Cartan-Based Abelian Projection for Yang-Mills Theory

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December 1, 2025

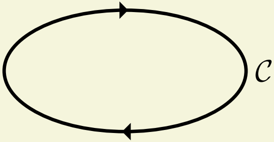
Introduction

- QCD is accepted as the correct theory to describe the strong force. Yet, the basic degrees of freedom, quarks and gluons, are never seen freely in nature.
- This is known as the *confinement problem*, and its origin is still an open question.
- Many ideas have been proposed to explain confinement. Among them, the Abelian dominance scenario is a very interesting one.
- Our objective is to study this scenario in SU(3) gauge theory.
- For that, we first have to ask: how do we define confinement?

How to measure the potential? Wilson loops

The Wilson loop operator

$$\mathcal{W}(\mathcal{C}) = \mathrm{Tr}\, \mathcal{P} \exp \left[i \oint_{\mathcal{C}} dz^{\mu}\, A_{\mu}(z) \right]$$



describes how a pair of quark-antiquark probes evolves in time.

Quantizing this operator and using the spectral decomposition

$$\langle \mathcal{W}(\mathcal{C}) \rangle = F_0 \, \exp[-t\, V(r)] \times (1 + \mathcal{O}(\exp[-t(E_2 - V(r))]),$$

we have access to the energy of the heavy quark pair, the **quark-antiquark potential**.

Lattice discretization

To have access to the interquark potential we need to know

$$\langle \mathcal{W}(\mathcal{C}) \rangle = \frac{1}{Z} \int \mathcal{D}A \, e^{-\frac{1}{4}\int d^4x \, \mathrm{Tr} \, F^{\mu\nu}(x) \, F_{\mu\nu}(x)} \left[\mathrm{Tr} \, \mathcal{P} e^{i \oint_{\mathcal{C}} dz^{\mu} \, A_{\mu}(z)} \right],$$

where

$$F_{\mu\nu}(x) = -i[D_\mu(x), D_\nu(x)], \quad D_\mu(x) = \partial_\mu + i A_\mu(x).$$

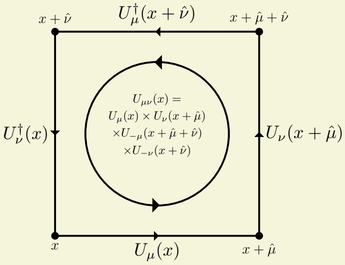
In the continuum this integral is ill-defined:

- There is no bare propagator without gauge fixing.
- Faddeev-Popov gauge-fixing procedure does not work in the IR regime.
- The running coupling is not small at low energies, i.e. perturbation theory is not applicable. **Non-perturbative techniques are necessary.**

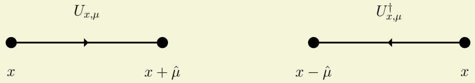
Lattice discretization

Using a lattice regulator the quantization of Yang-Mills theory is made clean and well-defined:

Lattice spacing a.



- The gauge field is the **link variable** $U_\mu(x) = \exp(ia\,A_\mu(x))$.



- $U_\mu(x)$ take values in the compact gauge group: the measure $dU_\mu(x)$ is well defined.
- Gauge invariant observables are traces of product of links: **Wilson loops**.

$$Z = \int \left(\prod_x \prod_\mu dU_\mu(x) \right) \exp\left\{ -\frac{\beta}{N_c} \mathrm{Re}\ \mathrm{Tr}\left[\mathbb{I} - U_{\mu\nu}(x)\right] \right\}, \quad \beta = \frac{N_c}{2g_s}.$$

Measuring the quark potential on the lattice

On the lattice: $\mathcal{W}(\mathcal{C}) = \mathrm{Tr}\ \left[\prod_{l\in\mathcal{C}} U_l\right]$.

$$\langle \mathcal{W}(\mathcal{C})\rangle = \int dP[U]\,\mathcal{W}(\mathcal{C}),$$

where

$$dP[U] = \frac{1}{Z}\mathcal{D}U\,e^{-S_g[U]},$$

is the probability distribution of the theory.

- Using a Markov-chain Monte Carlo we can generate configurations that respect the **exact** probability distribution $P[U]$.

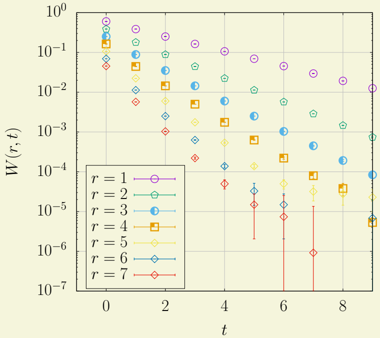
Numerical simulations!

$$U^{(1)} \to U^{(2)} \to \cdots \to U^{(N_{\mathrm{conf}})}$$

Measurements done by weighted means

$$\langle \mathcal{W}(\mathcal{C})\rangle \approx \frac{1}{N_{\mathrm{conf}}}\sum_{i=1}^{N_{\mathrm{conf}}} \mathcal{W}(\mathcal{C})\big|_{U^{(i)}}$$

subjected to statistical errors only.



Measuring the quark potential on the lattice

Taking rectangular time-like paths of size $R \times T$

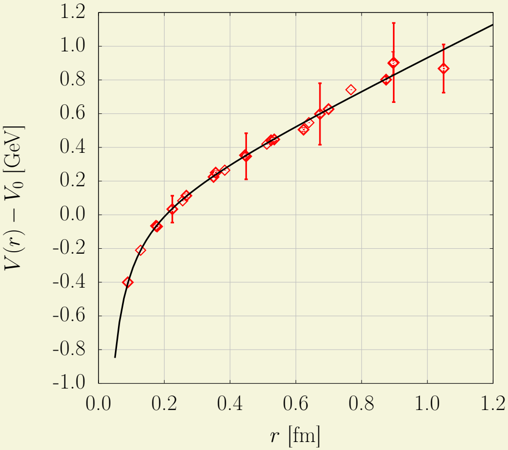
$$\log\langle \mathcal{W}(R,T)\rangle = -V(R)\,T + B.$$

The potential is of the form

$$V(r) = V_0 + \sigma\, r + \frac{\gamma}{r}.$$

Confinement is defined as the presence of a linear rising potential, which means a chromo-electric field squeezing into flux-tubes [Greensite, J. et al. **Physical Review D**, **75**, 034501 (2007)].

$$\sigma = \frac{1}{2}\int_{\substack{\text{cross}\\\text{section}}} d^2x\, E_k^a(x) E_k^a(x) \quad \text{string tension}$$



Static quark-antiquark potential for the SU(2) gauge theory.

Confinement scenarios

- Explain the origin of σ is to understand confinement.
- Among the various ideas proposed, we will analyze t' Hooft's Abelian projection.
- Special types of configurations, topological ones, may dominate the infrared regime of YM theory.
- On the lattice, these configurations can be isolated via a projection scheme

$$U_\mu(x) \xrightarrow{\text{law of projection}} \mathcal{U}_\mu(x),$$

  where $\mathcal{U}_\mu(x)$ is a projected link variable that depends on which kind of topological configuration we want to isolate.
- Abelian projection, center vortices, monopoles.
- Theses schemes usually go through a gauge fixing procedure (the links transform by gauge: $U'_\mu(x) = g(x)\, U_\mu(x)\, g^\dagger_\mu(x+\hat{\mu})$.)

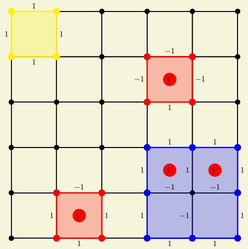
Success in the SU(2) case

A very interesting example are center vortices in SU(2) Yang-Mills theory.

- **Center vortices**: Lines of quantized flux of chromo-magnetic field.

$$\mathcal{B}(\mathcal{C})\,\mathcal{W}(\mathcal{C}') = z\,\mathcal{W}(\mathcal{C}')\,\mathcal{B}(\mathcal{C})$$

[G. 't Hooft, Nucl. Phys. B **138**, 1-25 (1978)].



Detection on the lattice? Maximal Center gauge [L. Del Debbio, et al., Phys. Rev. D **58** (1998), 094501]: Maximize

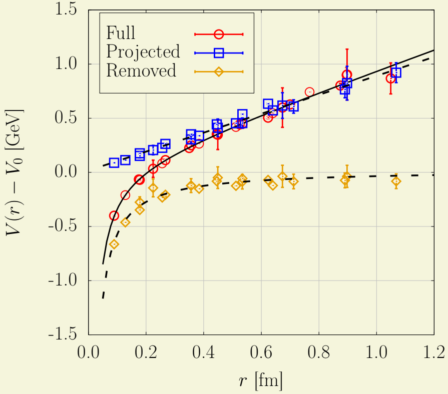
$$R = \sum_{x,\mu} |\mathrm{Tr}\; U^g_\mu(x)|^2.$$

We can generate three ensembles

- Untouched: $U_\mu(x)$,
- Center-projected: $Z_\mu(x) = \mathrm{sign}[\mathrm{Tr}\; U_\mu(x)]\,\mathbb{I}$,
- Center-removed: $R_\mu(x) = Z^\dagger_\mu(x)\,U_\mu(x)$.

Success in the SU(2) case

We can measure the static-quark potential on the three ensembles.



String tensions computed on the full and vortex-only ensemble.

β	σ	σ_v
2.2	0.251(4)	0.2335(4)
2.3	0.1521(6)	0.1371(1)
2.4	0.08151(1)	0.073414(1)
2.5	0.03945(2)	0.039786(1)

- Vortex-only ensemble capture the string tension.
- Vortex-removed configurations **shows no confinement**.

Abelian projected and magnetic monopoles configurations also give similar results.

Applying in the SU(3) case

Does this work in SU(3) theory? Let us start by checking Abelian dominance.

- Maximal Abelian gauge

$$R = \sum_{x,\mu,q=3,8} \mathrm{Tr}\, U_\mu(x)\lambda_q U_\mu^\dagger(x)\lambda_q.$$

- Abelian projection

$$U_\mu(x) \to U_\mu^A(x) = \mathrm{diag}(e^{i\phi_\mu^1(x)}, e^{i\phi_\mu^2(x)}, e^{i\phi_\mu^3(x)}),$$

where the angles $\phi_\mu^a(x) = \arg(U_\mu(x)_{aa})$ respect the constraint

$$\phi_\mu^1(x) + \phi_\mu^2(x) + \phi_\mu^3(x) = 0.$$

Values of string tension obtained from various types of topologicial configurations, [W. W. Tucker and J. D. Stack, Nucl. Phys. B Proc. Suppl. **106** (2002), 643-645].

β	Full	$U(1)^2$	$Z(3)$	Mon
5.9	0.068(3)	0.063(3)	0.060(3)	0.050(2)
6.0	0.050(1)	0.045(2)	0.040(2)	0.038(1)

The string tension is not fully reproduced by any of the topological d.o.f in the SU(3) theory.

What is wrong?

- $SU(N_c)$ has a special structure, encoded in the **Cartan subalgebra**, $[H_q, H_p] = 0$.
- Normalization convention: $H_q = \lambda_q/2\sqrt{N_c}$.
- Weights are eigenvalues of the Cartan matrices

$$H_q\,\vec{\beta}_i = \beta_i|_q \vec{\beta}_i.$$

- Roots are weights of the adjoint representation

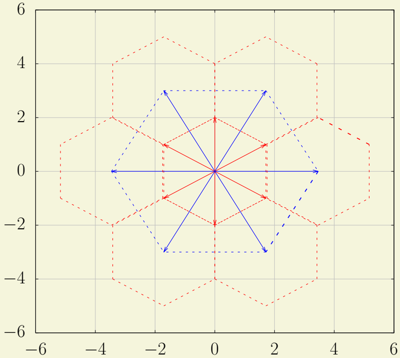
$$[H_q, E_{\vec{\beta}_{ij}}] = \beta_{ij}|_q\, E_{\vec{\beta}_{ij}}.$$

- Raising and lowering in weights space

$$E_{\vec{\beta}_{ij}}|n, \vec{\beta}_i\rangle = |n, \vec{\beta}_i + \vec{\beta}_{ij}\rangle.$$

$$\beta_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \beta_2 = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}, \beta_3 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

$$\beta_{12} = \begin{pmatrix} 2\sqrt{3} \\ 0 \end{pmatrix}, \beta_{13} = \begin{pmatrix} -2\sqrt{3} \\ 0 \end{pmatrix}, \beta_{23} = \begin{pmatrix} -\sqrt{3} \\ 3 \end{pmatrix}.$$



Cartan projection

- We can write: $A_\mu(x) = \sum_{q=1}^{N_c-1} \mathcal{A}_\mu^q \, H_q + \sum_{\vec{\alpha}} \mathcal{A}_\mu^{\vec{\alpha}} \, E_{\vec{\alpha}}$.
- $SU(3)$ Cartan projection: keep the **two independent fields** $\mathcal{A}_\mu^1$, $\mathcal{A}_\mu^2$.

$$A_\mu^A(x) = \mathcal{A}_\mu^q(x) \, H_q .$$

- Cartan projected link

$$U_\mu^A(x) = \mathrm{diag}\left\{ \exp\left[i \left(\frac{\mathcal{A}_\mu^1}{2\sqrt{3}} + \frac{\mathcal{A}_\mu^2}{6} \right) \right], \exp\left[i \left(-\frac{\mathcal{A}_\mu^1}{2\sqrt{3}} + \frac{\mathcal{A}_\mu^2}{6} \right) \right], \exp\left[i \left(-\frac{\mathcal{A}_\mu^2}{3} \right) \right] \right\} .$$

- Relating this with the usual form

$$\mathcal{A}_\mu^1(x) = \sqrt{3} \, \left(\phi_{\mu\nu}^1(x) - \phi_{\mu\nu}^2(x) \right) ,$$

$$\mathcal{A}_\mu^2(x) = 3 \, \left(\phi_{\mu\nu}^1(x) + \phi_{\mu\nu}^2(x) \right) .$$

Recovering the string tension

- The Wilson loop becomes

$$\mathcal{W}^{\text{Abel}}(\mathcal{C}) = \frac{1}{3}\left\langle \cos\left[\frac{1}{6}\sum_{x,\mu\in C}\left(\mathcal{A}_\mu^2(x) + \sqrt{3}\,\mathcal{A}_\mu^1(x)\right)\right] + \cos\left[\frac{1}{6}\sum_{x,\mu\in C}\left(\mathcal{A}_\mu^2(x) - \sqrt{3}\,\mathcal{A}_\mu^1(x)\right)\right] + \cos\left[\frac{1}{6}\sum_{x,\mu\in C}\mathcal{A}_\mu^2(x)\right]\right\rangle.$$

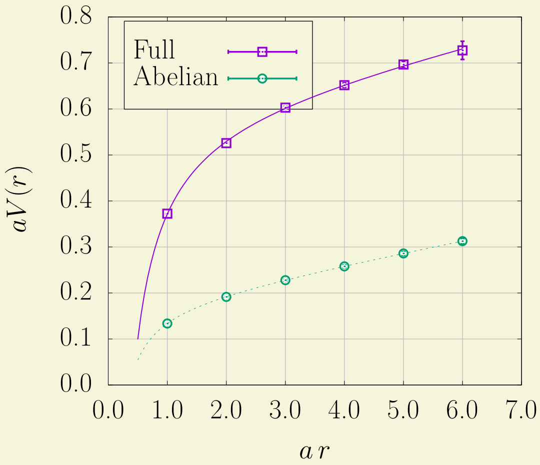
- We can compute the static-quark potential in this new projection and compare the string tension with those values of reference.

β	V	$\sqrt{\sigma^*}$	Measurements	$\sqrt{\sigma^{\text{Abel}}}$	$\sqrt{\sigma^{\text{Abel}}}/\sqrt{\sigma}$
5.70	8^4	0.396(2)	750	0.389(8)	**0.98(3)**
5.90	12^4	0.261(1)	100	0.2572(6)	**0.987(7)**
5.93	16^4	0.243(2)	590	0.2418(8)	**1.00(1)**
6.00	16^4	0.213(1)	300	0.212(4)	**1.00(5)**

The string tension is fully reproduced by the Cartan projection.

Recovering the string tension

- The string tension is obtained from a fit of the potential.
- Cartan projected data is more stable than that of the full theory.
- Cartan projected potential still shows a small Coulombic term.
- We show the potentials obtained from $\beta = 6.235$ on a $V = 26^4$ lattice.
- This points to a Cartan dominance of the infrared behaviour of the static-quark potential.



Interpretation

- How does $A_\mu(x) = \sum_{q=1}^{N_c-1} \mathcal{A}^p_\mu\, H_p + \sum_{\vec{\alpha}} \mathcal{A}^{\vec{\alpha}}_\mu\, E_{\vec{\alpha}}$ transform under $\Omega(x) = \exp[i\theta^q(x)\, H_q]$?

$$A'_\mu = \sum_{q=1}^{N_c-1} \mathcal{A}^p_\mu\, H_p + \sum_{\vec{\alpha}} e^{i\,\vec{\theta}\cdot\vec{\alpha}} \mathcal{A}^{\vec{\alpha}}_\mu\, E_{\vec{\alpha}}.$$

- $\mathcal{A}^q_\mu$ transform as neutral fields. $\mathcal{A}^{\vec{\alpha}}_\mu$ are charged with charge $\vec{\alpha}$.

$$\mathcal{L} = -\frac{1}{2}\sum_q \left[\mathcal{A}^q_\mu\, \partial^2\, \mathcal{A}^{\mu\, q} - \mathcal{A}^q_\mu \partial^\mu \partial^\nu \mathcal{A}^q_\nu\right] - \frac{1}{2}\sum_{p,q} \mathcal{A}^p_\mu\, \mathcal{A}^{\mu\, p} \left(\sum_{\vec{\alpha}} \alpha_p\, \alpha_q\, \mathcal{A}^{\vec{\alpha}}_\mu \mathcal{A}^{\mu\, \vec{\alpha}}\right) + \text{non-Cartan terms.}$$

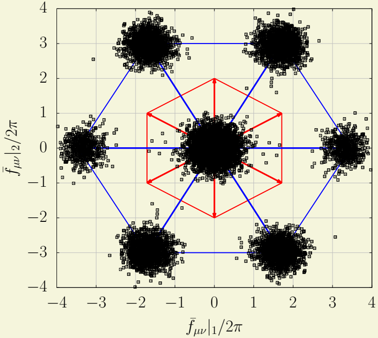
- In a gauge that minimizes the charged part, i.e. $\int d^4x \sum_{\text{off diagonal}} (A^a_\mu(x))^2$, there are no fluctuations in $\mathcal{A}^\alpha_\mu$.
- This is the continuum analogue of lattice MAG.
- This heuristic argument supports the Cartan projection.

Measuring the right monopole charges

- We can use this modified projection to define an Abelian field strength

$$f^q_{\mu\nu}(x) = \partial_\mu \mathcal{A}^q_\nu(x) - \partial_\nu \mathcal{A}^q_\mu(x).$$

- We can measure the field strength on the lattice.



- Reducing these points to the fundamental cell allows monopole currents detection.
- These have the magnetic charges taking values in the group roots.

β	V	ρ_{usual}	ρ_{cartan}	ratio
5.70	8^4	0.04069(8)	0.0609(3)	1.497(8)
5.90	12^4	0.0126(1)	0.0182(2)	1.44(2)
5.93	16^4	0.01064(2)	0.01535(6)	1.443(6)
6.00	16^4	0.00745(3)	0.01053(4)	1.413(8)

- For the details see Gustavo's talk on thursday.

Conclusion

- Usual Abelian projection scheme has a problem.
- We need to take into account the Cartan subalgebra structure.
- The projection onto the Cartan subalgebra returns two *independent fields.*
- The static-quark potential expressed in this projection reproduces the full $SU(3)$ string tension.
- We can define a scheme of detection of magnetic monopoles with charges taking values on the group roots.

These results point in the direction a Cartan dominance in SU(3) color confinement.

Tank you very much.