2 (in GeV) for the 4d case ($\beta = 2.2$ with volume 80^4). We find that $p^2 G(p^2)$ is best fitted by the form $p^2 G(p^2) = a - b[\log(1 + cp^2) + dp^2]/(1 + p^2)$, with



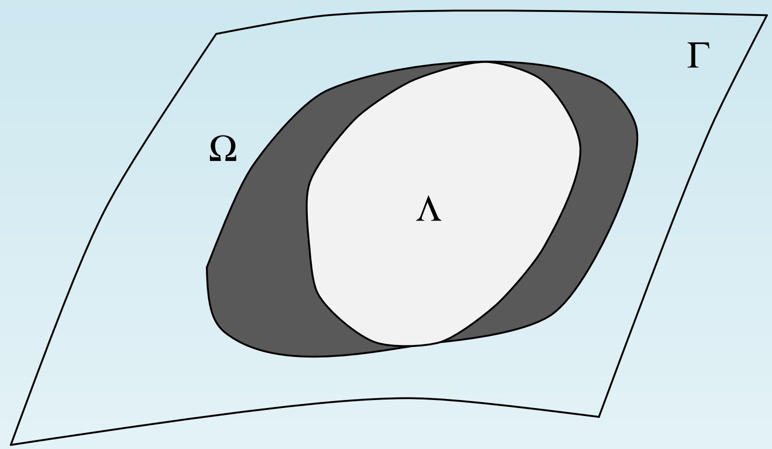
$a = 4.32(2)$,
$b = 0.38(1)\, GeV^2$,
$c = 80(10)\, GeV^{-2}$,
$d = 8.2(3)\, GeV^{-2}$.

In IR limit $p^2 G(p^2) \sim a$.

Attilio Cucchieri & T.M. (PRD, 2008)

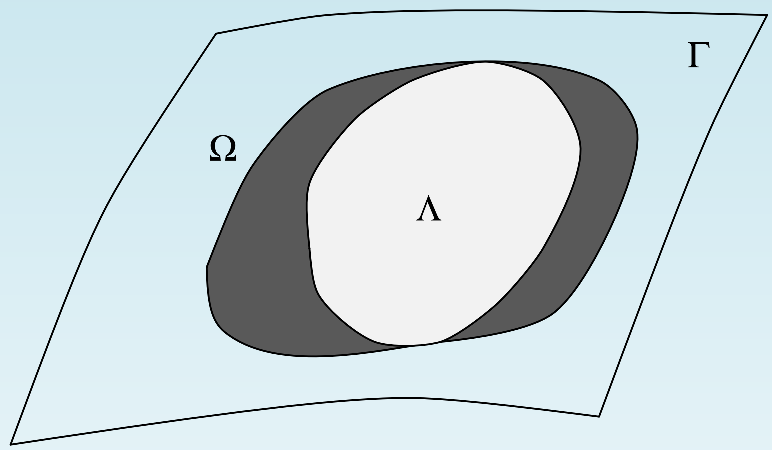
The Infinite-Volume Limit

As the infinite-volume limit is approached, the sampled configurations (inside Ω = region for which $\mathcal{M}$ is positive semi-definite) are closer and closer to the first Gribov horizon $\partial\Omega$



The Infinite-Volume Limit

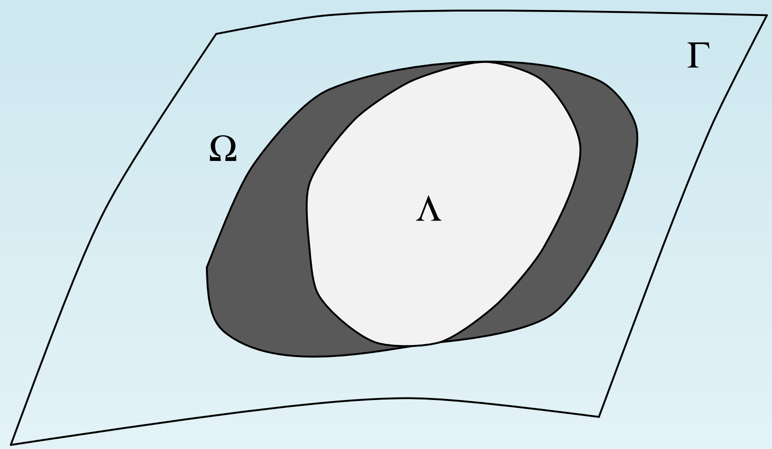
As the infinite-volume limit is approached, the sampled configurations (inside Ω = region for which $\mathcal{M}$ is positive semi-definite) are closer and closer to the first Gribov horizon $\partial\Omega$



Can we learn more about the geometry of this region?

The Infinite-Volume Limit

As the infinite-volume limit is approached, the sampled configurations (inside Ω = region for which $\mathcal{M}$ is positive semi-definite) are closer and closer to the first Gribov horizon $\partial\Omega$



Can we learn more about the geometry of this region?

Lattice simulation produces thermalized gauge configurations, but we can also “visit” nearby configs and extract info from them!

Issues

Simulation on large lattices (IR limit) is very costly. How to be more efficient?

Issues

Simulation on large lattices (IR limit) is very costly. How to be more efficient? How to disentangle approach to Gribov horizon (as lattice volume increases) and behavior of $G(p^2)$ (or $\lambda_{\min}$)?

Issues

Simulation on large lattices (IR limit) is very costly. How to be more efficient? How to disentangle approach to Gribov horizon (as lattice volume increases) and behavior of $G(p^2)$ (or $\lambda_{\min}$)?

Get insight from features of the lattice simulations themselves:

1) Educated guess of infinite-volume-limit behavior

2) Explore Gribov horizon by visiting neighboring (unsampled) configurations, get info about $\lambda_{\min}$

3) Simulate on effectively large lattices by "faking" periodic crystal and invoking Bloch's theorem

Issues

Simulation on large lattices (IR limit) is very costly. How to be more efficient? How to disentangle approach to Gribov horizon (as lattice volume increases) and behavior of $G(p^2)$ (or $\lambda_{\min}$)?

Get insight from features of the lattice simulations themselves:

1) Educated guess of infinite-volume-limit behavior

2) Explore Gribov horizon by visiting neighboring (unsampled) configurations, get info about $\lambda_{\min}$

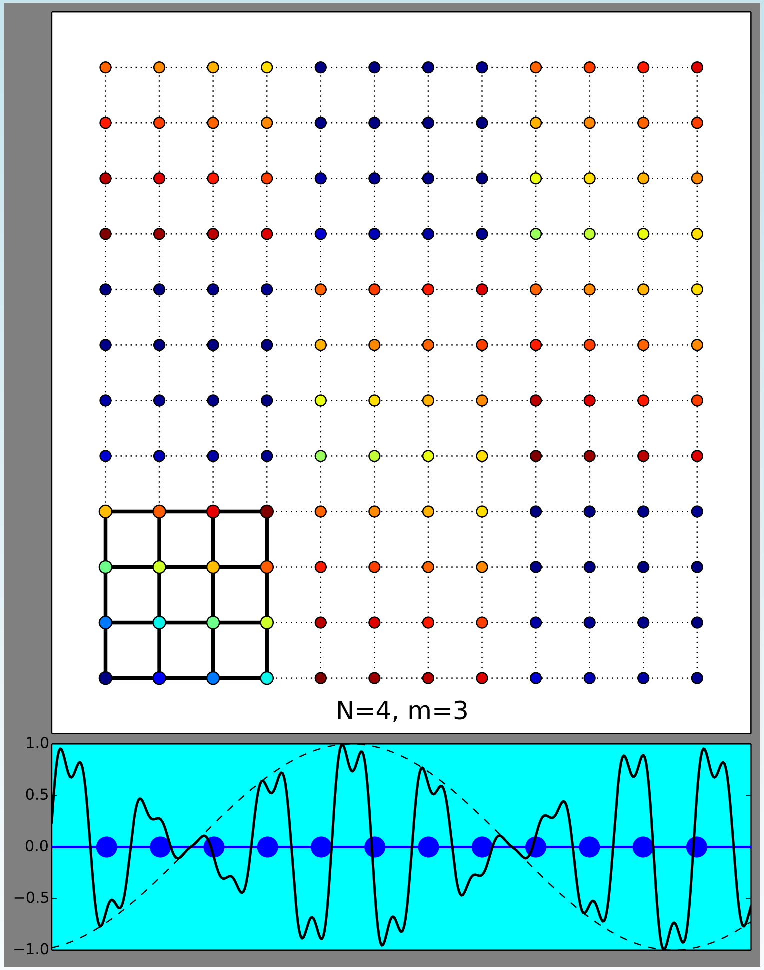
3) Simulate on effectively large lattices by "faking" periodic crystal and invoking Bloch's theorem

Also: Investigate the analytic structure of the propagators via "perturbative tecniques" (e.g. use rational approximants)

Large Lattices via Bloch's Theorem

Perform thermalization on small lattice, then replicate it and use Bloch's theorem to obtain gauge-fixing for much larger lattice

A. Cucchieri, TM, PRL 2017 & Universe 2025, 11(8), 273, 56 pp.



Periodic (Crystal) Potential in QM

For ideal crystalline solid in d dimensions, consider electrostatic potential $U(\vec{r})$ with the periodicity of the Bravais lattice, i.e. $U(\vec{r}) = U(\vec{r} + \vec{R})$ for any vector $\vec{R} = n_\mu \vec{a}_\mu$

1. Choose eigenstates of $\mathcal{H}$ to be also eigenstates of $\mathcal{T}(\vec{R})$
2. Then eigenstates $\psi(\vec{r})$ can be written as Bloch waves

$$\psi_{\vec{k}}(\vec{r}) \;=\; \exp\left(i\vec{k}\cdot\vec{r}\right) h_{\vec{k}}(\vec{r}) \;,$$

where the functions $h_{\vec{k}}(\vec{r})$ have the periodicity of the Bravais lattice, i.e. $h_{\vec{k}}(\vec{r} + \vec{R}) = h_{\vec{k}}(\vec{r})$

Periodic (Crystal) Potential in QM

For ideal crystalline solid in d dimensions, consider electrostatic potential $U(\vec{r})$ with the periodicity of the Bravais lattice, i.e. $U(\vec{r}) = U(\vec{r} + \vec{R})$ for any vector $\vec{R} = n_\mu \vec{a}_\mu$

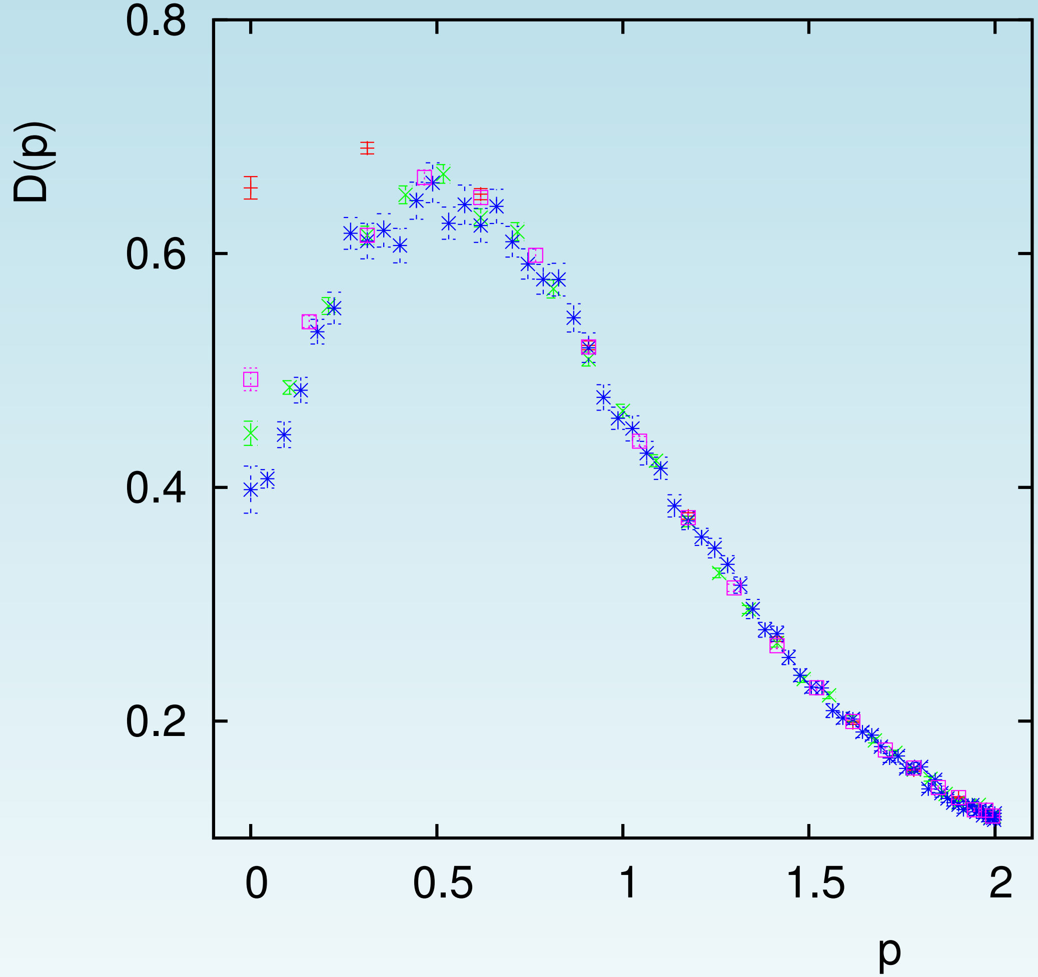
1. Choose eigenstates of $\mathcal{H}$ to be also eigenstates of $\mathcal{T}(\vec{R})$
2. Then eigenstates $\psi(\vec{r})$ can be written as Bloch waves

$$\psi_{\vec{k}}(\vec{r}) \,=\, \exp\left(i\vec{k}\cdot\vec{r}\right) h_{\vec{k}}(\vec{r}) \; ,$$

   where the functions $h_{\vec{k}}(\vec{r})$ have the periodicity of the Bravais lattice, i.e. $h_{\vec{k}}(\vec{r} + \vec{R}) = h_{\vec{k}}(\vec{r})$

Idea: infinite-volume limit in LQCD as periodic-potential problem, simplified by analogy with Bloch's theorem

Gluon Propagator: Volume Effects



Gluon propagator vs. lattice momentum for $V = 20^3$, 40^3, 60^3 and 140^3

Lattice Landau Gauge

Landau gauge is imposed on the lattice by minimizing the functional

$$\mathcal{E}[U;g] \,=\, \Re\,\mathrm{Tr}\sum_{x,\mu}[\mathbb{1} - U_\mu^g(x)]$$

with respect to $g(x) \in SU(N_c)$ for a fixed gauge configuration $U_\mu(x)$

Lattice Landau Gauge

Landau gauge is imposed on the lattice by minimizing the functional

$$\mathcal{E}[U;g] \, = \, \Re\,\mathrm{Tr}\sum_{x,\mu}[\mathbb{1} - U_\mu^g(x)]$$

with respect to $g(x) \in SU(N_c)$ for a fixed gauge configuration $U_\mu(x)$

Taking $g(x) = e^{i\tau\gamma(x)}$ with $\gamma(x) = \gamma^b(x)\, T_b \in \mathfrak{su}(N_c)$ fixed and $\tau \to 0$

$$\mathcal{E}[U;g] \,\approx\, \mathcal{E}[U;\mathbb{1}] \,+\, \tau\,\mathcal{E}'[U;\mathbb{1}](b,x)\,\gamma^b(x) + (\tau^2/2)\,\gamma^b(x)\,\mathcal{E}''[U;\mathbb{1}](b,x;c,y)\,\gamma^c(y)$$

$\Rightarrow\ \mathcal{E}''[U;\mathbb{1}] = \mathcal{M}[A]$ is a lattice discretization of Faddeev-Popov operator $-D\cdot\partial$

Lattice Landau Gauge

Landau gauge is imposed on the lattice by minimizing the functional

$$\mathcal{E}[U;g] = \Re\,\mathrm{Tr}\sum_{x,\mu}[\mathbb{1} - U^{g}_{\mu}(x)]$$

with respect to $g(x) \in SU(N_c)$ for a fixed gauge configuration $U_\mu(x)$

Taking $g(x) = e^{i\tau\gamma(x)}$ with $\gamma(x) = \gamma^b(x)\,T_b \in \mathfrak{su}(N_c)$ fixed and $\tau \to 0$

$$\mathcal{E}[U;g] \approx \mathcal{E}[U;\mathbb{1}] + \tau\,\mathcal{E}'[U;\mathbb{1}](b,x)\,\gamma^b(x) + (\tau^2/2)\,\gamma^b(x)\,\mathcal{E}''[U;\mathbb{1}](b,x;c,y)\,\gamma^c(y)$$

$\Rightarrow\ \mathcal{E}''[U;\mathbb{1}] = \mathcal{M}[A]$ is a lattice discretization of Faddeev-Popov operator $-D\cdot\partial$

At any local minimum (stationary solution) we have $\mathcal{E}' = 0\ \forall\,\gamma^b(x)$

$\Rightarrow\ \left(\nabla\cdot A^b\right)(x) = 0\ \forall\,x,b\,,$ where $A_\mu(\vec{x}) = \dfrac{1}{2\,i}\left[\,U_\mu(\vec{x}) - U^{\dagger}_{\mu}(\vec{x})\,\right]_{\rm traceless}$

Therefore, the (minimal) Landau gauge condition on the lattice reads

$$\left(\nabla\cdot A^b\right)(\vec{x}) = \sum_{\mu=1}^{d} A^b_\mu(\vec{x}) - A^b_\mu(\vec{x} - \hat{e}_\mu) = 0$$

Two-step Infinite-Volume Limit

Zwanziger suggests (NPB 1994) taking the infinite-volume limit in two steps

1) first, considering the $V \to +\infty$ limit for the gauge transformation $g(x)$

2) then, taking the same limit for the gluon field [i.e. the link variables $\{U_\mu(x)\}$]

How can one do that in practice?

Two-step Infinite-Volume Limit

Zwanziger suggests (NPB 1994) taking the infinite-volume limit in two steps

1) first, considering the $V \to +\infty$ limit for the gauge transformation $g(x)$

2) then, taking the same limit for the gluon field [i.e. the link variables $\{U_\mu(x)\}$]

How can one do that in practice?

Let us build the two-step limit directly from the link configuration, by "cloning" it to generate a bigger (extended) lattice(!!)

Two-step Infinite-Volume Limit

Zwanziger suggests (NPB 1994) taking the infinite-volume limit in two steps

1) first, considering the $V \to +\infty$ limit for the gauge transformation $g(x)$

2) then, taking the same limit for the gluon field [i.e. the link variables $\{U_\mu(x)\}$]

How can one do that in practice?

Let us build the two-step limit directly from the link configuration, by “cloning” it to generate a bigger (extended) lattice(!!)

$\Rightarrow$ $g(x)$ sees “infinite” volume while the one for $U_\mu(x)$ is still finite

The Extended Lattice

Setup:

1. Consider a d-dimensional link configuration $\{U_\mu(\vec{x})\} \in$ SU(N_c), defined on a lattice Λ_x with volume $V = N^d$ and periodic boundary conditions (PBC)

2. Replicate this configuration m times along each direction, yielding an extended lattice Λ_z with volume $m^d V$ and PBC

3. Indicate the points of Λ_z with $\vec{z} = \vec{x} + \vec{y}N$, where $\vec{x} \in \Lambda_x$ and $\vec{y}$ is a point on the index lattice Λ_y

4. By construction, $\{U_\mu(\vec{z})\}$ in Λ_z is invariant under translations by N (in any direction)

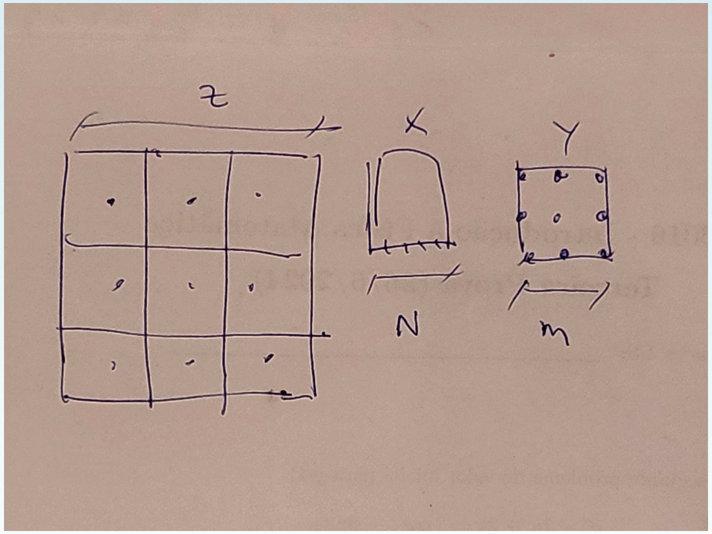
The Extended Gauge Transformation

Impose the minimal-Landau-gauge condition on Λ_z, i.e. consider the minimizing functional

$$\mathcal{E}_U[g] = -\frac{\Re\,\mathrm{Tr}}{d\,N_c\,m^d V} \sum_{\mu=1}^{d} \sum_{\vec{z}\in\Lambda_z} g(\vec{z})\,U_\mu(\vec{z})\,g(\vec{z}+\hat{e}_\mu)^\dagger$$

where $g(\vec{z})$ has periodicity mN, i.e. $g(\vec{z}+mN\hat{e}_\mu) = g(\vec{z})$ (PBC in Λ_z)

The two limits: first take $m \to +\infty$ and then $N \to +\infty$



Analogy with Bloch's Theorem

1. $\Lambda_y \Longleftrightarrow$ finite Bravais lattice with PBC
2. $\{U_\mu(\vec{z})\} \Longleftrightarrow$ periodic electrostatic potential $U(\vec{r})$

One can prove that:

- $g(\vec{z})$ can be written as $g(\vec{z}) = \exp\left(i\Theta_\mu \, z_\mu / N\right) h(\vec{z})$
- $h(\vec{z})$ has periodicity N, i.e. $h(\vec{z} + N\hat{e}_\mu) = h(\vec{z}) \;\Rightarrow\; h(\vec{x})$
- The matrices $\Theta_\mu = \tau^a \theta^a_\mu$ (with $a = 1, \ldots, N_c^2 - 1$) are elements of a Cartan sub-algebra of the SU(N_c) Lie algebra
- The matrices Θ_μ have eigenvalues $2\pi n_\mu / m$, with $n_\mu \in \mathcal{Z}$

The New Minimizing Functional

Due to the expression for $g(\vec{z})$ and to the cyclicity of the trace, the minimizing functional becomes

$$\mathcal{E}_U[h, \Theta_\mu] \; = \; -\frac{\Re \, \mathrm{Tr}}{d \, N_c \, V} \sum_{\mu=1}^{d} e^{-i\Theta_\mu / N} \, Q_\mu \, ,$$

$$Q_\mu \; = \; \sum_{\vec{x} \in \Lambda_x} h(\vec{x}) \, U_\mu(\vec{x}) \, h(\vec{x} + \hat{e}_\mu)^\dagger \, ,$$

i.e. the numerical minimization is still carried out on the original lattice Λ_x

Numerical Simulations

In the SU(N_c) case:

1. generate a thermalized d-dimensional link configuration $U_\mu(x)$ with periodicity N, i.e. $V = N^d$ with PBC
2. minimize $\mathcal{E}_U[h, \Theta_\mu]$ with respect to $h(x)$ and Θ_μ using two alternating steps:
   a) the matrices Θ_μ are kept fixed and one updates the matrices $h(\vec{x})$ by sweeping through the lattice
   b) the matrices Q_μ are kept fixed and one minimizes $\mathcal{E}_U[h, \Theta_\mu]$ with respect to the matrices Θ_μ, belonging to the corresponding Cartan sub-algebra
3. evaluate the gluon propagator using the extended gauge-fixed link variables $U_\mu^{(g)}(\vec{z}) = g(\vec{z})\, U_\mu(\vec{z})\, g(\vec{z} + \hat{e}_\mu)^\dagger$

The SU(2) Case

In the $SU(2)$ case (one-dimensional Cartan sub-algebra) we can write

$$\Theta_\mu \equiv (v^\dagger \sigma_3 v) \alpha_\mu$$

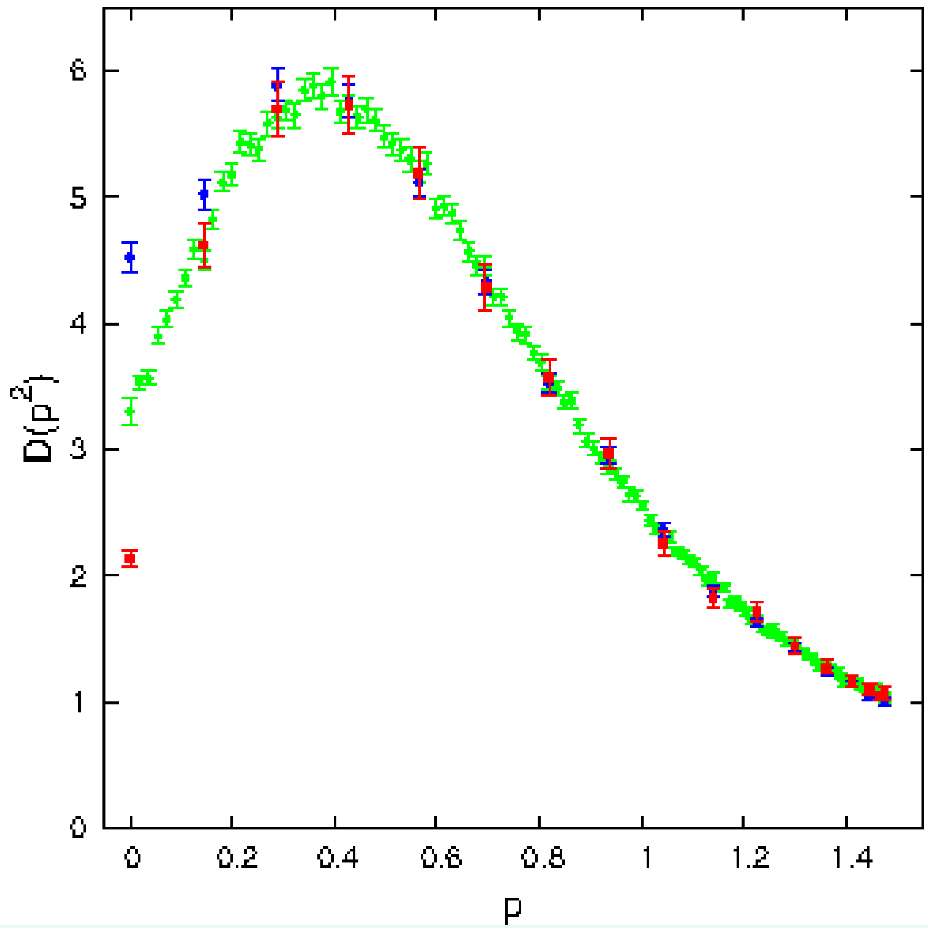
with $v \in$ SU(2) and eigenvalues $\pm\alpha_\mu = \pm 2\pi n_\mu / m$

Then, in the new minimizing functional

$$\exp\left(-i\Theta_\mu/N\right) \, = \, v^\dagger \, \exp[-2\pi i \sigma_3 n_\mu/(mN)] \, v$$

Also, the matrices Q_μ are proportional to SU(2) matrices

Results: $3D$ Gluon Propagator



The gluon propagator $D(p^2)$ as a function of the lattice momentum p at $\beta = 3.0$ for the Λ_x lattice volumes $V = 32^3$ ($+$) and 256^3 ($*$) and for the Λ_z lattice volume $V = 32^3 \times 8^3 = 256^3$ ($\square$)

Back to the Minimizing Problem

As mentioned earlier, the minimizing problem is simplified as a consequence of $g(\vec{z}) = \exp\left(i\Theta_\mu z_\mu/N\right) h(\vec{x})$, since the solution for the extended-lattice problem is obtained from minimizing a similar functional on the small one

Back to the Minimizing Problem

As mentioned earlier, the minimizing problem is simplified as a consequence of $g(\vec{z}) = \exp(i\Theta_\mu z_\mu/N)\, h(\vec{x})$, since the solution for the extended-lattice problem is obtained from minimizing a similar functional on the small one

For the gauge-transformed link variable $U^g_\mu(z)$ we have

$$U_\mu(g;\vec{z}) = e^{i\Theta_\nu z_\nu/N}\, U_\mu(h;\vec{x})\, e^{-i\Theta_\mu/N}\, e^{-i\Theta_\nu z_\nu/N}$$

$$= e^{i\Theta_\nu y_\nu} \left[e^{i\Theta_\nu x_\nu/N}\, U_\mu(h;\vec{x})\, e^{-i\Theta_\mu/N}\, e^{-i\Theta_\nu x_\nu/N} \right] e^{-i\Theta_\nu y_\nu}$$

where we used that $\vec{z} = \vec{x} + \vec{y}\vec{N}$

Back to the Minimizing Problem

As mentioned earlier, the minimizing problem is simplified as a consequence of $g(\vec{z}) = \exp(i\Theta_\mu z_\mu / N)\, h(\vec{x})$, since the solution for the extended-lattice problem is obtained from minimizing a similar functional on the small one

For the gauge-transformed link variable $U^g_\mu(z)$ we have

$$U_\mu(g;\vec{z}) = e^{i\Theta_\nu z_\nu/N}\, U_\mu(h;\vec{x})\, e^{-i\Theta_\mu/N}\, e^{-i\Theta_\nu z_\nu/N}$$

$$= e^{i\Theta_\nu y_\nu} \left[e^{i\Theta_\nu x_\nu/N}\, U_\mu(h;\vec{x})\, e^{-i\Theta_\mu/N}\, e^{-i\Theta_\nu x_\nu/N} \right] e^{-i\Theta_\nu y_\nu}$$

where we used that $\vec{z} = \vec{x} + \vec{y}\vec{N}$

Note that the central (local) part of the above expression is the same for all “cells” and that different domains (=cells) are related by a global “rotation” (determined by $\vec{y}$), applied to each cell

Gauge-Configuration Domains

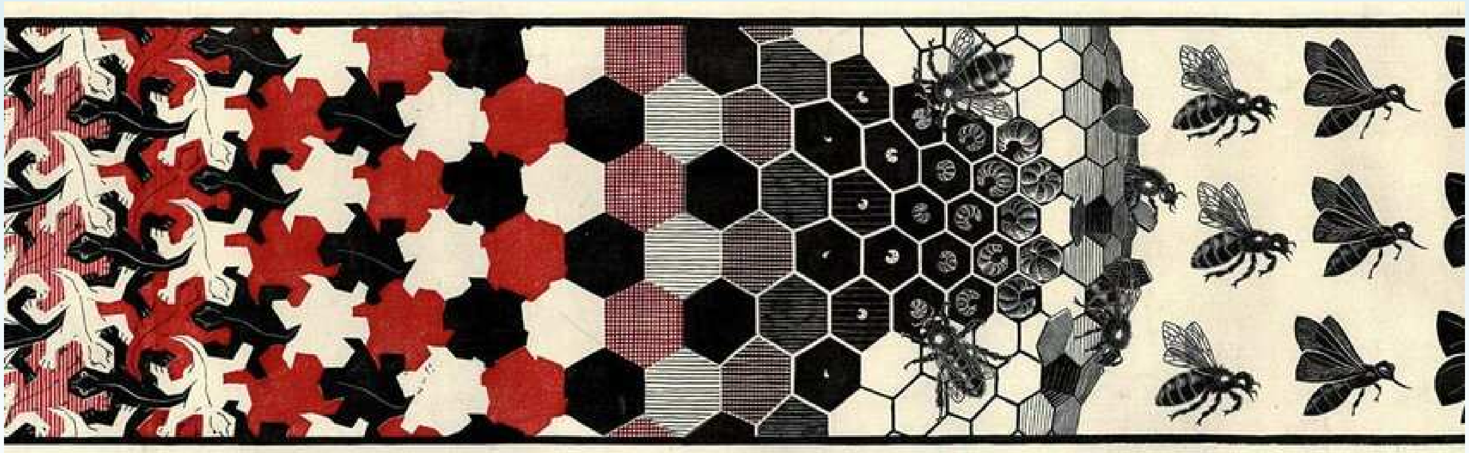
The local part of $U^g_\mu(z)$ is gauge-fixed (i.e. transverse) but is no longer periodic, while the original gauge configuration $U_\mu(z)$ was periodic, but not transverse

Gauge-Configuration Domains

The local part of $U^g_\mu(z)$ is gauge-fixed (i.e. transverse) but is no longer periodic, while the original gauge configuration $U_\mu(z)$ was periodic, but not transverse

Gauge-field configurations within cells are rotated, transformed by global group elements defined by the cell index $\vec{y}$, in a manner reminiscent of Escher's work (Metamorphosis I, II, III), so that the full configuration on the extended lattice has the required $m \times N$ periodicity

A pattern of domains emerges!



Color Magnetization

One can define a (gluon-field) color magnetization

$$A_\mu^b \;=\; \frac{1}{N^d}\,\sum_{\vec{x}} A_\mu^b(\vec{x})$$

which is related to the gluon propagator at zero momentum as

$$D(0) \;=\; \frac{N^d}{d(N_c^2-1)}\sum_{b,\mu}\langle |A_\mu^b|^2\rangle$$

Quantity $\mathcal{A} \;=\; \sum_{b,\mu}\langle |A_\mu^b|\rangle / d(N_c^2-1)$ considered by Zwanziger (in Landau gauge, this should vanish at least as fast as $1/N$).

Color Magnetization

One can define a (gluon-field) color magnetization

$$A^b_\mu \;=\; \frac{1}{N^d} \sum_{\vec{x}} A^b_\mu(\vec{x})$$

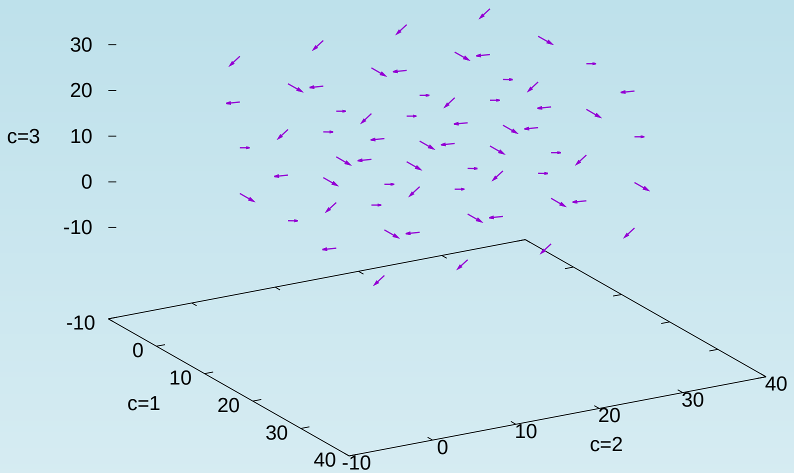
which is related to the gluon propagator at zero momentum as

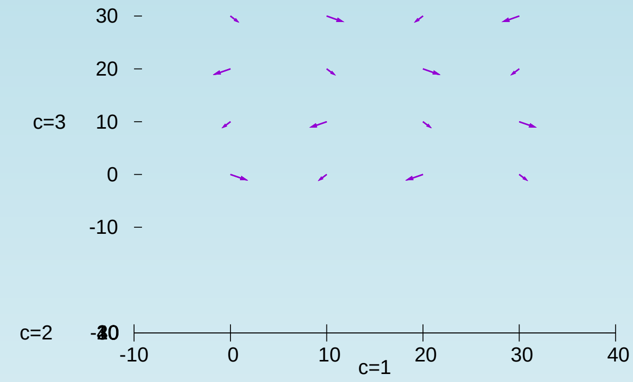
$$D(0) \;=\; \frac{N^d}{d(N_c^2-1)} \sum_{b,\mu} \langle |A^b_\mu|^2 \rangle$$

Quantity $\mathcal{A} \;=\; \sum_{b,\mu} \langle |A^b_\mu| \rangle / d(N_c^2-1)$ considered by Zwanziger (in Landau gauge, this should vanish at least as fast as $1/N$).

$\Rightarrow$ Let us look for the average color magnetization in each cell and try to relate it to the domains mentioned above

Average Cell Magnetization



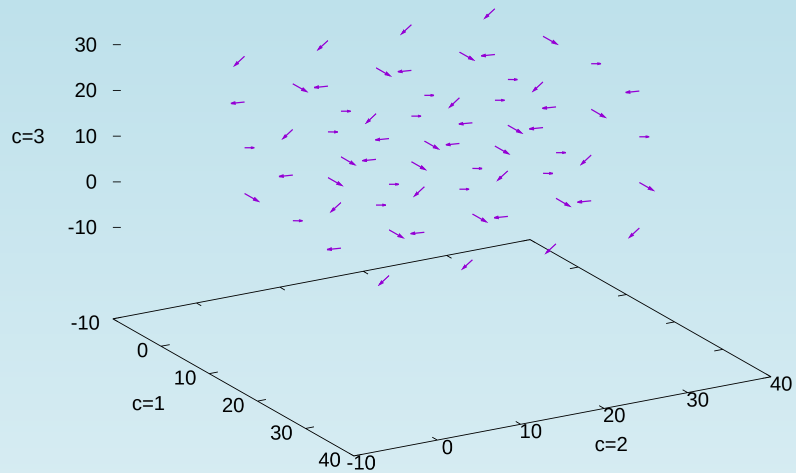


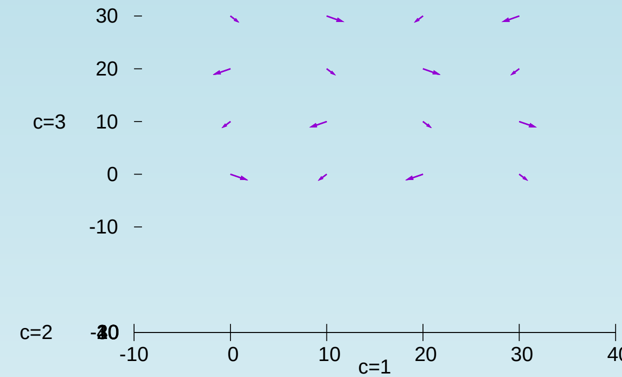
Average color “magnetization” in each cell

$$M_\mu^c(\vec{y}) \;=\; \frac{1}{N^d} \sum_{\vec{x}} A_\mu^c(\vec{z})$$

for the pure-$SU(2)$ case and lattice volume $V = (60 \times 4)^3$

Average Cell Magnetization





Average color “magnetization” in each cell

$$M_\mu^c(\vec{y}) \;=\; \frac{1}{N^d} \sum_{\vec{x}} A_\mu^c(\vec{z})$$

for the pure-$SU(2)$ case and lattice volume $V = (60 \times 4)^3$

A new type of domain wall?

Conclusions

- Numerical results (in the gluon sector) obtained using large lattice volumes can also be obtained using small lattice volumes with extended gauge transformations
- From the physical point of view:
  1. the information encoded in a thermalized configuration does not depend much on the lattice volume V
  2. the properties of the Landau-gauge Green's functions are essentially set by the gauge-fixing procedure and the size of V matters!
- Limitation: the allowed momenta seem to be fixed by the lattice discretization on the original lattice Λ_x, no way to obtain "big-volume" momenta?
- Interesting properties regarding "magnetization" domains!