Gross-Neveu-Yukawa model at five loops

York Schröder

(UBB Chillán)

recent work with John Gracey, Andreas Maier, Peter Marquard
[arXiv:2507.22594]

and earlier work with
T. Luthe, J. Möller, C. Studerus

WONPAQCD Valparaíso, Dec 2025

Motivation

- LHC: era of precision QFT
  - ▷ large SM backgrounds
  - ▷ dominant effects: mainly QCD
  - ▷ high precision required for BSM searches

- theory working horse: renormalizable QFTs
  - ▷ perturbative expansions $\Rightarrow$ Feynman diagrams
  - ▷ higher-loop effects important (e.g. $g_\mu - 2$; m_q; H production; ...)

- lots of machinery developed recently
  - ▷ high automatization of complicated perturbative calculations
  - ▷ algebraic handling of Feynman diagrams
  - ▷ reduction of Feynman integrals to masters
  - ▷ numerical and/or algebraic determination of masters
  - ▷ renormalization (of QCD) under control at 5 loops

Motivation

- idea: recycling
  - ▷ apply same techniques to other (than SM and QCD) models
  - ▷ important applications e.g. in CM theory [see talk by Vladimir Juricic]
  - ▷ consider interacting Dirac fermions ψ in 2+1 dimensions
  - ▷ undergo variety of quantum phase trans (QPTs) towards ordered states
  - ▷ with different symm breaking patterns

- general universal critical behavior $\Leftrightarrow$ class of relativistic GNY models
  - ▷ models emerge as low-energy EFTs in CM
  - ▷ but: strongly coupled systems in D=2+1!
  - ▷ order parms $\Leftrightarrow$ bosonic fields ϕ (with corresponding nr of components + symms)
  - ▷ Yukawa coupling between ψ and ϕ

- captured by universal critical exponents
  - ▷ near cont phase trans, free energy exhibits scaling form
  - ▷ specific heat / corr length have power-law behavior
  - ▷ characterized by universal critical exponents
  - ▷ e.g. thermal PT: $\xi \sim |t|^{-\nu}\,(1 + c|t|^{\omega} + \ldots)$ at 'distance' $t = (T - T_c)/T_c$
  - ▷ ν, ω universal; depend on symm and dimensionality $\Rightarrow$ def univ class

Motivation

- describe quantum critical points of interacting Dirac semimetals
  - ▷ N flavors of chiral Dirac fermions [main interest N=2, graphene]
  - ▷ chiral Ising model [$\phi \in \mathbb{R}$, with Z_2 symm]
  - ▷ chiral XY model [U(1) breaking, $\phi \in \mathbb{C}$]
  - ▷ chiral Heisenberg model [SU(2) breaking, $\phi \in \mathbb{R}^3$]

- different models argued to describe certain QPTs in graphene (N=2)
  - ▷ QCP of semimetal-insulator trans in graphene
  - ▷ but also: N=1, semimetal-insulator trans in system of spinless fer on hex lattice
  - ▷ emergent SuSy (topological insulators, N=1/2 or 1/4)
  - ▷ replica limit ($N \to 0$) for PT in Weyl semimetals

Setup

- Gross-Neveu model (for reference only)

$$\mathcal{L}_{GN} = \bar{\psi}\partial\!\!\!/\psi + g\phi\bar{\psi}\psi + \frac{1}{2}\phi^2$$

  - ▷ ϕ not dynamic; int out $\Rightarrow$ 4-fermion coupling $\sim g^2$
  - ▷ renormalizable in $D = 2 + \epsilon$, asymptotically free
  - ▷ known to 4 loops [Gracey/Luthe/YS 2016]

- Gross-Neveu-Yukawa model (chiral Ising; $\phi \in \mathbb{R}$)

$$\mathcal{L}_{GNY} = \bar{\psi}\partial\!\!\!/\psi + g\phi\bar{\psi}\psi + \frac{1}{2}\phi(m^2 - \partial_\mu^2)\phi + \lambda\phi^4$$

  - ▷ same univ class as GN for 2<D<4
  - ▷ renormalizable in $D = 4 - \epsilon$
  - ▷ analyzed at 4 loops [Zerf/Mihaila/Marquard/Herbut/Scherer 2017]
  - ▷ now known to 5 loops [Gracey/Maier/Marquard/YS 2025]

Renormalization

- $\psi_0 = \sqrt{Z_\psi}\psi_R \ , \quad \phi_0 = \sqrt{Z_\phi}\phi_R \ , \quad$ etc.

$$\mathcal{L}_{GNY,R} = Z_\psi \bar{\psi}\partial\!\!\!/\psi + Z_{\phi\bar{\psi}\psi}\mu^{\frac{\epsilon}{2}} g\phi\bar{\psi}\psi + \frac{1}{2}\phi(Z_{\phi^2}m^2 - Z_\phi\partial_\mu^2)\phi + Z_{\phi^4}\mu^\epsilon\lambda\phi^4$$

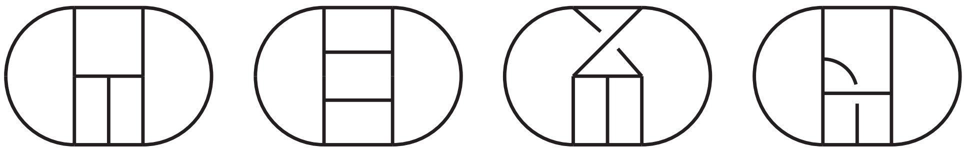
  - ▷ rename squared Yukawa coupling $g^2 \equiv y$
  - ▷ RG scale (μ) dep follows from μ-independence of bare params

$$y = y_0\mu^{-\epsilon}Z_\psi^2 Z_\phi Z_{\phi\bar{\psi}\psi}^{-2} \ , \quad \lambda = \lambda_0\mu^{-\epsilon}Z_\phi^2 Z_{\phi^4}^{-1} \ , \quad m^2 = m_0^2\mu^{-\epsilon}Z_\phi Z_{\phi^2}^{-1}$$

- evaluate all five RG constants $Z_x(y,\lambda)$: absorb divergences in dim reg, $\overline{\text{MS}}$
  - ▷ *QGRAF*: generate diags
  - ▷ *FORM*: Fy rules, Lorentz algebra, spinor traces, project onto 1-scale massive tadpoles
  - ▷ *Mathematica*: identify topologies, mapping to common family
  - ▷ *C++*: IBP reduction, Laporta algorithm, mapping onto master integrals
  - ▷ *C++*: solve difference equations for masters, evaluate numerically
  - ▷ *Mathematica*: plug in numerical masters, identify transcendental content

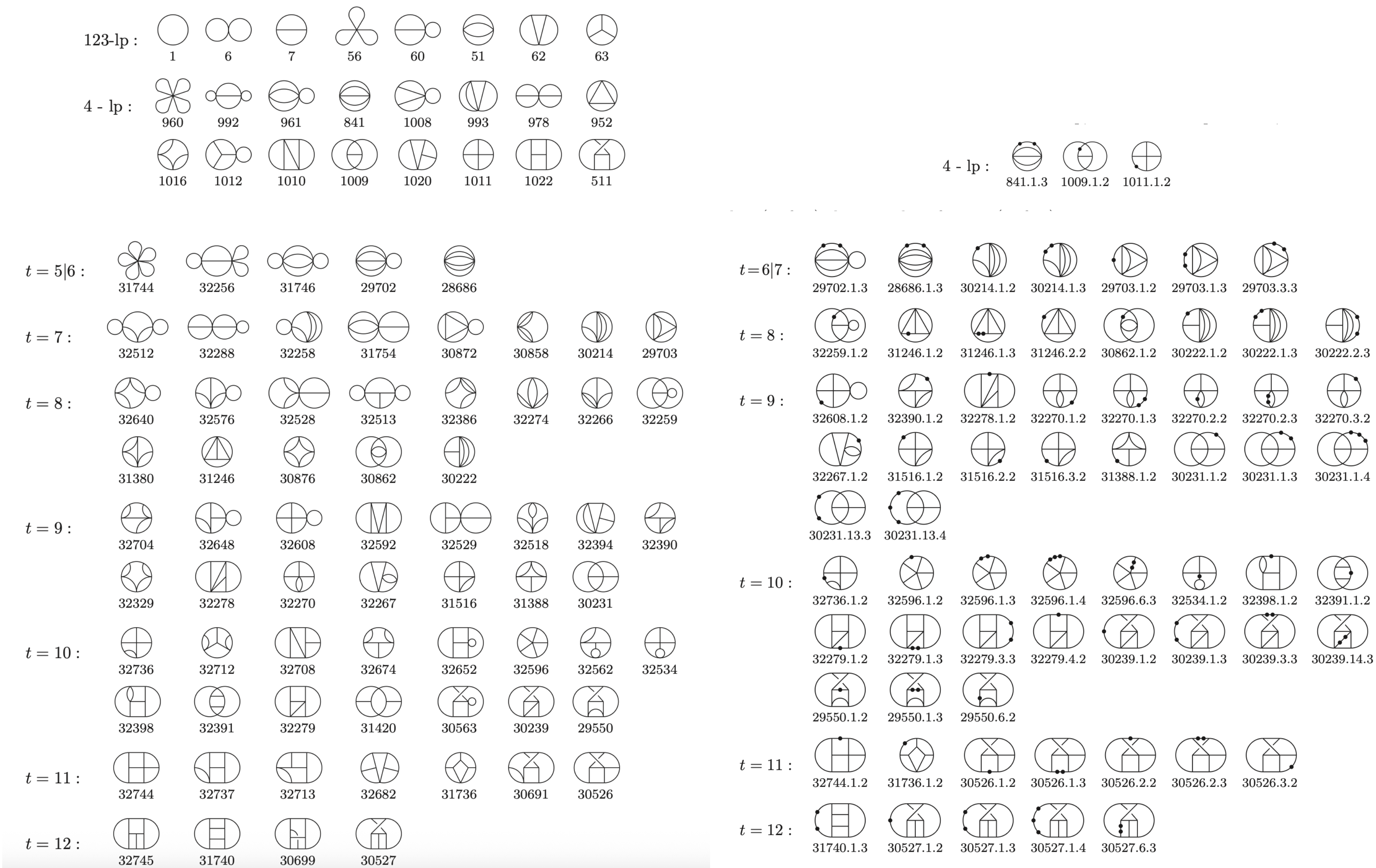
Renormalization techniques

- project all Fy integrals onto common family of scalar massive vacuum ints
  - ▷ exact decomposition of propagators ($m \in \{0, m, M\}$) [Chetyrkin/Misiak/Münz 1998]
  - ▷ $\frac{1}{(k-p)^2+m^2} = \frac{1}{k^2+M^2} + \frac{2kp-p^2+M^2-m^2}{(k^2+M^2)((k-p)^2+m^2)}$
  - ▷ recursively lower degree of UV div
  - ▷ other IR regularization schemes: e.g. (local/global) R* [Chetyrkin 1984]



- map all integrals to minimal set: IBP [Chetyrkin/Tkachov 1981; Laporta 2000]
  - ▷ systematically use linear rels $0 = \int d^dk\, \partial_{k_\mu} f_\mu(k)$
  - ▷ key idea: lexicographic ordering among all loop integrals [Laporta 2000]
  - ▷ arrive at rep in terms of irreducible ($\equiv$ master) integrals: $\sum_i \mathrm{rat}_i(d)\,\mathrm{Master}_i(d)$
  - ▷ obtain 1+1+3+13+110 masters at 12345-loop

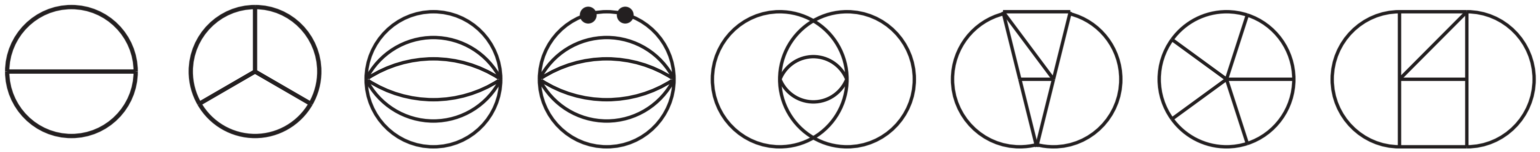
Single-scale master integrals to 5 loops



Integral evaluation

- evaluate this minimal set of single-scale master integrals
  - ▷ numerical solution of difference equations via factorial series [Laporta 2000]
  - ▷ perform IBP reduction with symbolic power x on one line
  - ▷ derive difference equation for generalized master $I(x) \equiv \int \frac{1}{D_1^x D_2 \ldots D_N}$
  - ▷ generic form: $\sum_{j=0}^{R} p_j(x) I(x+j) = F(x)$
- typically, want $I(1)$; solve the difference equation
  - ▷ explicitly (e.g. if 1st order; or if soln nested sum)
  - ▷ numerically (very general setup) [Laporta 2000]
  - ▷ solve via factorial series $I(x) = I_0(x) + \sum_{j=1}^{R} I_j(x)$,
  - ▷ where $I_j(x) = \mu_j^x \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_j)}$
  - ▷ need boundary condition for fixing, say, $a_j(0)$: use decoupling at large x
  - ▷ $I(x) = \int_{k_1} g(k_1)/(k_1^2+1)^x \;\Rightarrow\; I(x) \sim (1)^x x^{-d/2} g(0)$
- deep expansions (ϵ^{20}) at 5 loops (132 masters) at high precision (>250 digits) [with Luthe 2016]

Sample results (4d)



$$I_{28686.1.1} = +(-3)\,\epsilon^0 + (-\frac{3}{2})\,\epsilon^1 + (\frac{13}{24})\,\epsilon^2 + (-\frac{1267}{1440})\,\epsilon^3 + (-\frac{4193}{3456})\,\epsilon^4 + \\ +135.95072868792871461956492733702218574897992953584\,\epsilon^5 + \ldots$$

$$I_{28686.1.3} = +(0)\,\epsilon^0 + (\frac{3}{2})\,\epsilon^1 + (-\frac{1}{2})\,\epsilon^2 + (-\frac{443}{360})\,\epsilon^3 + (\frac{95}{216})\,\epsilon^4 + \\ -38.292059175062436961881799538284449799148385376441\,\epsilon^5 + \ldots$$

$$I_{30862.1.1} = +(-\frac{3}{5})\,\epsilon^0 + (-\frac{27}{10})\,\epsilon^1 + (-\frac{4\zeta_3}{5} - \frac{421}{60})\,\epsilon^2 + (-\frac{12\zeta_2^2}{25} + \frac{24\zeta_3}{5} + \frac{211}{24})\,\epsilon^3 + (\frac{72\zeta_2^2}{25} - 98\zeta_3 + \frac{32\zeta_5}{5} + \frac{12959}{48})\,\epsilon^4 + \\ +1143.1838307558764599466030303839590323268318605888\,\epsilon^5 + \ldots$$

$$I_{30231.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (\frac{3\zeta_3}{5})\,\epsilon^2 + (\frac{9\zeta_2^2}{25} + \frac{21\zeta_3}{5} + 3\zeta_5)\,\epsilon^3 + (-36H_2\zeta_3 + \frac{12\zeta_2^3}{7} + \frac{63\zeta_2^2}{25} - \frac{21\zeta_3^2}{5} + 27\zeta_3 - \frac{24\zeta_5}{5})\,\epsilon^4 \\ -531.32391547725635267943444561495368318398901378435\,\epsilon^5 + \ldots$$

$$I_{32596.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (0)\,\epsilon^2 + (0)\,\epsilon^3 + (-14\zeta_7)\,\epsilon^4 + \\ +235.07729596783467131454388080950411779239347239580\,\epsilon^5 + \ldots$$

[Wheel: Broadhurst 1985]

$$I_{32279.3.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (0)\,\epsilon^2 + (0)\,\epsilon^3 + (-\frac{441\zeta_7}{40})\,\epsilon^4 + \\ +181.78223928612340820790788236018642961198741994209\,\epsilon^5 + \ldots$$

[Zigzag: Broadhurst/Kreimer 1995; Brown/Schnetz 2012]

Beta functions, anomalous dimensions, critical exponents

- procedure outlined above gives all five RG constants $Z_x(y,\lambda)$, analytically, to 5 loops
- extract beta functions of couplings ($x \in \{y,\lambda\}$)

$$\beta_x \equiv \partial_{\ln(\mu)}\, x \;=\; -\epsilon x + \sum_{L=1}^{5} \beta_x^{(L)}(y,\lambda) + \dots$$

- extract anomalous dimensions ($x \in \{\psi,\phi,\phi^2\}$)

$$\gamma_x \equiv \partial_{\ln(\mu)}\, Z_x \;=\; \sum_{L=1}^{5} \gamma_x^{(L)}(y,\lambda) + \dots$$

- extract RG fixed points from $\beta_x(y_*,\lambda_*) = 0$
  - ▷ get FPs as series in ϵ, 5 orders
  - ▷ choose the stable positive non-Gaussian (Wilson-Fisher) FP
- substitute FP into anom dims to get critical exponents
  - ▷ $\eta_x \equiv \gamma_x(y_*,\lambda_*)$ for $x \in \{\psi,\phi,\phi^2\}$; trade the latter for $\frac{1}{\nu} \equiv 2 - \eta_\phi + \eta_{\phi^2}$
  - ▷ successfully checked vs all-order large-N results [Gracey 91-94; Vasiliev et al. 93]

Analysis: Graphene case $N = 2$

- aim: from $\underline{4-\epsilon}$ results for $\{\eta_\psi, \eta_\phi, \frac{1}{\nu}\}$, provide refined estimates for critical exponents, in $\underline{\text{3d}}$
  - ▷ naively setting $\epsilon = 1$ is problematic: questionable convergence ($\epsilon^1 \sim \epsilon^5$)

- employ resummation techniques
  - ▷ canonical Padé approximants $P_{[M,L-M]}(d) = (\sum_p a_p\, d^p)/(1 + \sum_q b_q d^q)$ (need to ensure continuity 4⟶3)
  - ▷ add info from our 2016 4-loop $\underline{2+\epsilon}$ GN [idea: Mihaila/Scherer 18]
  - ▷ two-sided diagonal [L/L+1] and [L+1/L] Padé approximants
  - ▷ interpolating polynomial $I_{[L,L+1]}(d) = \sum_m \eta_m^{(2d)}\,(d-2)^m + \sum_{n>m} c_n^{(4d)}\,(d-2)^n$ (note: no singularities, unlike Padé)

- evaluate reliability of 3d predictions $P(3)$ and $I(3)$
  - ▷ compare with other methods
  - ▷ repeat the exercise for different N

Analysis: Graphene case $N = 2$

TABLE II. Summary of previous exponent estimates for $N = 2$.

Method and source	η_ψ	η_ϕ	$1/\nu$
Large N [17,31–36]	0.044	0.743	0.952
Monte Carlo [17]		0.754(8)	1.00(4)
Monte Carlo [18]	0.38(1)	0.62(1)	1.20(1)
Functional renormalization group [25]	0.032	0.760	0.982
Functional renormalization group [25]	0.033	0.767	0.978
Functional renormalization group [25]	0.032	0.756	0.982
Functional renormalization group [26]	0.0276	0.7765	0.994(2)
Four-loop $d = 2$ naive Padé [37]	0.082	0.745	0.931
Three-loop $d = 4$ naive Padé [38]	0.0740	0.672	1.048
Conformal bootstrap [28]	0.044	0.742	0.880
Monte Carlo [20]		0.65(3)	1.2(1)
Monte Carlo [21]		0.54(6)	1.14(2)
Four-loop $d = 4$ naive Padé [39]	0.0539	0.7079	0.931
Four-loop $d = 4$ naive Padé [39]	0.0506	0.6906	0.945
Four-loop two-sided Padé [40]	0.042	0.735	1.004
Four-loop interpolating polynomial [40]	0.043	0.731	0.982
Four-loop Padé-Borel [40]	0.043(12)	0.704(15)	0.993(27)
Monte Carlo [22]	0.05(2)	0.59(2)	1.0(1)
Conformal bootstrap [29]	0.04238(11)	0.7329(27)	0.998(12)
Monte Carlo [24]	0.043(12)	0.72(6)	1.07(12)
Conformal bootstrap [30]		0.7339(26)	0.998(12)
Functional renormalization group [27]	0.032	0.760	0.982

- ▷ in chronological order, 1991 – 2025
- ▷ highest precision to date: conformal bootstrap [Poland et al. 23/24]

Analysis: Graphene case $N = 2$

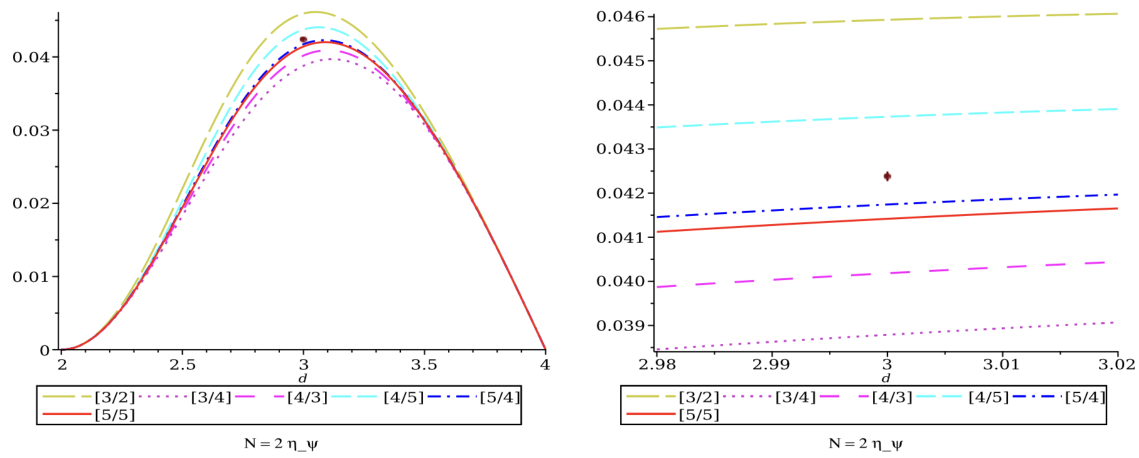


FIG. 2. Two-sided Padé approximants for η_ψ with $N = 2$ in $2 \leq d \leq 4$ (left panel) and $2.98 \leq d \leq 3.02$ (right panel).

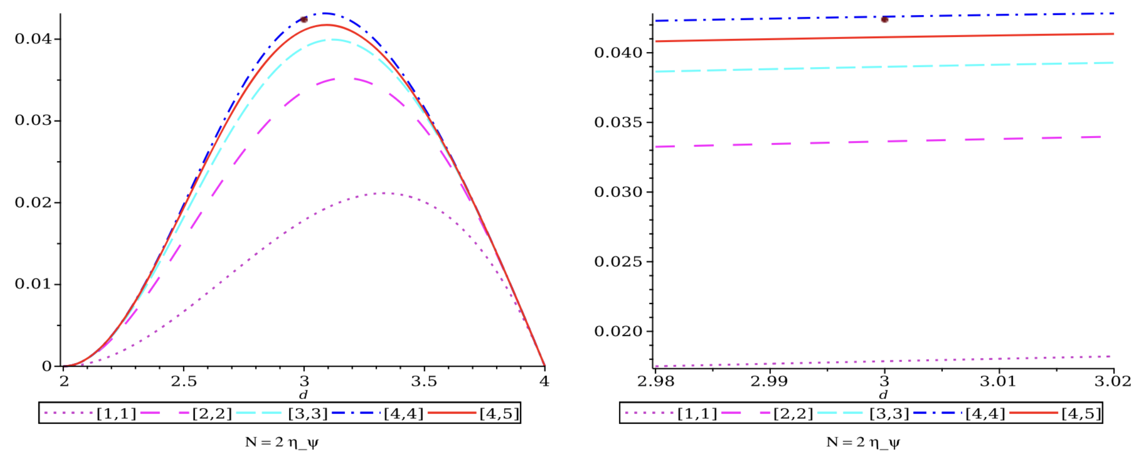


FIG. 5. Interpolating polynomials for η_ψ with $N = 2$ in $2 \leq d \leq 4$ (left panel) and $2.98 \leq d \leq 3.02$ (right panel).

▷ similar patterns for η_ϕ and $\frac{1}{\nu}$, and other $N \Rightarrow$ motivation for 5-loop GN?

Conclusions

- recent advance in methods allows high-loop renormalization of quantum field theories
  - ▷ highly automated computer-algebra approaches
  - ▷ improved IBP algorithms (finite fields, . . .)
  - ▷ new insight into functional content of Feynman integrals
- difference equations (+ lots of CPU) are powerful enough to achieve 5 loops
  - ▷ non-trivial algorithmic fine-tuning
  - ▷ more? memory seems to become an issue
- applications to a host of QFTs in various fields
  - ▷ phenomenology / QCD+SM
  - ▷ mathematical physics/ SUSY
  - ▷ CM theory / effective models / universality classes
  - ▷ $\Rightarrow$ consistent exponents for graphene from ϵ-exp, large N, FRG and MC
- applications in various space-time dimensions
  - ▷ 2d, 3d, 4d, . . . , fractional, . . .

Fractional dimensions

- difference eqs carry full information on d
  - ▷ expand massive tadpoles e.g. around $d=\frac{10}{3}-2\epsilon$?
- motivation: renormalization in critical dimension (where [g]=0)
  - ▷ e.g. O(N) scalar ϕ^n theory: $\partial_\mu\phi\,\partial^\mu\phi+g\phi^n$ [Gracey 2017]
  - ▷ critical dim $D_n=\frac{2n}{n-2}\Rightarrow D_{\{3,4,5,6,7,\dots,\infty\}}=\{6,4,\frac{10}{3},3,\frac{14}{5},\dots,2\}$
  - ▷ near FP (nontriv zero of Beta fct), RG fcts carry info on phase transitions
- study RG fcts in non-integer dimensions
  - ▷ renormalization 'as usual', dim. reg. natural (angular ints?!)
  - ▷ fewer diagrams (e.g. ϕ^5: LO 2pt diag has 3 loops)
  - ▷ numbers: $\frac{p}{q}\to\Gamma(\frac{p}{q})$ at LO; $\zeta(s)\to$ Dirichlet β fct at NLO ($\in$HPL(i)) [Hager 2002]
- sample results in $d=\frac{10}{3}-2\epsilon$ [with Luthe]
  - ▷ as usual we normalize by $1/J^{loop}$; here, $J\sim\Gamma(1-\frac{d}{2})=-\frac{3}{2}\Gamma(\frac{1}{3})+\dots$
  - ▷ 2-loop sunset $=-2.4882241714632542542\,\epsilon^0-8.6116306893649818141\,\epsilon^1+\dots$
  - ▷ 3-loop merc. $=-0.0305966721641989027\,\epsilon^0+0.0149523122719722312\,\epsilon^1+\dots$
  - ▷ 4-loop non-pl $=+0.0006880032418228675\,\epsilon^0+0.0023853718027957011\,\epsilon^1+\dots$
  - ▷ etc.