UNIVERSITY OF HELSINKI

6th Workshop on Nonperturbative Aspects of QCD

thermodynamics of deconfined matter from QCD inequalities and the lattice

Pablo Navarrete

University of Helsinki, Finland

with: T. Gorda, R. Paatelainen, L. Sandbote, K. Seppänen

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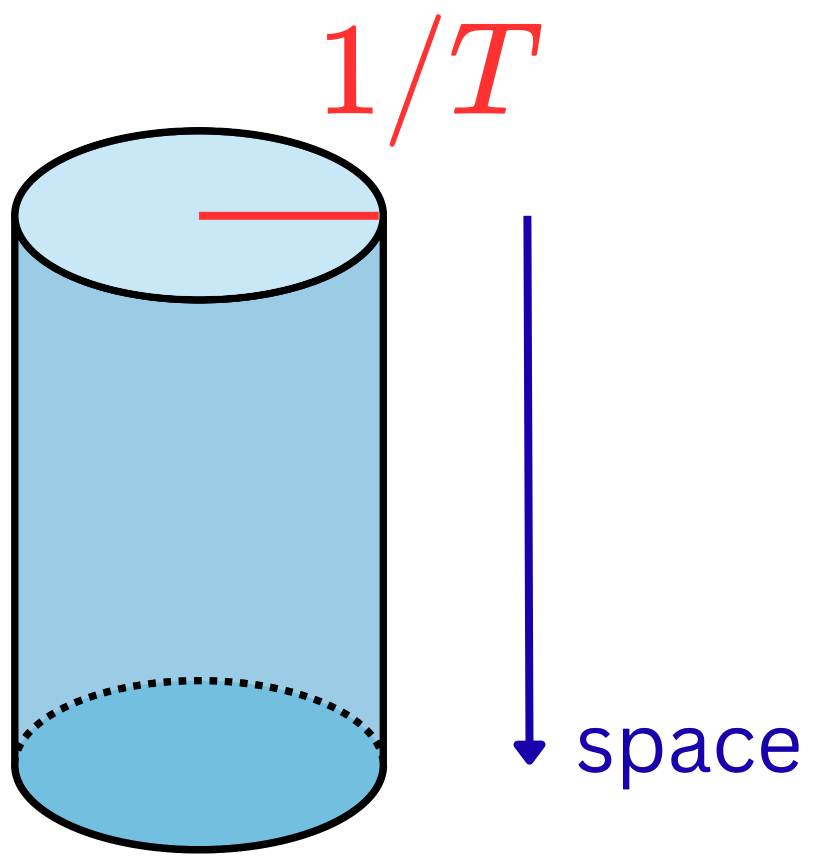
01 December 2025

thermal QCD

put system at finite temperature and density

→ in equilibrium: euclidean time $t = i\tau$; extent $1/T$

→ quark of flavor j carries chemical potential μ_j



[e.g. LeBellac, Laine/Vuorinen, etc.]

thermal QCD

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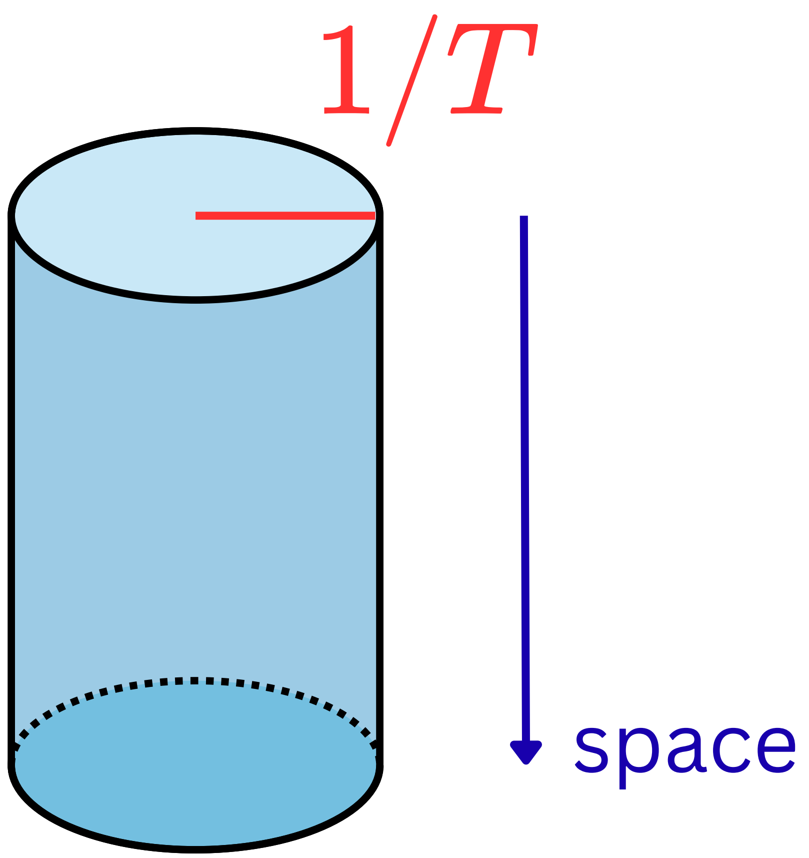


- euclidean path integral:

[e.g. LeBellac, Laine/Vuorinen, etc.]

$$Z(T, \mu_{\rm j}) = \int \mathcal{D}A\, e^{-S[A]} \prod_{j=1}^{N_f} \det[\mathcal{D}(\mu_j)]$$

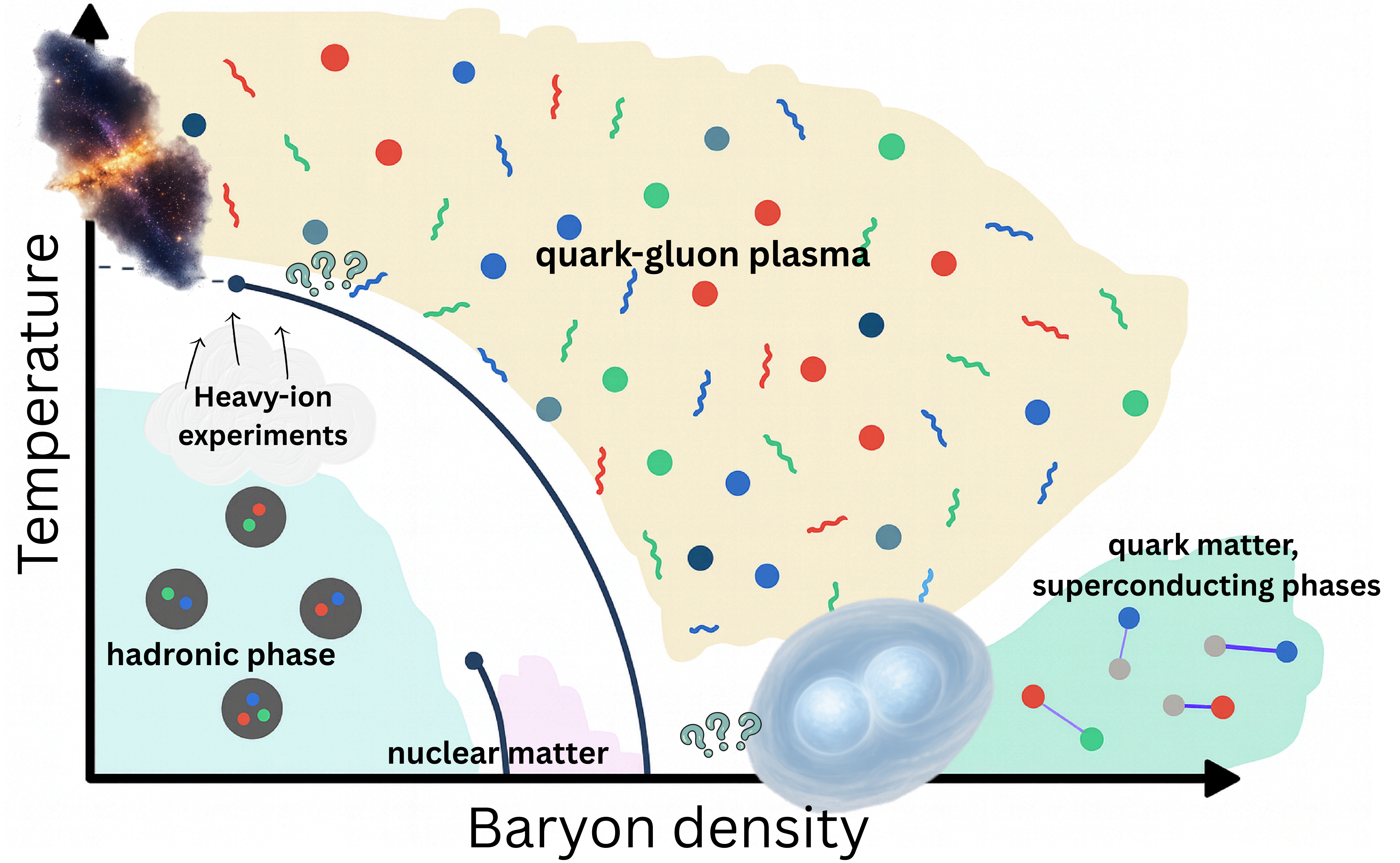
- gluon action

$$S[A] = \int_0^{1/T} d\tau \int_V \frac{1}{2} \mathrm{tr}\, F_{\mu\nu} F_{\mu\nu}$$

- Dirac operator

$$\mathcal{D}(\mu_j) = \not{D} + m_j + \mu_j \gamma^0$$

QCD phase diagram



QCD thermodynamics

thermodynamics from $p(T, \mu_j) \sim \log Z(T, \mu_j)$

- equation of state
- entropy density
- etc.

recall $Z = \mathrm{tr}\, e^{-\frac{1}{T}\left(\hat{H} - \mu \hat{N}\right)}$

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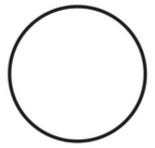
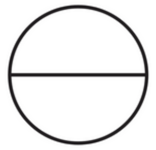
- perturbation theory: $p =$  $+$  $+ \cdots$

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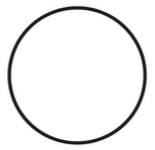
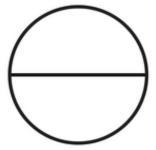
- perturbation theory: $p =$  $+$  $+ \cdots$

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$$n_j = \frac{\langle \hat{N}_j \rangle}{V} = \frac{\partial p}{\partial \mu_j}, \qquad \hat{N}_j \equiv \int_V \bar{\psi}_j \gamma^0 \psi_j$$

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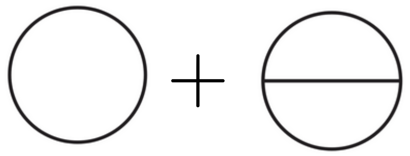
- perturbation theory: $p =$  $+ \cdots$

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fermion Sign Problem:

$$\gamma^5 \mathcal{D}(\mu_j)\gamma^5 = \mathcal{D}^\dagger(-\mu_j) \quad \Longrightarrow \quad \det \mathcal{D}(\mu_j) \in \mathbb{C}$$

"γ^5 − hermiticity"

complex Boltzmann weight

~~Monte Carlo importance sampling~~

phase-quenched QCD

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(assume $\theta_{\rm QCD} = 0$)

- positive measure: $e^{-S[A]} \prod_{j=1}^{N_f} |\det[\mathcal{D}(\mu_j)]| \geq 0$

no Sign Problem!

$$\longrightarrow \quad Z_{\rm PQ}(T, \mu_j) \geq Z(T, \mu_j)$$

[Cohen, **PRL** ‘03]

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- QCD pressure bounded from above

$$\boxed{p_{\rm PQ}(T, \mu_j) \geq p(T, \mu_j)}$$

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$$|\det \mathcal{D}(\mu_j)| = \sqrt{\det \mathcal{D}(\mu_j)\ \det \mathcal{D}(-\mu_j)}$$

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[lattice: Abbott et al., **PRL** ‘24]

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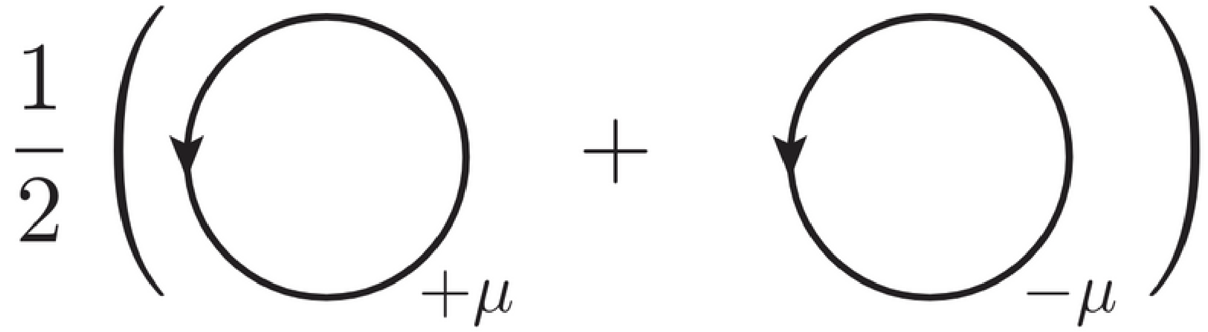
perturbation theory: $\alpha_s(T, \mu_j) \ll 1$

$$\alpha_s \equiv \frac{g^2}{4\pi}$$

$$\sqrt{\det \mathcal{D}(\mu_j)} = \exp\left\{\frac{1}{2}\text{tr}\,\log \mathcal{D}(\mu_j)\right\}$$

- PQ Feynman rule



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$$\Delta p \equiv p_{\mathrm{PQ}} - p$$

- we know $\Delta p \geq 0$

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$\text{quark with } (+\mu) = \text{anti-quark with } (-\mu)$

- perturbatively, $\text{tr}\left(T^a T^b T^c\right)_{\text{fund.}} \neq \text{tr}\left(T^a T^b T^c\right)_{\text{anti-fund.}}$

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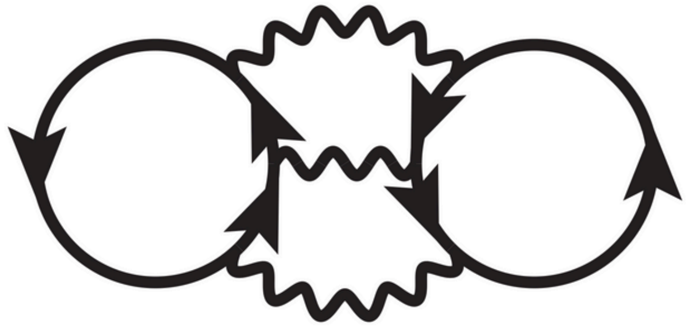
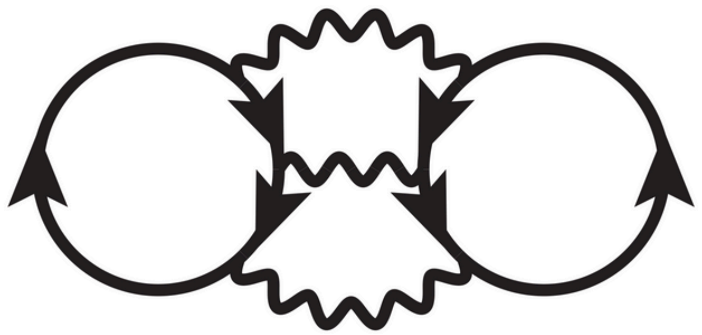
[Moore and Gorda, **JHEP** ‘23]

$(T=0)$

$\Delta p =$  $+$  $+ \; O(\alpha_s^4 \log \alpha_s)$

<u>fully finite</u> at $T=0$

[**PN**, Paatelainen, Seppänen, **PRD** ‘24]

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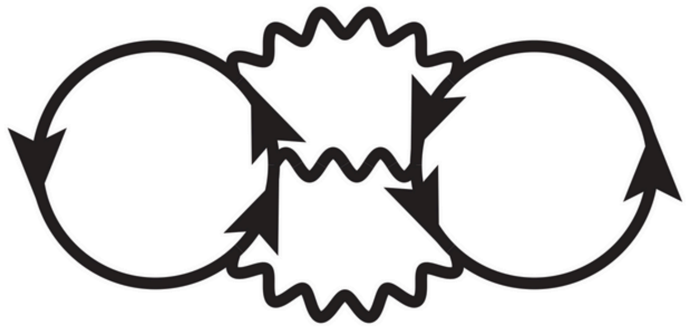
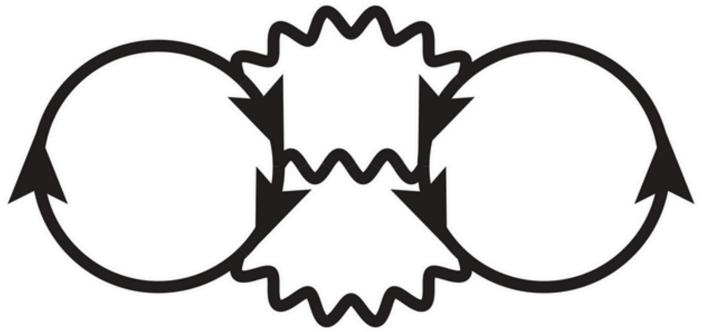
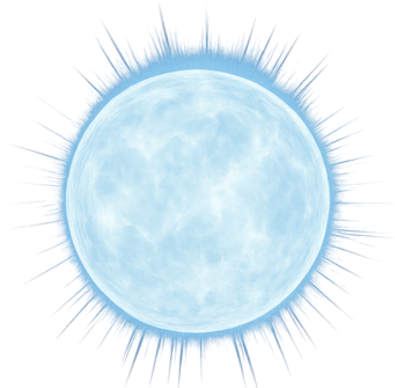
$\Delta p =$  $+$  $+ \; O(\alpha_s^4 \log \alpha_s)$

caveat: potentially large quark-pairing contributions $\Delta p_{\text{pairing}} \sim \mu^2 \Delta_{\mathrm{PQ}}^2, \quad \Delta_{\mathrm{PQ}} \sim g^{-5} e^{-\frac{1}{\sqrt{2}} c/g}$

[Fujimoto, **PRD** '23]

evaluating Δp



setup: u, d and s quarks in beta equilibrium at finite T

$$\mu_u = \mu_d = \mu_s, \qquad \mu_{\mathrm{B}} = 3\mu_u$$

$$d \leftrightarrow u + e^- + \bar{\nu}_e$$
$$s \leftrightarrow u + e^- + \bar{\nu}_e$$
$$\bullet\ \bullet\ \bullet$$

$$T_c^{\mathrm{PQ}} \sim 40 - 110\ \mathrm{MeV}$$

$(\mu_{\mathrm{B}} \sim 2.5 - 3.0\ \mathrm{GeV})$

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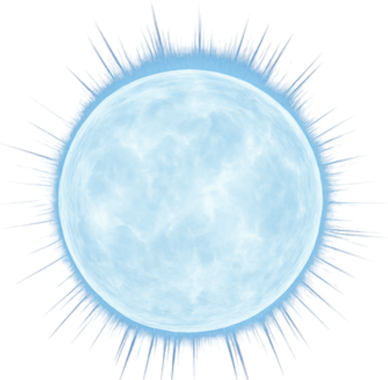

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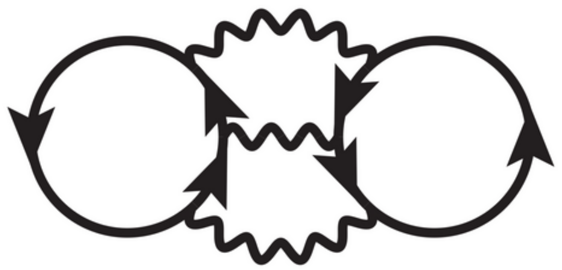
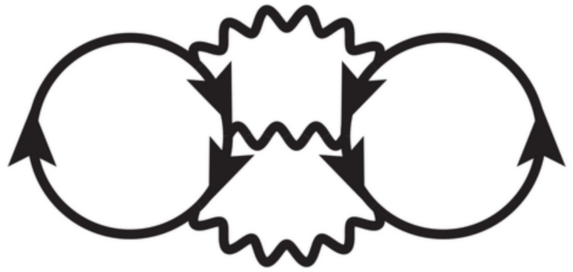
$\Delta p =$  $+$  $+$ ~~$\Delta p_{\rm pairing}$~~

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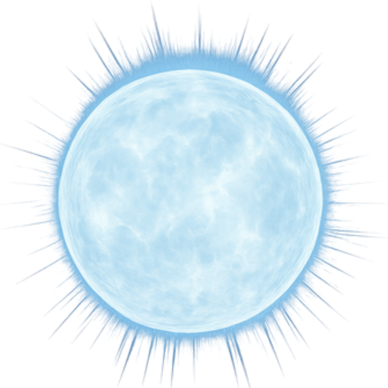

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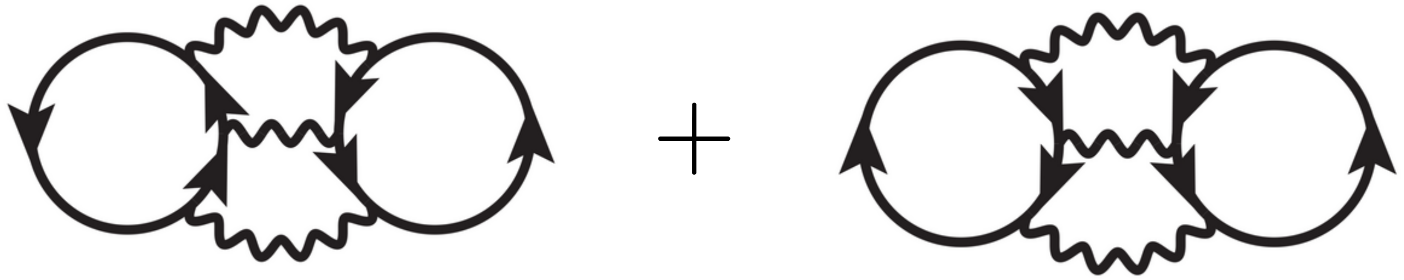
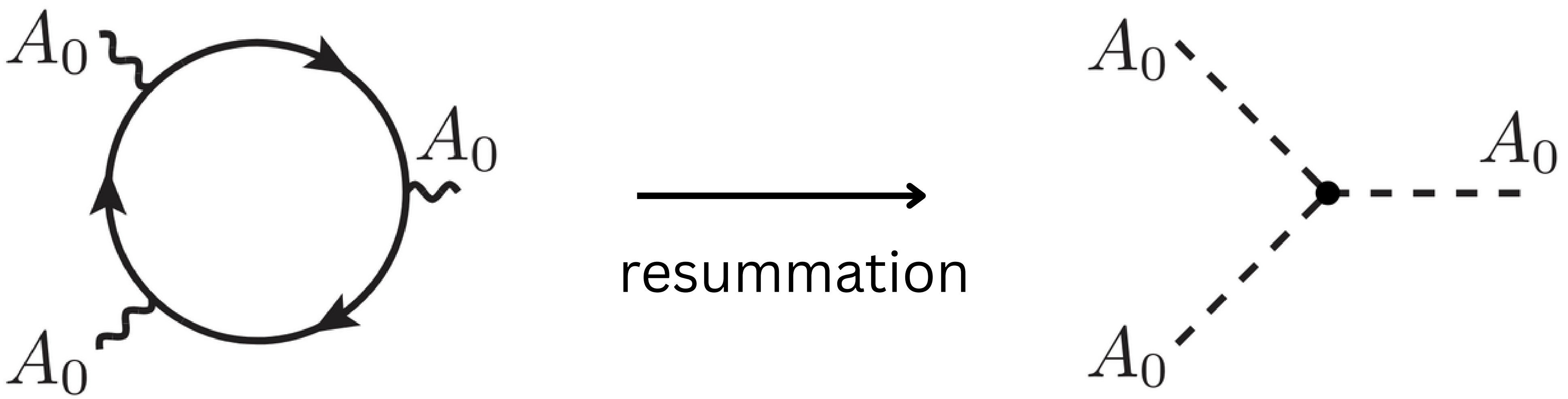
$\Delta p =$ 

- static low-momentum gluons at $T > 0$ source infrared divs.

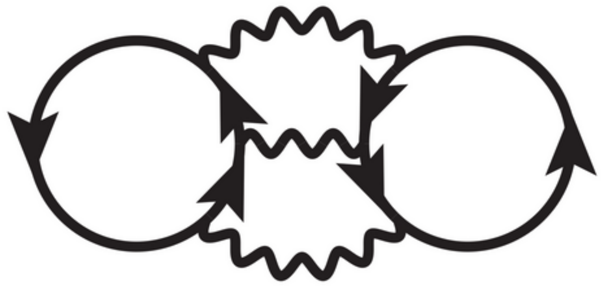
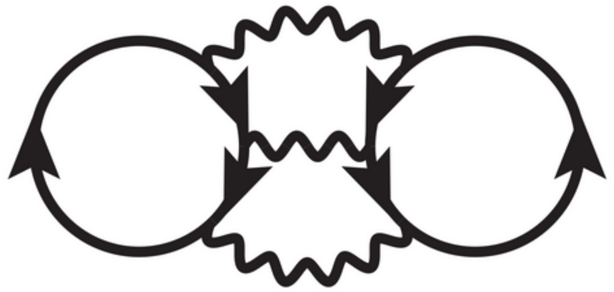
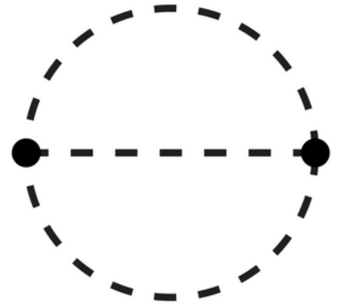


"electrostatic QCD"

$$\sim \; i\bar{\gamma}\, \mathrm{tr}\, (A_0)^3$$

[Hart et al. '03]

evaluating Δp

$\Delta p =$  $+$  $+$ 

$$= \alpha_s^3 (c_1 + c_2 \log \alpha_s)$$

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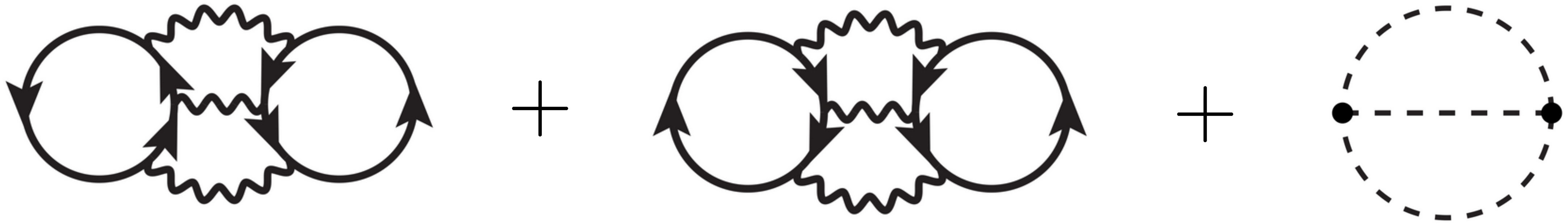
$\Delta p =$ 

$$= \alpha_s^3 (c_1 + c_2 \log \alpha_s)$$

[Catani et al.; **JHEP** '08]

[Capatti et al.; **PRL** '19]

- evaluated with <u>thermal Loop Tree Duality</u>:

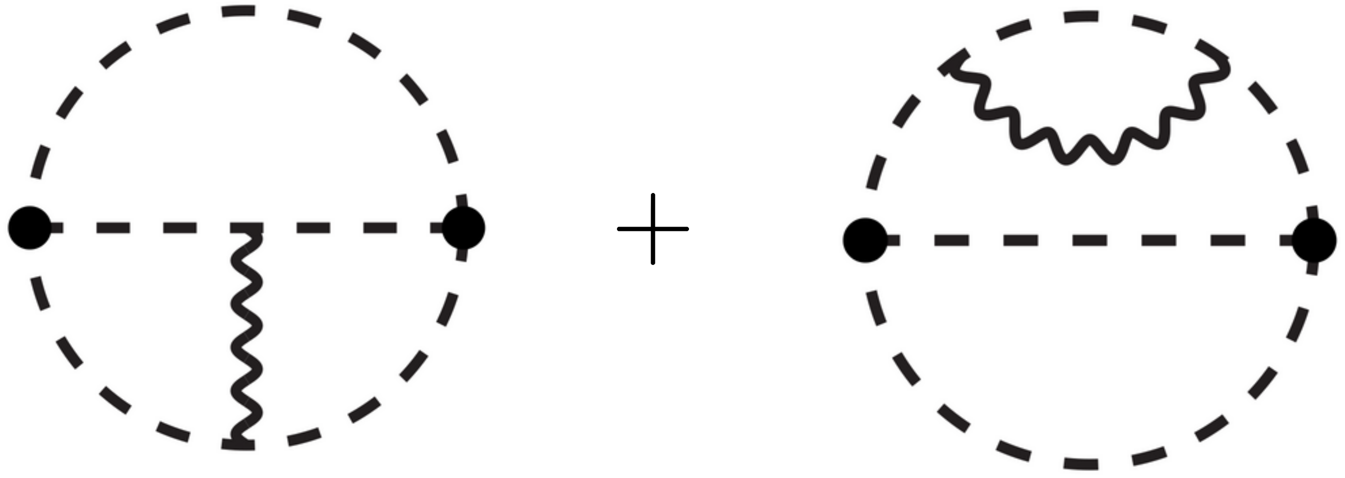
numerical Monte Carlo framework

[**PN**, Paatelainen, Seppänen; **PRD** '24]

[Kärkkäinen, **PN**, Nurmela, Paatelainen, Seppänen, Vuorinen; **PRL** '25]

evaluating Δp

$$\Delta p\Big|_{O(\alpha_s^{7/2})} = \quad + $$



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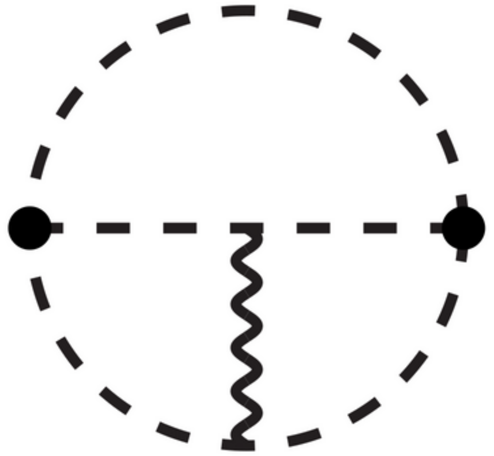
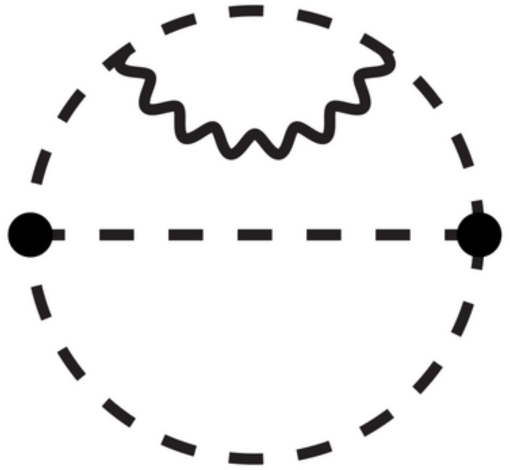
$$\Delta p\Big|_{O(\alpha_s^{7/2})} =$$  $+$ 

- full result to $O(\alpha_s^{7/2})$:

$$\Delta p = \alpha_s^3(c_1 + c_2 \log \alpha_s) + c_3\alpha_s^{7/2}$$

- check nonperturbative condition $\Delta p \geq 0$

[Gorda, **PN**, Sandbote, Paatelainen, Seppänen; **2511.09627**]

conclusions

- thermal QCD relevant for compact stars, heavy-ion collisions and cosmology

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- try to get the best of lattice simulations and perturbation theory

$$p(T, \mu_{\mathrm{B}}) = p_{\mathrm{PQ}} - \Delta p$$

p_{PQ} → lattice

Δp → pert. theory

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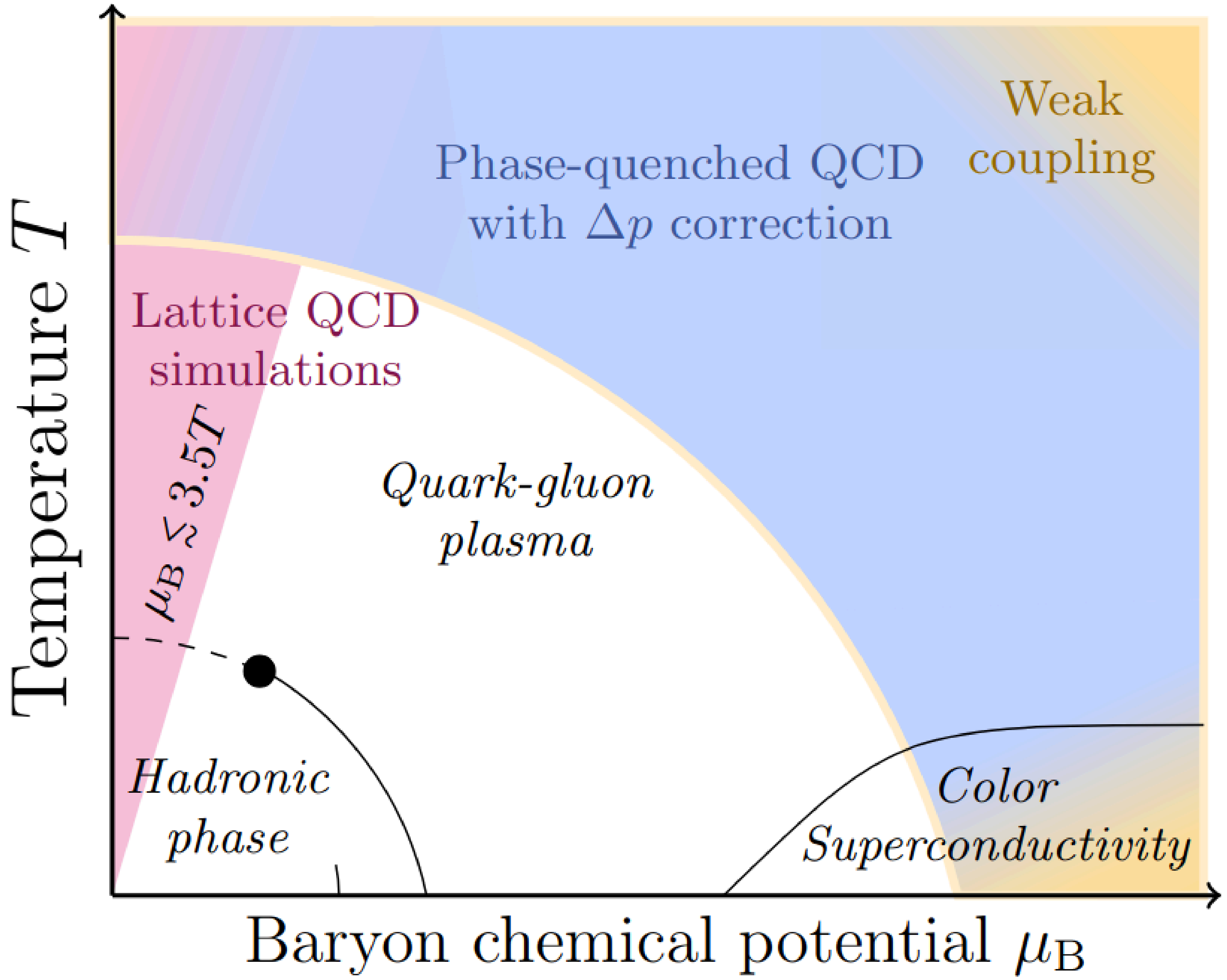
p_{PQ} → lattice; Δp → pert. theory

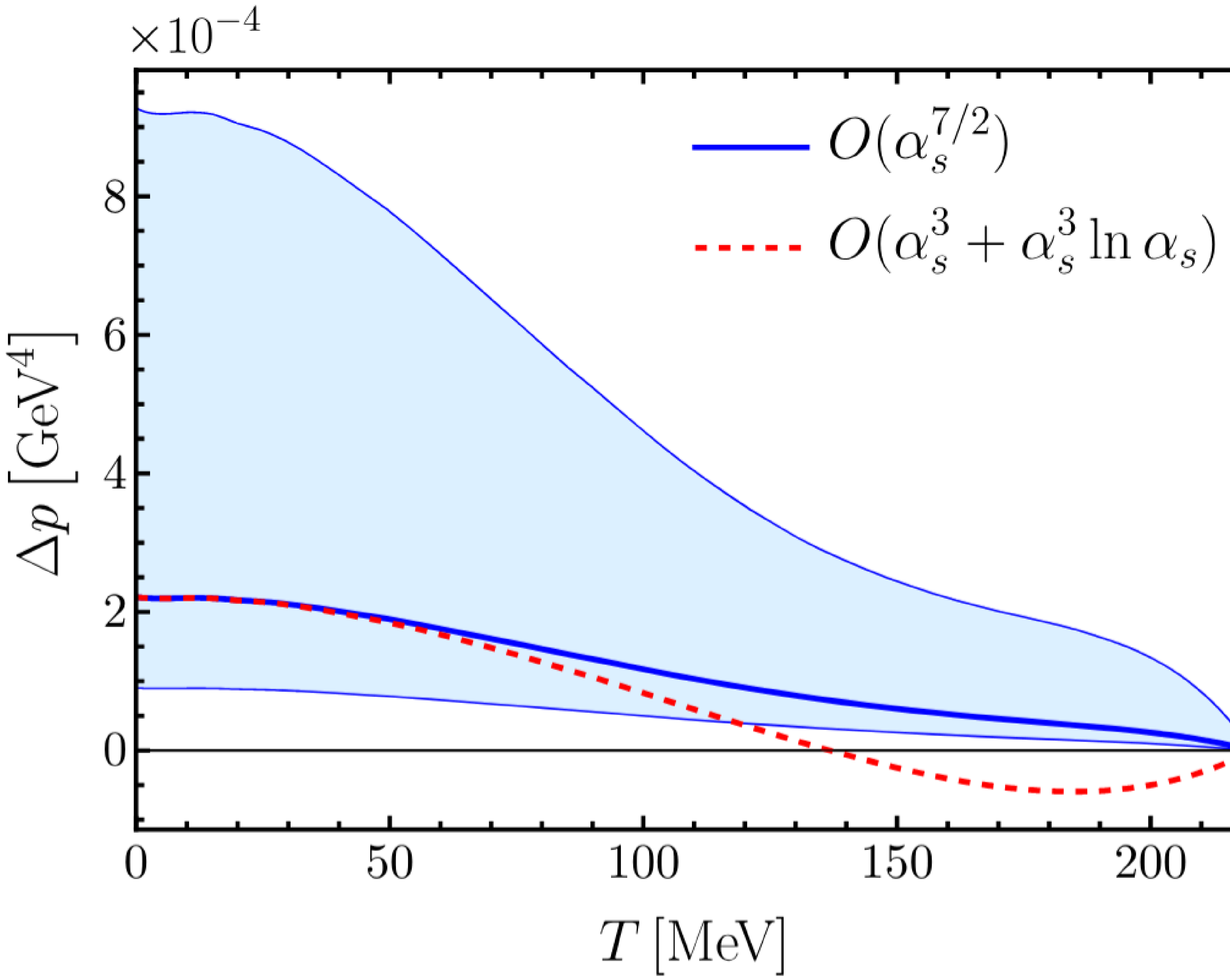


- room for systematic improvements on Δp

$$O(\alpha_s^4) \longrightarrow \text{five loops!}$$

Backup

results



$\times 10^{-4}$
$O(\alpha_s^{7/2})$
$O(\alpha_s^3 + \alpha_s^3 \ln \alpha_s)$
Δp [GeV^4]
8
6
4
2
0
0
50
100
150
200
T [MeV]

thermal Loop Tree Duality

📌 **goal: render a Feynman (sum)-integral locally finite and integrable in** $D = 4$

(i) subtract ultraviolet divergences → well-known from vacuum QFT (R-operation")

(ii) subtract infrared divergences → of thermal origin (use effective theories)

(iii) integrate energy components via Residue theorem (original "Loop Tree Duality")

(iv) numerical Monte Carlo integration

$$\Gamma = \underbrace{(\Gamma - \mathrm{CT}_\Gamma)}_{R(\Gamma)} + \mathrm{CT}_\Gamma$$

CT_Γ → defines the renormalization scheme

$\mathrm{CT}_\Gamma \downarrow D = 4 - 2\varepsilon$

quark pairing

ground state of QCD at high density is superconducting

(2SC, CFL phases, etc.)

[Alford, Rajagopal, Wilczek + others, late '90s]

- condensation of Cooper pairs

$$p_{\text{pairing}}(T \ll \mu) \sim \mu^2 \Delta^2, \qquad \Delta \sim g^{-5} e^{-c/g}$$

exponential suppression

$\mu \gtrsim 1\,\text{GeV}$

[Son, **PRD** '99]

$$\boxed{\Delta p = p_{\text{PQ}} - p + \Delta p_{\text{pairing}}}$$

- 2-flavor isospin case: pairing enhancement $\Delta_{\text{PQ}} \sim g^{-5} e^{-\frac{1}{\sqrt{2}} c/g}$

[Fujimoto, **PRD** '23]

PQ theory: potentially large pairing effects; however

$T_c^{\text{PQ}} = 40 - 110\ \text{MeV} \longrightarrow$ condensate melts

$(\mu_{\text{B}} = 2.5 - 3\ \text{GeV})$

[Gorda, **PN**, Sandbote, Paatelainen, Seppänen; **2511.09627**]