In-medium Finite Energy Sum Rules: advances and perspectives

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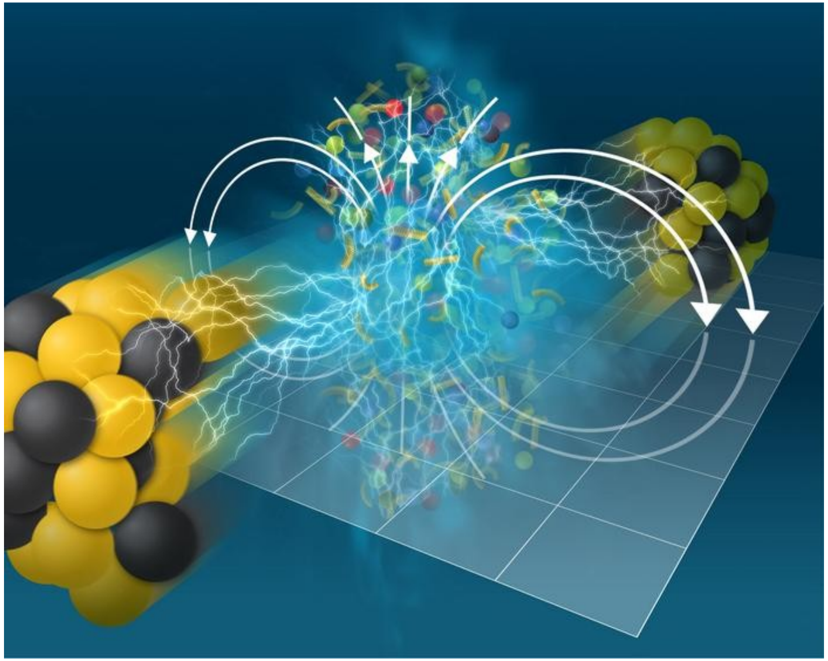
Motivation:

Extreme density,
Extreme temperature
Extreme magnetic field



$$\rho_B \text{ up to } \sim 10\rho_0 \quad \text{(compact stars)}$$

$$T \text{ up to } \sim 0.5 \text{ GeV} \sim 10^{12} \text{ K} \quad \text{(LHC)}$$

$$B \text{ up to } \sim 0.3 \text{ GeV}^2/e \sim 10^{19} \text{ G} \quad \text{(LHC)}$$

Some alternative model to obtain medium modification of parameters

Outline

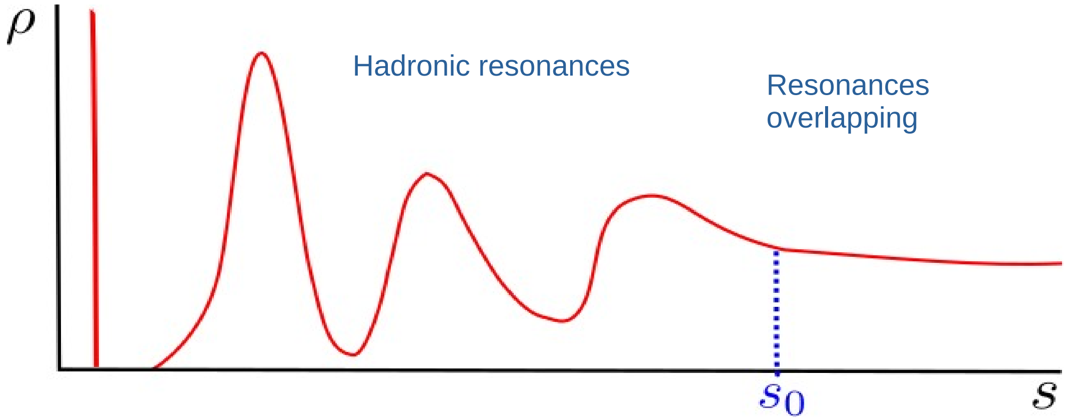
- Introduction to finite energy sum rules (FESR)
- In-medium sum rules
- Advantages and disadvantages
- some properties of FESR - magnetic field and chemical potential
- Summary and outlook

The spectral function

Two current correlator $$\Pi_{\mu\nu}(x-y) = i\langle 0|TJ_\mu(x)J_\nu^\dagger(y)|0\rangle$$

Fourier transformation $$\Pi_{\mu\nu}(q) = q_\mu q_\nu \Pi_L(q^2) + (g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi_T(q^2)$$

Spectral function $$\rho(s) = \frac{1}{\pi}\mathrm{Im}\Pi(s+i\epsilon)$$

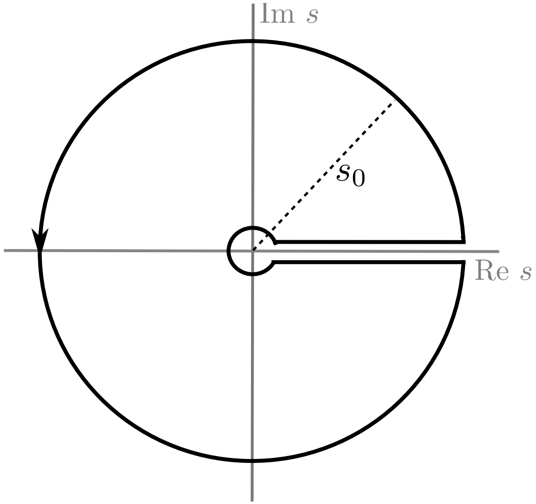


$s_0 \rightarrow$ Hadronic continuum threshold

FESR Quark-hadron duality $\Pi^{\text{Had}} \leftrightarrow \Pi^{\text{QCD}}$

Cauchy's theorem

$$\frac{1}{\pi}\int_0^{s_0} ds\, s^N\, \text{Im}\Pi^{\text{had}}(s+i\epsilon) = \frac{-1}{2\pi i}\oint_{|s|=s_0} ds\, s^N\, \Pi^{\text{QCD}}(s)$$



Operator Product Expansion (OPE)

Wilson coefficients

condensates

$$\Pi^{\text{QCD}}(x,y) = \Pi^{\text{pQCD}}(x-y) + \sum_{n>0} C_n(x-y)\langle\Omega| : \mathcal{O}_n(x+y) : |\Omega\rangle$$

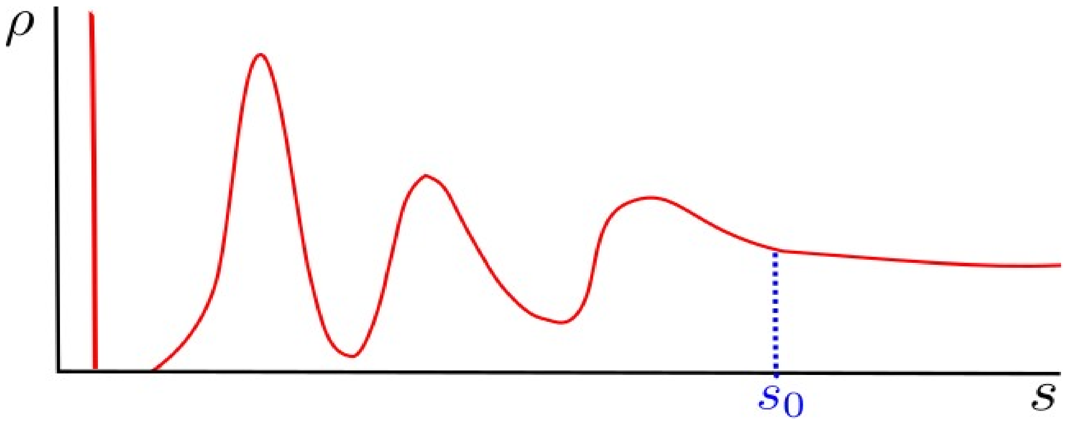
In-medium FESR

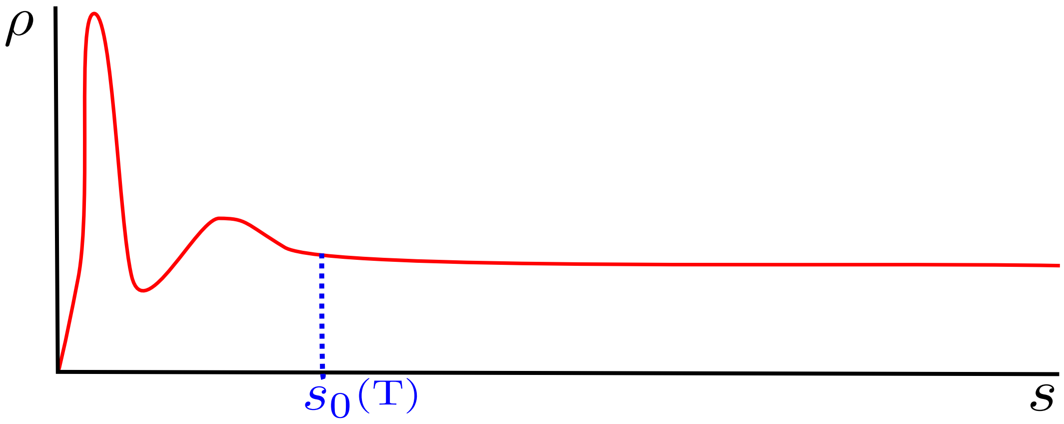
$$\Pi(p; T, B, \mu_q) = \Pi_{\text{pQCD}}(p, \mu; T, B, \mu_q) + \sum_{n>0} C_n(p,\mu)\,\langle : \mathcal{O}_n(T, B, \mu_q) :\rangle$$

The good of FESR

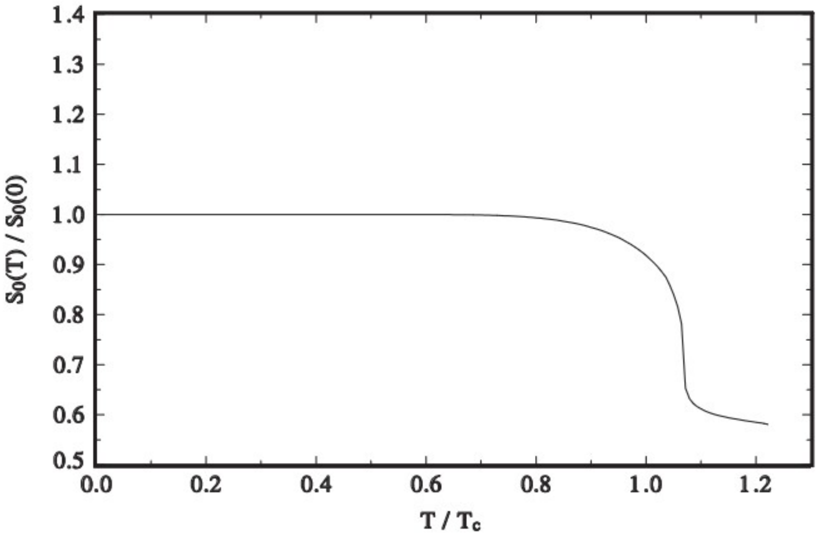
- Hadronic threshold acts as a deconfinement order parameter
- The kernel s^N cuts the OPE series (in-vacuum, no radiative corrections, low current masses)
- Contour integrals can be performed before loop integration $\rightarrow$skip full form factor calculation
- More equations $\rightarrow$ more parameters to find $\rightarrow$ more information
- No unphysical Borel mass parameter $\rightarrow$ no need of window of applicability

Hadronic threshold acts as a deconfinement order parameter

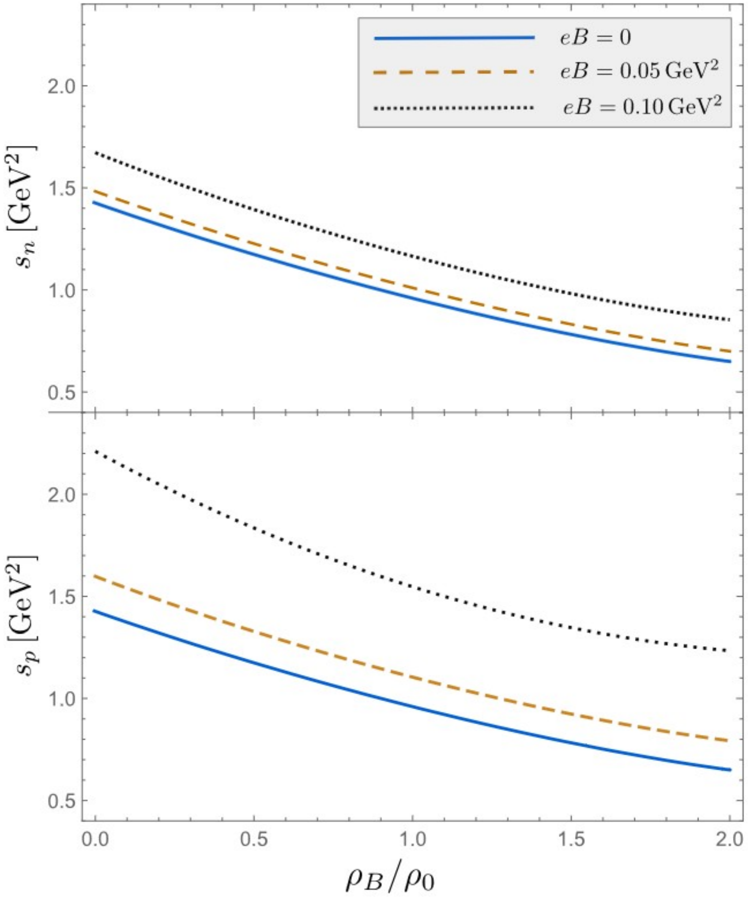




In-medium hadronic thresholds



Dominguez, Loewe, Rojas, Zhang
PRD **81**, 014007 (2010)



- hadronic threshold related with Polyakov loop

Carlomagno, Loewe,
PRD **95**, 036003 (2017)

Dominguez, Loewe, C.V., Zamora
PRD **108**, 074024 (2023)

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In momentum space, high *s* expansion

$$\Pi^{\mathrm{QCD}}(s) = c_0 \ln(-s) + \sum_{n>0} c_{2n} \frac{\langle \mathcal{O}_{2n} \rangle}{s^n} + \text{regular terms}$$

$$\frac{-1}{2\pi i} \oint_{s_0} ds s^N \, \Pi^{\mathrm{QCD}}(s) = \frac{c_0 \, s_0^{N+1}}{N+1} + c_{2(N+1)} \langle \mathcal{O}_{2(N+1)} \rangle$$

Integrated Wilson coefficient in general

$$\sim C \left(1 - \frac{4m^2}{s_0}\right) \frac{\langle \mathcal{O}_{2n} \rangle}{s_0^{n-N-1}}$$

Raya,Villavicencio,Eur. Phys. J. C **85**, 13 (2025)

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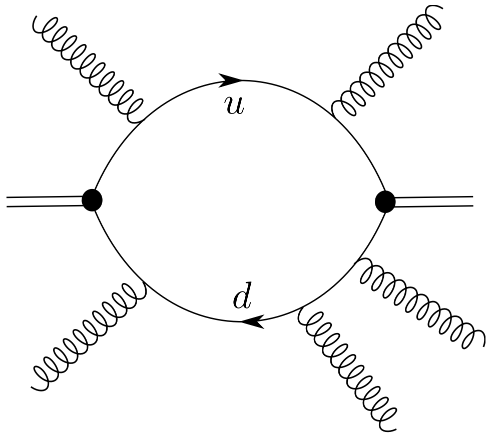
example: one-loop multi-gluon condensate using Feynman parameters



$$I_n^N(s_0) = \frac{-1}{2\pi i}\oint_{C(s_0)} ds\, s^N \int_0^1 dx\, \frac{f(x)}{[s - M^2(x)]^n}$$

$$M^2(x) \equiv \frac{m_u^2}{1-x} + \frac{m_d^2}{x}$$

$$I_{N+1}^N(s_0) = -\int_0^1 dx\, f(x)\, \theta\left(s_0 - M^2(x)\right) = -\int_{x_-}^{x_+} dx\, f(x)$$

$$x_\pm = \frac{1}{2}\left[1 + \frac{m_d^2}{s_0} - \frac{m_u^2}{s_0} \pm \sqrt{1 - 2\left(\frac{m_d^2}{s_0} + \frac{m_u^2}{s_0}\right) + \left(\frac{m_d^2}{s_0} - \frac{m_u^2}{s_0}\right)^2}\,\right].$$

Dominguez, Loewe, Villavicencio, Phys. Rev. D **98**, 034015 (2018)

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The bad of FESR

- Strong dependence on Hadronic continuum threshold → scale dependance
- Some care when Hadronic threshold diminishes too much → OPE “convergence”

The ugly of FESR

- Too many equations! → to extract all the info is an art

Example: charmonium using AA, VV, AP, channels

$$\mu = 3 \text{ GeV} \qquad \Rightarrow \qquad m_c = 1 \text{ GeV}$$

$$\sqrt{s_V} = 2.70 \text{ GeV}, \quad \sqrt{s_A} = 2.62 \text{ GeV}, \quad \sqrt{s_P} = 2.55 \text{ GeV}$$

$$\sqrt{s_0} > 3.1\text{GeV}$$

$$f_{J/\Psi} = 0.30 \text{ GeV}, \quad f_{\eta_c} = 0.25 \text{ GeV}, \quad \langle \frac{\alpha_s}{\pi} GG \rangle = 0.035 \text{ GeV}^4$$

$$\mu = m_c, \qquad \Rightarrow \qquad m_c = 1.27 \text{ GeV}$$

$$\sqrt{s_V} = 3.32 \text{ GeV}, \quad \sqrt{s_A} = 3.13 \text{ GeV}, \quad \sqrt{s_P} = 3.20 \text{ GeV}$$

$$f_{J/\Psi} = 0.28 \text{ GeV}, \quad f_{\eta_c} = 0.23 \text{ GeV}, \quad \langle \frac{\alpha_s}{\pi} GG \rangle = 0.022 \text{ GeV}^4$$

$$\mu = 1 \text{ GeV} \qquad \Rightarrow \qquad m_c = 1.42 \text{ GeV}$$

$$\sqrt{s_V} = 3.72 \text{ GeV}, \quad \sqrt{s_A} = 3.50 \text{ GeV}, \quad \sqrt{s_P} = 2.57 \text{ GeV}$$

$$\sqrt{s_0} \lesssim 3.6\text{GeV}$$

$$f_{J/\Psi} = 0.28 \text{ GeV}, \quad f_{\eta_c} = 0.23 \text{ GeV}, \quad \langle \frac{\alpha_s}{\pi} GG \rangle = 0.022 \text{ GeV}^4$$

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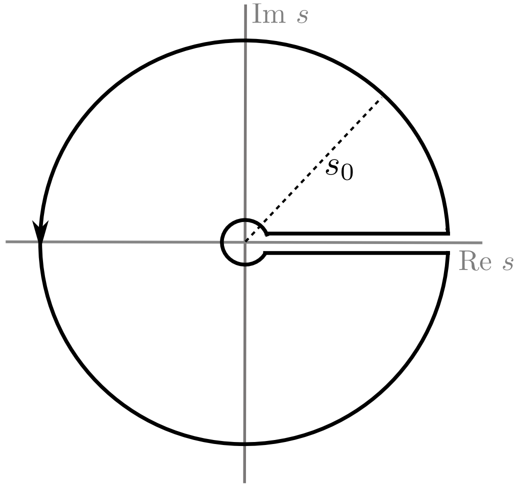
The "scattering term"

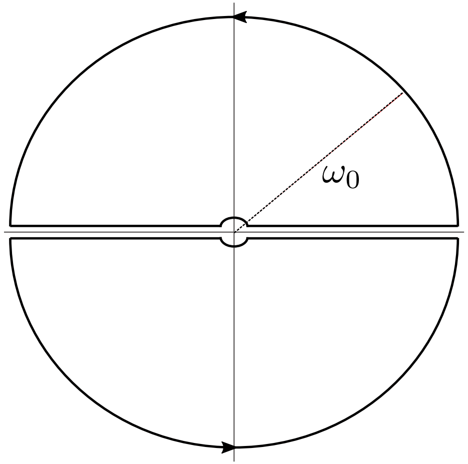
$$\Pi(\omega, \boldsymbol{p}^2) = \Pi^{\text{even}}(\omega^2, \boldsymbol{p}^2) + \omega \Pi^{\text{odd}}(\omega^2, \boldsymbol{p}^2) \qquad \omega^2 = s$$

$$\int_0^{s_0} \frac{ds}{\pi} s^{N-1} \, \text{Im}\Pi^{\text{had}}(s + i\epsilon, \boldsymbol{p}^2)$$
$$= -\oint_{|s|=s_0} \frac{ds}{2\pi i} s^{N-1} \, \Pi^{\text{QCD}}(s, \boldsymbol{p}^2) + \operatorname*{Res}_{s \to 0} \Pi^{\text{QCD}}(s, \boldsymbol{p}^2)$$



Alternative integration path $p_0 = \omega$ instead of $p^2 = s$

$$\int_{-\omega_0}^{\omega_0} \frac{d\omega}{\pi} \omega^{n+1} \text{Im}\,\Pi^{\text{had}}(\omega + i\epsilon, \boldsymbol{p}^2)$$
$$= -\oint_{\omega_0} \frac{d\omega}{2\pi i} \omega^{n+1} \Pi^{\text{QCD}}(\omega, \boldsymbol{p}^2) + \operatorname*{Res}_{\omega=0} \left[\omega^{n+1} \Pi^{\text{QCD}}(\omega, \boldsymbol{p}^2)\right]$$



The “scattering term”

Bochkarev, Shaposhnikov, Nucl. Phys. B **268**, 220–252 (1986).
Dominguez, Loewe, Phys. Lett. B **233**, 201–204 (1989).
Furnstahl, Hatsuda, Lee, Phys. Rev. D **42**, 1744–1756 (1990)

appears at finite temperature and/or chemical potential

careful with these limits:

$$m_u = m_d, \qquad \boldsymbol{p} = 0$$

scattering term can be obtained in rest frame keeping different masses

Raya,Villavicencio,Eur. Phys. J. C **85**, 13 (2025)

Operator mixing

$$\Pi(s) = \Pi_{\mathrm{pQCD}}(s,\mu) + \sum_{n>0} C_n(s) \langle : \mathcal{O}_n : \rangle$$

normal ordered condensates

Problems with the chiral limit: $\ln(-s/m_q^2)$ terms

$$\text{limits} \;\; m_u \to 0,\; m_d \to 0 \;\neq\; m_d \to 0,\; m_u \to 0 \;\neq\; m_u = m_d \to 0$$

Solution:

$$\Pi(s) = \tilde{\Pi}_{\mathrm{pQCD}}(s,\mu) + \sum_{n>0} \tilde{C}_n(s,\mu) \langle \mathcal{O}_n \rangle(\mu)$$

non-normal ordered condensates

$$\langle : \bar{q}\,\Gamma\, q : \rangle = \langle \bar{q}\,\Gamma\, q \rangle + \mathrm{tr} \int \frac{d^d k}{(2\pi)^d}\, \Gamma\, S_q(k,\mu;G)$$

Gluon background field

→ This replaces $\ln(-s/m_q^2) \to \ln(-s/\mu^2)$

In-medium operator mixing

$$\langle : \bar{q}\,\Gamma\, q : \rangle = \langle \bar{q}\,\Gamma\, q \rangle + \mathrm{tr} \int \frac{d^d k}{(2\pi)^d}\, \Gamma\, S_q(k, \mu; G, T, B, \mu_q)$$

At finite magnetic field: removes log terms

$$eB^2 \ln(-s/m_q^2) \quad \rightarrow \quad eB^2 \ln(-s/\mu^2)$$

Gubler, Hattori, Lee, Oka, Ozaki, Suzuki, Phys. Rev. D **93**, 054026 (2016).

At finite chemical potential:

- Silver Blaze behavior in pion channels
- Predicted condensate values will be incorrect if not considered

Raya,Villavicencio,Eur. Phys. J. C **85**, 13 (2025)

At finite temperature: probably must be considered

Some properties of in-medium FESR

Finite magnetic field

QCD sector → B-expansion $\left(\frac{eB}{s_0(B)}\right)^n$ must be treated as the gluon background

eB for light quarks can reach values $\gtrsim 0.1\ \mathrm{GeV}^2$

Hadronic threshold increases with *B*

Hadron sector → Sum of Landau levels with Euler-Maclaurin formula

Some properties of in-medium FESR

Finite chemical potential → phase transition in pion channels

$\mu_q > \mu_c = \sqrt{s_A}/2$ In vacuum $\mu_c \sim 0.4\ \text{GeV}$

$f_\pi^2 = \frac{\mu^2}{2\pi^2}(1+a)$ $v_\pi^2 = \frac{\frac{2}{5} - a}{1+a}$ $\langle \bar{q}q \rangle \sim m$ $m_\pi^2 \sim m^2$ $a = \frac{4\pi^2}{9} \frac{\langle \theta_g \rangle}{s_A\, \mu^2}$

Raya, Villavicencio, Eur. Phys. J. C **85**, 13 (2025)

Apparently describes Color-Flavor-Locked phase → $SU(3)_{c+L+R}$ models for high μ

$f_\pi \sim 0.2\mu$ $v_\pi^2 \sim \frac{1}{3}$ $\langle \bar{q}q \rangle \sim m$ $m_\pi^2 \sim m m_s$

Some results

Axial-Axial, Axial-Pseudoscalar, Pseudoscalar Pseudoscalar channels

nucleon – nucleon correlator

masses, decay constantsm, hadronic thresholds, condensates of pion at finite *B*

nucleon – Axial current - nucleon correlator

nucleon axial coupling constant at binite *B* and ρ_B

Summary

- All the in-medium and vacuum parameters obtained are consistent with other approaches
- Contour integration before loop integration → skips full correlator calculation
  → low mass expansion not necessary
- Scattering term and in-medium operator mixed **must** be incorporated
- Magnetic field expansion can reach high values → considered as operator expanded
- For $\mu > \mu_c$ the model (apparently) describes $SU(3)_{c+L+R}$ models

Outlook

A lot of incoming projects

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