Unitarization of the Proton-Proton Scattering Amplitude via a Non-Perturbative Two-Gluon Exchange Model at TeV Energies[1]

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• One of the main approaches to studying the hadronic cross section at high energies is based on Regge theory;

• In a simple scenario, the scattering amplitude is governed by a pole at $j = \alpha_{\mathbb{P}}(t)$, leading to:

$$\mathcal{A}(s,t) \propto s^{\alpha_{\mathbb{P}}(t)} \tag{1}$$

where $\alpha_{\mathbb{P}}(t)$ denotes the Pomeron trajectory;

- Pomeron treated as a colorless state carrying vacuum quantum numbers

• In the 1970s, Low and Nussinov independently proposed interpreting the Pomeron as a two-gluon exchange (2GE);

• The corresponding amplitude, later formalized through perturbation theory, is expressed as:

$$\mathcal{A}(s,t) = is\frac{8}{9}n_p^2\alpha_s^2\left[T_1 - T_2\right] \tag{2}$$

where T_1 and T_2 represent the contributions in which the gluons couple to the same quark or to different quarks, respectively;

• This model exhibits singularities at $-t = 0$, arising from the pole of the gluon propagator at $q^2 = 0$;

- Landshoff and Nachtmann (LN) proposed modifying the gluon propagator in the infrared region so that it remains finite at $q^2 = 0$;

- Within this framework, the Pomeron behaves as a $C = +1$ photon–like exchange, with amplitude:

$$i\beta_0^2(\bar{u}\gamma_\mu u)(\bar{u}\gamma^\mu u) \tag{3}$$

where β_0 characterizes the strength of the Pomeron–quark coupling and is given by:

$$\beta_0^2 = \frac{1}{36\pi^2}\int d^2k\,\left[g^2 D(k^2)\right]^2 \tag{4}$$

- The expression above converges only if the gluon propagator is non-perturbative;

- For a more accurate description of high-energy experimental data, it is essential to reggeize the amplitude;

- Consider a particle of mass M and spin j exchanged in the t-channel of a scattering process. If the amplitude behaves as $\mathcal{A}(s,t) \propto s^{\alpha(t)}$, the exchanged particle is said to be reggeized.

- Phenomenologically, this reggeization can be implemented through the replacement:

$$s \;\rightarrow\; s^{\alpha(t)}$$

- In our analysis, the amplitude in the LN model takes the form:

$$\mathcal{A}(s,t) = i\, s^{\alpha_{\mathbb{P}}(t)} \frac{1}{\tilde{s}_0} \frac{8}{9} n_p^2 \left[\tilde{T}_1 - \tilde{T}_2\right] \tag{5}$$

The quantities $\tilde{T}_1$ and $\tilde{T}_2$ are given by:

$$\tilde{T}_1 = \int_0^s d^2k\,\bar{\alpha}\left(\tfrac{q}{2}+k\right) D\left(\tfrac{q}{2}+k\right) \bar{\alpha}\left(\tfrac{q}{2}-k\right) D\left(\tfrac{q}{2}-k\right) \left[G_p(q,0)\right]^2 \tag{6}$$

$$\tilde{T}_2 = \int_0^s d^2k\,\bar{\alpha}\left(\tfrac{q}{2}+k\right) D\left(\tfrac{q}{2}+k\right) \bar{\alpha}\left(\tfrac{q}{2}-k\right) D\left(\tfrac{q}{2}-k\right) G_p\left(q,k-\tfrac{q}{2}\right) \left[2G_p(q,0) - G_p\left(q,k-\tfrac{q}{2}\right)\right] \tag{7}$$

- In the amplitude, $\alpha_{\mathbb{P}}(t) = 1+\epsilon+\alpha'_{\mathbb{P}}t$ represents the Pomeron trajectory in the LN model, and $\tilde{s}_0 \equiv s_0^{\alpha_{\mathbb{P}}(t)-1}$. The function $G_p(q,k)$ corresponds to the convolution of the proton wave functions:

$$G_p(q,k) = \int d^2p\,d\alpha\,\psi^*(\alpha,p)\,\psi(\alpha,p-k-\alpha q) \tag{8}$$

where $\psi(\alpha,p)$ denotes the amplitude for a quark to carry transverse momentum p and longitudinal fraction α;

- By setting $k = 0$ in the previous expression, the result reduces to the well-known proton elastic form factor $F_1(q^2)$;

- The terms $\tilde{T}_1$ and $\tilde{T}_2$ encodes the non-perturbative QCD contributions to the amplitude;

- The total cross section $\sigma_{\text{tot}}(s)$ and the differential cross section $d\sigma/dt$ are given by:

$$\sigma_{\text{tot}}(s) = \frac{\operatorname{Im}\mathcal{A}(s, t = 0)}{s} \tag{9}$$

$$\frac{d\sigma}{dt}(s, t) = \frac{|\mathcal{A}(s, t)|^2}{16\pi s^2} \tag{10}$$

- The non-perturbative dynamics of QCD induces an effective mass for the gluon $m(q^2)$;

- The Schwinger–Dyson equations indicate that a finite propagator is associated with the dynamical generation of mass for gluons;

- Lattice QCD studies indicate that:
  - The momentum dependence of the gluon mass appears in both $SU(2)$ and $SU(3)$;
  - The gluon propagator remains finite in the infrared region;

- Intrinsically related to the dynamical gluon mass is the QCD effective charge $\bar{\alpha}(q^2)$;

- The effective charge is expressed as:

$$\frac{1}{\bar{\alpha}(q^2)} = b_0 \ln\left(\frac{q^2 + m^2(q^2)}{\Lambda^2}\right) \qquad (11)$$

where $b_0 = \beta_0/4\pi = (33 - 2n_f)/12\pi$ is the first coefficient of the QCD β-function and Λ a QCD dimensional parameter;

- The propagator $D_{\mu\nu} = -ig_{\mu\nu}D(q^2)$ has a scaling factor given by:

$$D^{-1}(q^2) = \left[q^2 + m^2(q^2)\right] bg^2 \ln\left[\frac{q^2 + 4m^2(q^2)}{\Lambda^2}\right] \qquad (12)$$

- Functional forms for $m(q^2)$ and for the non-perturbative propagator $D_{\mu\nu}$ were obtained by Cornwall via the Pinch technique;

- In Euclidean space, the dynamical mass is given by:

$$m^2(q^2) = m_g^2 \left[\frac{\ln\left(\frac{q^2+4m_g^2}{\Lambda^2}\right)}{\ln\left(\frac{4m_g^2}{\Lambda^2}\right)} \right]^{-12/11} \quad (13)$$

where $b = b_0/4\pi$ and $m_g^2 = m^2(0)$;

- Through non-linear Schwinger–Dyson equations, we can obtain a more general version of the previous equation:

$$m_{\log}^2(q^2) = m_g^2 \left[\frac{\ln\left(\frac{q^2+\rho m_g^2}{\Lambda^2}\right)}{\ln\left(\frac{\rho m_g^2}{\Lambda^2}\right)} \right]^{-1-\gamma_1} \quad (14)$$

where $\gamma_1 = -6(1 + c_2 - c_1)/5$, with c_1 and c_2 being parameters related to the gluon self-energy;

- The parameters ρ and m_g control the behavior of the dynamical mass in the infrared region;

- A second possibility for the dynamical mass in asymptotic regions is given by a power law:

$$m_{pl}^2(q^2) = \frac{m_g^4}{q^2 + m_g^2} \left[\frac{\ln\left(\frac{q^2+\rho m_g^2}{\Lambda^2}\right)}{\ln\left(\frac{\rho m_g^2}{\Lambda^2}\right)} \right]^{\gamma_2 - 1} \qquad (15)$$

where $\gamma_2 = (4 + 6c_1)/5$;

- The values $\rho = 4$, $\gamma_1 = 0.084$ and $\gamma_2 = 2.36$ were fixed, since these minimize χ^2/ν, where ν is the number of degrees of freedom;

- The effective charge can be rewritten in terms of the dynamical mass functions $m_i(q^2)$ as:

$$\bar{\alpha}_i(q^2) = \frac{1}{b_0 \ln\left(\frac{q^2+4m_i^2(q^2)}{\Lambda^2}\right)}, \qquad i = \log,\ \mathrm{pl} \tag{16}$$

- Using the previous results, we obtain expressions that ensure the convergence of the integrals $\tilde{T}_1$ and $\tilde{T}_2$:

$$\frac{1}{\bar{\alpha}_i(q^2)\, D(q^2)} = b_0 \left[q^2 + m_i^2(q^2)\right] \ln\left(\frac{q^2 + 4m_i^2(q^2)}{\Lambda^2}\right) \tag{17}$$

where in the propagator expression we take $g^2 = 4\pi\, \bar{\alpha}_i(q^2)$;

- In this model, a good description of the differential and total cross sections requires:
  - A reggeized amplitude;
  - An alternative implementation of the convolution of the proton wave function;

- As shown previously, for $k^2 = 0$, the wave function reduces to:

$$G_p(q,0) = F_1(q^2) = \exp\left[-\left(\sum_{n=1}^{N_a} a_n |t|^n\right)\right] \qquad (18)$$

where $-t = q^2$;

- Using χ^2 minimization, we find that $N_a = 2$ provides a good description of $d\sigma/dt$ for both the ATLAS and TOTEM datasets;

- Following the philosophy of maintaining the smallest possible number of free parameters, we adopt the expansion with $N_a = 2$;

- In total, the model contains four free parameters:

$$m_g, \quad \epsilon, \quad a_1, \quad a_2$$

- Whenever possible, we set $n_f = 3$ and $\Lambda = 284$ MeV for the following reasons:
  - These values are consistent with those used in calculations of strong-interaction processes;
  - Our primary objective is to investigate the behavior of the dynamical gluon mass;

- As presented previously:

$$\beta_0^2 = \frac{1}{36\pi^2} \int d^2k \left[g^2 D(k^2)\right]^2$$

- Since the phenomenological values of the dynamical gluon mass are known, we can use them to compute the corresponding quantities[2];

$$\beta_{0,\,\mathrm{log},\,\mathrm{ATLAS}} = 2.33^{+0.39}_{-0.30}\,\mathrm{GeV}^{-1} \qquad \beta_{0,\,\mathrm{pl},\,\mathrm{ATLAS}} = 2.13^{+0.33}_{-0.25}\,\mathrm{GeV}^{-1}$$

$$\beta_{0,\,\mathrm{log},\,\mathrm{TOTEM}} = 2.04^{+0.28}_{-0.22}\,\mathrm{GeV}^{-1} \qquad \beta_{0,\,\mathrm{pl},\,\mathrm{TOTEM}} = 1.91^{+0.22}_{-0.19}\,\mathrm{GeV}^{-1}$$

[2] G. B. Bopsin, E. G. S. Luna, A. A. Natale, and M. Peláez, Phys. Rev. D **107** (2023) 114011

- We aim to evaluate the amplitude:

$$\mathcal{A}(s,t) = i\, s^{\alpha_{\mathbb{P}}(t)} \frac{1}{\tilde{s}_0} \frac{8}{9} n_p^2 \left[\tilde{T}_1 - \tilde{T}_2\right]$$

where:

$$\tilde{T}_1 = \int_0^s d^2k\, \bar{\alpha}\left(\tfrac{q}{2}+k\right) D\left(\tfrac{q}{2}+k\right) \bar{\alpha}\left(\tfrac{q}{2}-k\right) D\left(\tfrac{q}{2}-k\right) \left[G_p(q,0)\right]^2$$

$$\tilde{T}_2 = \int_0^s d^2k\, \bar{\alpha}\left(\tfrac{q}{2}+k\right) D\left(\tfrac{q}{2}+k\right) \bar{\alpha}\left(\tfrac{q}{2}-k\right) D\left(\tfrac{q}{2}-k\right) G_p\left(q, k-\tfrac{q}{2}\right) \left[2G_p(q,0) - G_p\left(q, k-\tfrac{q}{2}\right)\right]$$

- Here, $\bar{\alpha}$ and D denote the effective charge and propagator scaling factor, respectively, both depending on q^2 and on the chosen dynamical mass model.

- The inverse of the product of the effective charge and propagator is:

$$\frac{1}{\bar{\alpha}_i(q^2)D(q^2)} = b_0\left[q^2 + m_i^2(q^2)\right]\ln\left(\frac{q^2 + 4m_i^2(q^2)}{\Lambda^2}\right)$$

- For each *ensemble*, we consider both dynamical gluon mass model:

$$m_{\log}^2(q^2) = m_g^2\left[\frac{\ln\left(\frac{q^2+\rho m_g^2}{\Lambda^2}\right)}{\ln\left(\frac{\rho m_g^2}{\Lambda^2}\right)}\right]^{-1-\gamma_1} \qquad m_{\rm pl}^2(q^2) = \frac{m_g^4}{q^2+m_g^2}\left[\frac{\ln\left(\frac{q^2+\rho m_g^2}{\Lambda^2}\right)}{\ln\left(\frac{\rho m_g^2}{\Lambda^2}\right)}\right]^{\gamma_2-1}$$

- To apply eikonal unitarization, we first express the scattering amplitude $H_{pp}(s,b)$ in terms of the Born amplitude $\chi_{pp}(s,b)$;

- The physical scattering amplitude in momentum space is obtained through the inverse Fourier-Bessel transform of $H_{pp}(s,b)$:

$$A_{pp}(s,t) = s\int_0^\infty b\,db\,J_0(bq)\,H_{pp}(s,b) \tag{19}$$

- The explicit form of $H[\chi(s,b)]$ depends on the chosen unitarization scheme and follows from the solution of the unitarity equation:

$$\mathrm{Im}H(s,b) = \frac{1 \pm \sqrt{1-(1+\rho^2)\,G_{\mathrm{in}}(s,b)}}{1+\rho^2} \tag{20}$$

- The eikonal unitarization corresponds to choosing the solution with the negative square root in the unitarity equation;

- This leads to the eikonal expression:

$$H(s,b) = i\left[1 - e^{\,i\chi(s,b)}\right] \tag{21}$$

- Substituting we get the eikonal scattering amplitude:

$$A_{\text{eik}}(s,t) = is\int_0^\infty b\,db\,J_0(bq)\,\left[1 - e^{\,i\chi(s,b)}\right] \tag{22}$$

- Numerical evaluation of the integrals is performed with `scipy` (`python`) using non-adaptive methods (`fixed_quad`);
  - Adaptive methods produced significant numerical instabilities;
- First, we reproduce the fits of $\sigma_{\rm tot}$ and $d\sigma/dt$ reported in the reference paper as a validation of our code and methodology;
- Minimization of free parameters is implemented using `iminuit`;
  - Other libraries such as `PYROOT` did not satisfy the programming requirements;
- Future steps:
  - Numerical eikonalization of the amplitude;

- Fit performed using the least-squares method with `iminuit`;

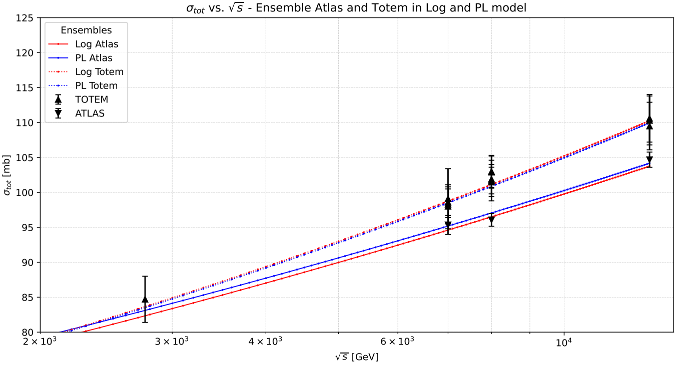

Figura: Total cross section minimized via `iminuit`.

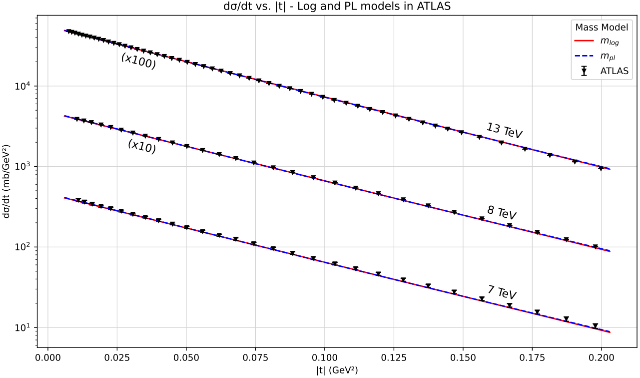

Figura: Differential cross section (ATLAS) minimized via `iminuit`.

Reproduction of Results — $d\sigma/dt$ (TOTEM)

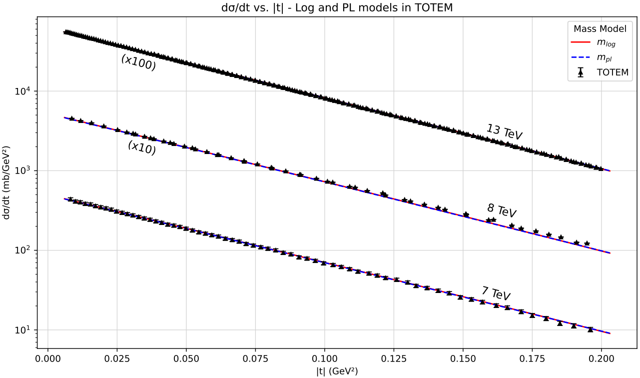

Figura: Differential cross section (TOTEM) minimized via `iminuit`.

- Developments already implemented:
  - Reproduction of the amplitude at Born level;
  - Adaptation of the code for numerical eikonalization;
  - Construction of the required equations and integrals;
    - Integrals currently evaluated via mid-point Riemann sum;

- Work to be carried out:
  - Fit of $d\sigma/dt$ in the eikonal model using data at $\sqrt{s} = 7$, 8, and 13 TeV for ATLAS and TOTEM;
    - Using χ^2 minimization;
  - Determination of the free parameters and computation of the eikonalized $\sigma_{\text{tot}}(s)$ via `iminuit`;

Thank you!