Hyperasymptotic Approximation to the mass of the lightest gluelump

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work in collaboration with
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[JHEP10(2025)176]

WONPAQCD, 01 - 05 December 2025

Outline

Motivation

- QCD perturbation theory for many observables produces a formal series

$$S_{\text{pert}}(\alpha_X) = \sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}$$

which is asymptotic, with large-order behaviour

$$p_n^{(X)} \sim n! \left(\frac{\beta_0}{2\pi d}\right)^n \left(1 + \mathcal{O}(1/n)\right).$$

- Divergence is tied to renormalon singularities in the Borel plane.
- The inverse Borel transform from the approximate Borel sum is ill defined. Here the associated error is not quantified.
- To define nonperturbative (NP) quantities (masses, condensates, gluelumps) consistently, the perturbative series must be handled with exponential accuracy.

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  - Allows us to have a control on the power corrections.
  - Add a NP power correction in a systematic way. Combined expansion of perturbation series and NP terms (Hyperasymptotic expansion [Proc.Roy.Soc.London A,430(1990)]).

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  - Add a NP power correction in a systematic way. Combined expansion of perturbation series and NP terms (Hyperasymptotic expansion [Proc.Roy.Soc.London A,430(1990)]).
  - The mixing between perturbative and NP effects may hinder estimating the real size of NP effects
- Hyperasymptotic expansions provide a systematic framework to deal with these issues.

OPE as the Organizing Principle

Consider a dimensionless observable with a hard scale $Q \gg \Lambda_{\rm QCD}$:

$$\text{Observable}\left(\frac{Q}{\Lambda_{\rm QCD}}\right) = S_{\rm pert}(\alpha_X(Q)) + \sum_d C_{O,d}(\alpha_X(Q))\,\frac{\langle O_d\rangle}{Q^d}\,,$$

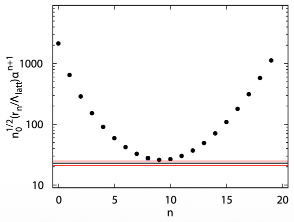
$$S_{\rm pert}(\alpha_X(Q)) = \sum_{n=0}^{\infty} p_n^{(X)}\,\alpha_X^{n+1}(Q).$$

- O_d: local or non-local operators, $\langle O_d\rangle \sim \Lambda_{\rm QCD}^d$ (up to anomalous dimensions).
- The same dynamics generating the NP power corrections $\sim (\Lambda_{\rm QCD}/Q)^d$ produces the renormalon structure of $S_{\rm pert}$.

Superasymptotics

What is the optimal truncation order?

$$\sum_{n=0}^{N} r_n \alpha^{n+1}$$



- For factorially divergent series, convergence, plateau, divergence
- Truncate in the plateau: Minimize $|r_n \alpha^{n+1}|$
- Superasymptotics[1]

$$N_{\text{optimal}} \sim \frac{\#}{\alpha}$$

- No fixed order. Exponentially suppressed ambiguity $\sim \alpha^{1/2} e^{\frac{-\#}{\alpha}}$
- Hyperasymptotic analysis disentangles the truncated perturbative sum from the NP piece with power accuracy.

[1] M. V. Berry et al. Proc. R. Soc. A 430, 653 (1990)

Borel Transform and PV Prescription

Given $S_{\text{pert}}(\alpha) = \sum_{n=0}^{\infty} p_n \, \alpha^{n+1}$, its Borel transform is

$$B[O](t) = \sum_{n=0}^{\infty} \frac{p_n}{n!} \, t^n .$$

The (formal) Borel sum is $O_{\text{Borel}}(\alpha) = \int_0^\infty dt \, e^{-t/\alpha} \, B[O](t)$.

- Renormalons $\Rightarrow$ singularities at $t_d = \frac{2\pi d}{\beta_0}$. These singularities are determined by the OPE (up to the normalization) and linked to the asymptotic behavior of perturbation theory
- For $d > 0$ these lie on the integration contour: ambiguity in O_{Borel}.
- Singularities in the real axis! $\rightarrow$ Principal Value (PV) prescription:

$$O_{\text{PV}}(\alpha) = \text{PV} \int_0^\infty dt \, e^{-t/\alpha} \, B[O](t).$$

which is Scale and Scheme independent (Ayala, Llobregat, Pineda; Takaura)

Divergent Series and Large-Order Behaviour

For an IR renormalon associated to dimension $d > 0$ one expects

$$p_n^{(as)} = Z_X^O \left(\frac{\mu}{Q}\right)^d \frac{\Gamma(n+b)}{\Gamma(b)} \left(\frac{\beta_0}{2\pi d}\right)^n \left(1+\frac{c_1}{n+b}+\frac{c_2}{(n+b)(n+b-1)}+\dots\right)$$

with b related to anomalous dimensions.

- This behaviour implies S_{pert} has zero radius of convergence.
- Nevertheless, it is highly informative: the same parameters Z_X^O, d, b, c_i enter the NP sector.

Hyperasymptotic Expansion: General Structure

$S_{\rm PV}$ will be computed truncating the hyperasymptotic expansion in a systematic way. This means truncating as follows:

$$S_{\rm PV}(Q) = S_P(Q;\mu) + \Omega(\mu) + \sum_{n=N_P+1}^{N'_P} \left(p_n - p_n^{(as)}\right) \alpha_X^{n+1}(\mu) + \Omega'(\mu) + \cdots ,$$

where S_P is the superasymptotic sum,

$$S_P \equiv \sum_{n=0}^{N_P(|d_{\rm min}|)} p_n \alpha^{n+1}(\mu) \equiv S_{|d|=0}$$

where N_P is chosen around the minimal term:

$$N_P \simeq \frac{2\pi|d|}{\beta_0 \alpha(\mu)} (1 - c\,\alpha(\mu)), \qquad c = \mathcal{O}(1).$$

Hyperasymptotic Expansion: General Structure

- The truncation error is of order

$$\delta S \sim \sqrt{\alpha(\mu)}\, \exp\left[-\frac{2\pi|d|}{\beta_0 \alpha(\mu)}\right],$$

  which is of NP size (power-suppressed in Q).
- This is the superasymptotic approximation.

For an IR renormalon at $d > 0$ one can write schematically Ω_d, that are the terminants completing the contribution of the renormalon at $u = \frac{\beta_0 t}{4\pi} = d/2$,

$$\Omega_d(\mu; Q) = Z_X^O \sum_{j=0}^{\infty} c_j \, \Delta\Omega(b-j),$$

$$\Delta\Omega(b-j) = \frac{1}{\Gamma(b-j)} \int_0^\infty dt\, t^{b-j-1} e^{-t} \left(1 + t\, \frac{\beta_0 \alpha_X(\mu)}{2\pi d}\right)^{-1},$$

with b related to anomalous dimensions, and $c_0 = 1$, $c_1 = s_1$, $c_2 = \frac{1}{2}\frac{b}{b-1}(s_1^2 - 2s_2)$, ... are determined purely by the β-function coefficients.

Infrared (IR) Renormalons

- Located at $u = d/2 > 0$ in the Borel plane.
- Associated with NP power corrections $\sim (\Lambda_{\mathrm{QCD}}/Q)^d$.
- For $d > 0$ one finds (as a series of $\alpha(\mu)$)

$$\Omega_{d>0} \sim \sqrt{\alpha(\mu)} \left(\frac{\mu}{Q}\right)^d \left(\frac{\beta_0 \alpha(\mu)}{4\pi}\right)^{-b} \exp\left[-\frac{2\pi d}{\beta_0 \alpha(\mu)}\right] (1 + \mathcal{O}(\alpha(\mu))).$$

- Needed to correctly define NP parameters (e.g. pole mass, gluelump mass) in the PV scheme.

Ultraviolet (UV) Renormalons

For a UV renormalon at $u = d/2 < 0$ we have asymptotically

$$p_n^{(X)} \xrightarrow[n\to\infty]{} Z_X^O \left(\frac{\mu}{Q}\right)^d \frac{\Gamma(n+b_0+1)}{\Gamma(b_0+1)} \left(\frac{\beta_0}{2\pi d}\right)^n \left(1+\dots\right),$$

and the associated terminant behaves as

$$\Omega_{d<0} \sim (-1)^{N_P+1}\sqrt{\alpha(\mu)}\left(\frac{Q}{\mu}\right)^{|d|}\left(\frac{\beta_0\alpha(\mu)}{4\pi}\right)^{-b_0}\exp\left[-\frac{2\pi|d|}{\beta_0\alpha(\mu)}\right].$$

- UV renormalons do not correspond to genuine NP operators but control alternating large-order behaviour.
- For $\mu \sim Q$ and $Q \gg \Lambda_{\mathrm{QCD}}$, they are highly suppressed but can be systematically included.

IR vs. UV Renormalons: Summary

IR Renormalons ($d > 0$)

- $u = d/2 > 0$.
- Related to NP power corrections $\sim (\Lambda_{\text{QCD}}/Q)^d$.
- Dominant ambiguity in many QCD observables.
- Must be treated to define NP constants (e.g. Λ_B^{PV}).

UV Renormalons ($d < 0$)

- $u = d/2 < 0$.
- Control oscillatory large-order tail of perturbation theory.
- No direct NP operator counterpart.
- Hyperasymptotics allows their systematics when needed.

Renormalization Group and Λ_X

The RG-invariant scale in scheme X can be written as

$$\Lambda_X = \mu \exp\left[-\frac{2\pi}{\beta_0 \alpha_X(\mu)}\right] \left(\frac{\beta_0 \alpha_X(\mu)}{2\pi}\right)^{-b} \left[1 + \sum_{j\geq 1} s_j^{(X)} \left(\frac{\beta_0 \alpha_X(\mu)}{2\pi}\right)^j\right],$$

$$b = \frac{\beta_1}{2\beta_0^2}.$$

- This expression governs the scaling of NP terms and of terminants with $\alpha_X(\mu)$.
- Provides the bridge between perturbative $\alpha_X(Q)$ and the physical scale Λ_{QCD}.

Lattice Scheme and Asymptotics

- In the lattice scheme, one typically uses $\alpha_L(a)$ defined at the lattice spacing a.
- In practice, $\alpha_L(a)$ is smaller than $\alpha_{\overline{\text{MS}}}(1/a)$ at the same scale.
- Asymptotic behaviour of the series (renormalon dominance) often sets in at relatively low orders (e.g. $n \sim 6 - 7$).
- This makes lattice observables (static energies, gluelumps, plaquette) ideal laboratories to test hyperasymptotic ideas.

Gluelumps and EFT Picture

- Gluelump: static adjoint colour source attached to gluonic excitations such that the full state is a colour singlet.
- In the case of bound states mass of heavy gluinos within the EFT:

$$M_{H,\tilde{g}} = m_{\tilde{g}}^{\mathrm{PV}} + \Lambda_H^{\mathrm{PV}} + \mathcal{O}(1/m_{\tilde{g}}^{\mathrm{PV}}).$$

- In the case of B meson mass in HQEFT:

$$M_B = m_{\mathrm{PV}} + \overline{\Lambda}_{\mathrm{PV}} + \mathcal{O}(1/m_{\mathrm{PV}}).$$

- On the lattice (Wilson action), the gluelump energy of a static adjoint source attached to glue yield

$$\Lambda_H^L(a) = \delta m_A^{L,\mathrm{PV}}(a) + \Lambda_H^{\mathrm{PV}} + \mathcal{O}(a^2).$$

Here $\delta m_A^{L,\mathrm{PV}}$ is the adjoint static self-energy in the lattice scheme, defined using the PV Borel prescription.

Hyperasymptotic Form of $\delta m_A^{L,\mathrm{PV}}$

The hyperasymptotic approximation reads schematically

$$\delta m_A^{L,\mathrm{PV}}(a) = \underbrace{\frac{1}{a}\sum_{n=0}^{N_P} c_{A,n}\,\alpha_L^{n+1}(a)}_{\delta m_A^{(P)}(1/a)} + \frac{1}{a}\,\Omega_A(1/a;a)$$

$$+\frac{1}{a}\sum_{n=N_P+1}^{N'}\left[c_{A,n}-c_{A,n}^{(\mathrm{as})}\right]\alpha_L^{n+1}(a)+\mathcal{O}(a^2).$$

- N_P chosen by the minimal term criterion; N' defines a "window" where asymptotic subtractions are applied.
- Ω_A is the terminant associated with the leading IR renormalon of the adjoint self energy.
- Lattice data for $\Lambda_H^L(a)$ can then be fitted to extract $\Lambda_H^{\mathrm{PV}} \equiv \Lambda_B^{\mathrm{PV}}$.

Heavy Quarkonium Hybrids

Heavy Quarkonium Hybrids: Unique place to study the behavior of QCD dynamics under the influence of a static octet colour source.

Simplified setup compared with glueballs.

The energy of a static quark and a static antiquark in a colour singlet configuration admits an OPE using pNRQCD (Pineda, Brambilla):

$$E_s(r) = 2m_{\rm PV} + V_s^{\rm PV}(r;\nu_{us}) + \delta E_{s,us}^{\rm PV}(r;\nu_{us})\,. \tag{2}$$

The energy of a static quark and a static antiquark in a colour octet configuration follows a similar pattern,

$$E_H(r) = 2m_{\rm PV} + V_o^{\rm PV}(r;\nu_{us}) + \delta E_{o,us}^{\rm PV}(r;\nu_{us})\,. \tag{3}$$

Heavy Quarkonium Hybrids

If we consider lattice analyses, the following "observables" show up:

$$E^L_{\Sigma_g^+}(r;a) = V_s^L(r;a) + \mathcal{O}(r^2)\,, \tag{4}$$

$$E^L_H(r;a) = V_o^L(r;a) + \Lambda_H^L + \mathcal{O}(r^2)\,. \tag{5}$$

In this work, rather than considering each static energy independently, we consider the following combination:

$$E_{\Pi_u} - E_{\Sigma_g^+} = V_A^{\rm PV} + \Lambda_H^{\rm PV} + \delta E_{A,us}^{(2){\rm PV}}\,. \tag{6}$$

We can fully work in the $\overline{\rm MS}$ scheme, where we have used the following definitions: $V_A^{\rm PV} = V_o^{\rm PV} - V_s^{\rm PV}$ and $\delta E_{A,us}^{(2){\rm PV}} = \delta E_{o,us}^{(2){\rm PV}} - \delta E_{s,us}^{\rm PV}$.

Lattice

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Static potentials ($\mathcal{O}(\alpha^4)$)

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Renormalon Normalizations

- The large-order behaviour of the relevant perturbative series depends on the renormalon normalizations and the updated analyses gives:

$$Z_m = -\tfrac{1}{2} Z_{V_s} = \{0.604(17),\, 0.551(20)\} \quad (n_f = 0, 3),$$
$$Z_{V_o} = \{0.136(8),\, 0.121(13)\}, \qquad Z_A = \{-1.343(36),\, -1.224(43)\}.$$

- These enter directly in the asymptotic templates $c_{A,n}^{(as)}$ and in the prefactor of Ω_A.

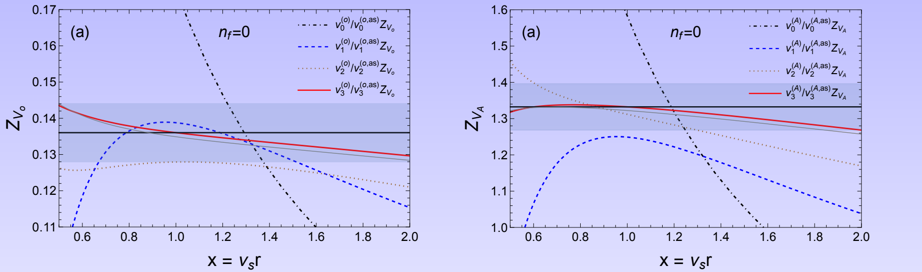

Figure: *Determination of Z_{V_o} and Z_{V_A} with $n_f = 0$ using $v_n^{(o,a)}/v_n^{(as)} Z_V$ as a function of $x = \nu_s r$ and for different values of n in the $\overline{MS}$ scheme. The gray continuous line is $v_3^{(o)}/v_3^{(o,as)} Z_{V_o}$ without the ultrasoft logarithmically related term. The black horizontal line is our final prediction and the blue band our final error estimate.*

Λ_H from PV scheme

Nothing fundamentally wrong but renormalization scale/scheme dependent $\Rightarrow$
Alternative: PV summation prescription + Hyperasymptotics:

- Independent of scale and scheme of the strong coupling constant
- Controlled approximation to the exact result

Determination of $\Lambda_B^{\rm PV}$ from the lattice gluelump energy (lattice scheme)

$$\Lambda_B^{\rm PV} = \Lambda_B^L(a) - \delta m_A^{(P)}(1/a) - \frac{1}{a}\Omega_A(1/a;a) - \sum_{N_P+1}^{N'=3N_P} \frac{1}{a}[c_{A,n} - c_{A,n}^{(\rm as)}]\alpha_L^{n+1}(a) + \mathcal{O}(a^2)\,.$$

Determination of $\Lambda_B^{\rm PV}$ from the static hybrid energy ($\overline{\rm MS}$ scheme)

$$\Lambda_B^{\rm PV} = (E_{\Pi_u}(r) - E_{\Sigma_g^+}(r)) - V_{A,P}(r) - \frac{1}{r}\Omega_{V_A} - \delta V_A^{\rm RG}(r) \quad (7)$$

$$- \sum_{n=N_P+1}^{3N_P/N_{max}} (V_n^{(A)} - V_n^{(A,\rm as)})\alpha_{\overline{\rm MS}}^{n+1}(\nu_s) - \delta E_{A,us}^{(2)\rm PV}(r;\nu_{us}) + o(r^2) \quad (8)$$

Final Result for the Lightest Gluelump

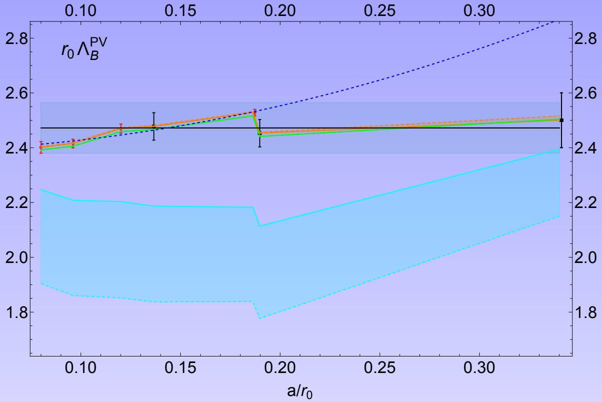

Figure: *We have truncated at different orders in the hyperasymptotic expansion:* $\Lambda_B^L(a) - \delta m_A^{(P)}(1/a)$ *(cyan band),* $\Lambda_B^L(a) - \delta m_A^{(P)}(1/a) - \frac{1}{a}\Omega_A$ *(orange band),* $\Lambda_B^L(a) - \delta m_A^{(P)}(1/a) - \frac{1}{a}\Omega_A - \sum_{N_P+1}^{13} \frac{1}{a}[c_{A,n} - c_{A,n}^{(\mathrm{as})}]\alpha_L^{n+1}$ *(green line). The dashed blue line is a fit to* $\Lambda_B^L(a) - \delta m_A^{(P)}(1/a) = \Lambda_B + Ka^2$.

Final Result for the Lightest Gluelump

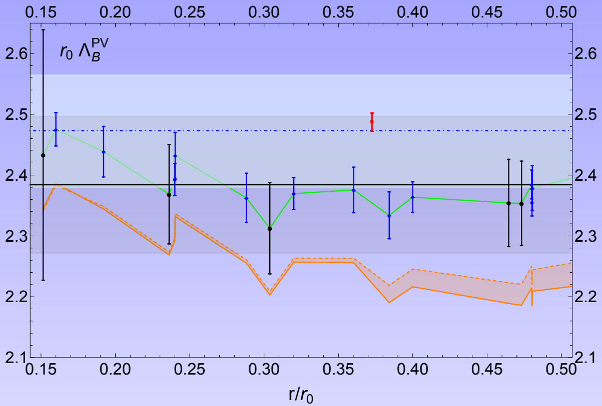

Figure: *We have truncated at different orders in the hyperasymptotic expansion:* $E^L_{\Pi_u}(r) - E^L_{\Sigma_g^+}(r) - V^{(P)}_{A,P}(r) - \frac{1}{r}\Omega_{V_A}$ *(orange band),* $E^L_{\Pi_u}(r) - E^L_{\Sigma_g^+}(r) - V^{(P)}_{A,P}(r) - \frac{1}{r}\Omega_{V_A} - \delta V_{A,\mathrm{RG}}(r) + \sum_{N_P+1}^{N'=N_{max}} \frac{1}{a}[c_{A,n} - c^{(\mathrm{as})}_{A,n}]\alpha^{n+1}_{\overline{MS}} - \delta E^{(2)\mathrm{PV}}_{o,us}(r;\nu_{us})$ *(green line).*

Final Result for the Lightest Gluelump and Conclusions

For $n_f = 0$ (quenched), we have obtained two independent hyperasymptotic determinations (lattice gluelump energy and static hybrid energy)

$$\Lambda_B^{\rm PV} = 2.47(9)\, r_0^{-1}\,, \qquad \Lambda_B^{\rm PV} = 2.38(11)\, r_0^{-1}\,,$$

Final Result: combined in quadrature

$$\Lambda_B^{\rm PV} = 2.44(7)\, r_0^{-1}, \qquad r_0^{-1} \approx 400 \text{ MeV}.$$

This is a renormalization-group invariant and renormalization-scale independent determination in the PV summation scheme.

Final Result for the Lightest Gluelump and Conclusions

We have devised an hyperasymptotic expansion applicable to QCD observables. We use the PV prescription of the Borel integral (scheme/scale independence). Analytic control of the error.

- Smooth connection with perturbation theory
- Parametric control of the error
- We get good agreement for the ground state hybrid potential up to relatively long distances $\rightarrow$ spectrum and properties of some heavy quarkonium hybrid states.
- The values we have obtained of the gluelump masses can be directly put in first principle computations of the hybrid spectrum when solving the Schroedinger equations.

Error Budget and Systematics

- **Perturbative:**
  - Variation of N_P and of the window N'.
  - Truncation of the expansion of Ω_A in α_L.
- **Renormalon inputs:**
  - Uncertainties in $Z_m, Z_{V_s}, Z_{V_o}, Z_A$ and β-function coefficients.
- **Lattice:**
  - Discretization effects $\mathcal{O}(a^2)$.
  - Finite-volume effects and continuum extrapolation.
- **Scheme/scale:**
  - Choices of renormalization scale μ around $\mu \sim 1/a$ or $\mu \sim 1/r$.

Additional References

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