

UNIVERSITY OF HELSINKI



6th Workshop on Nonperturbative Aspects of QCD

thermodynamics of deconfined matter from QCD inequalities and the lattice

Pablo Navarrete

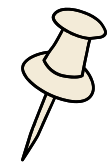
University of Helsinki, Finland

with: T. Gorda, R. Paatelainen, L. Sandbote, K. Seppänen

[arXiv:2511.09627](https://arxiv.org/abs/2511.09627)

01 December 2025

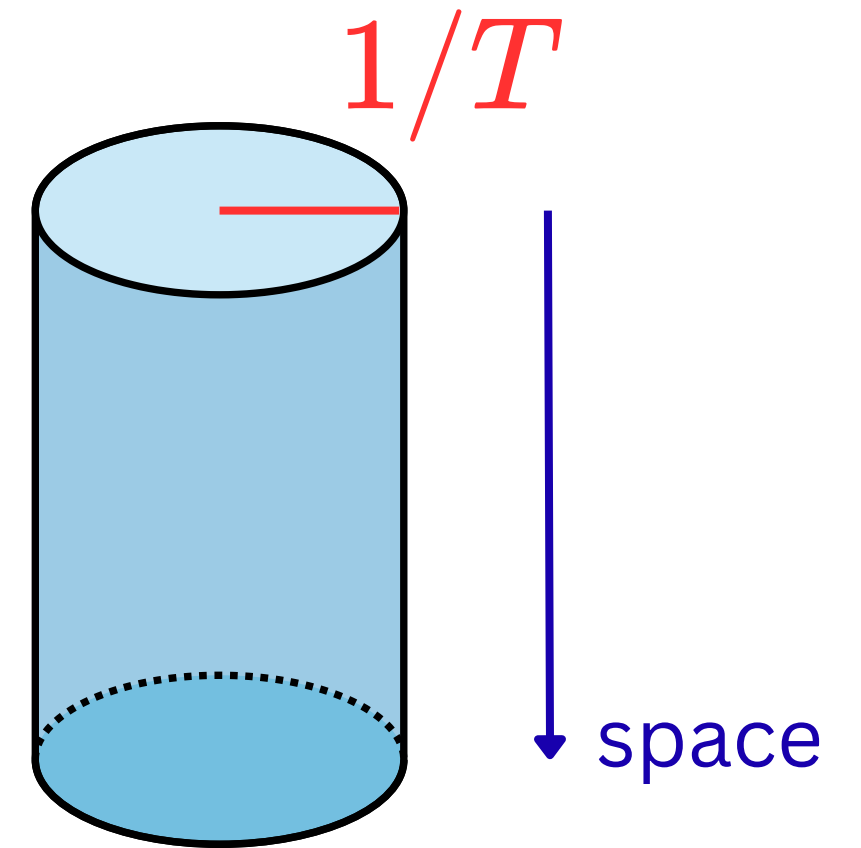
thermal QCD



put system at finite temperature and density

→ in equilibrium: euclidean time $t = i\tau$; extent $1/T$

→ quark of flavor j carries chemical potential μ_j



[e.g. LeBellac, Laine/Vuorinen, etc.]

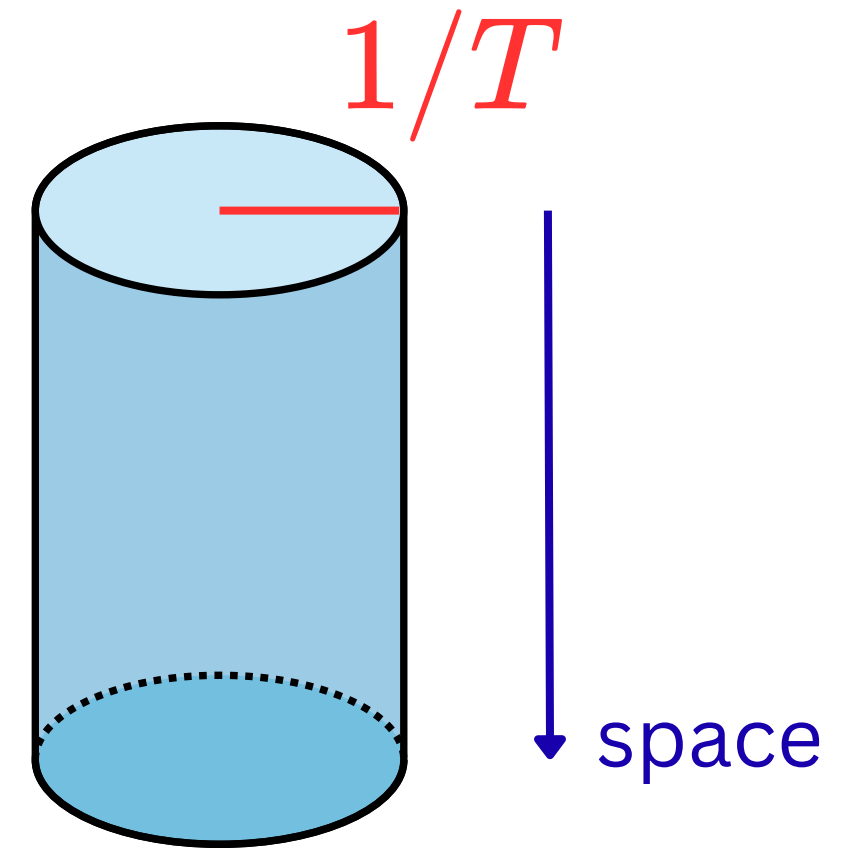
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euclidean path integral:

[e.g. LeBellac, Laine/Vuorinen, etc.]

$$Z(T, \mu_j) = \int \mathcal{D}A e^{-S[A]} \prod_{j=1}^{N_f} \det[\mathcal{D}(\mu_j)]$$

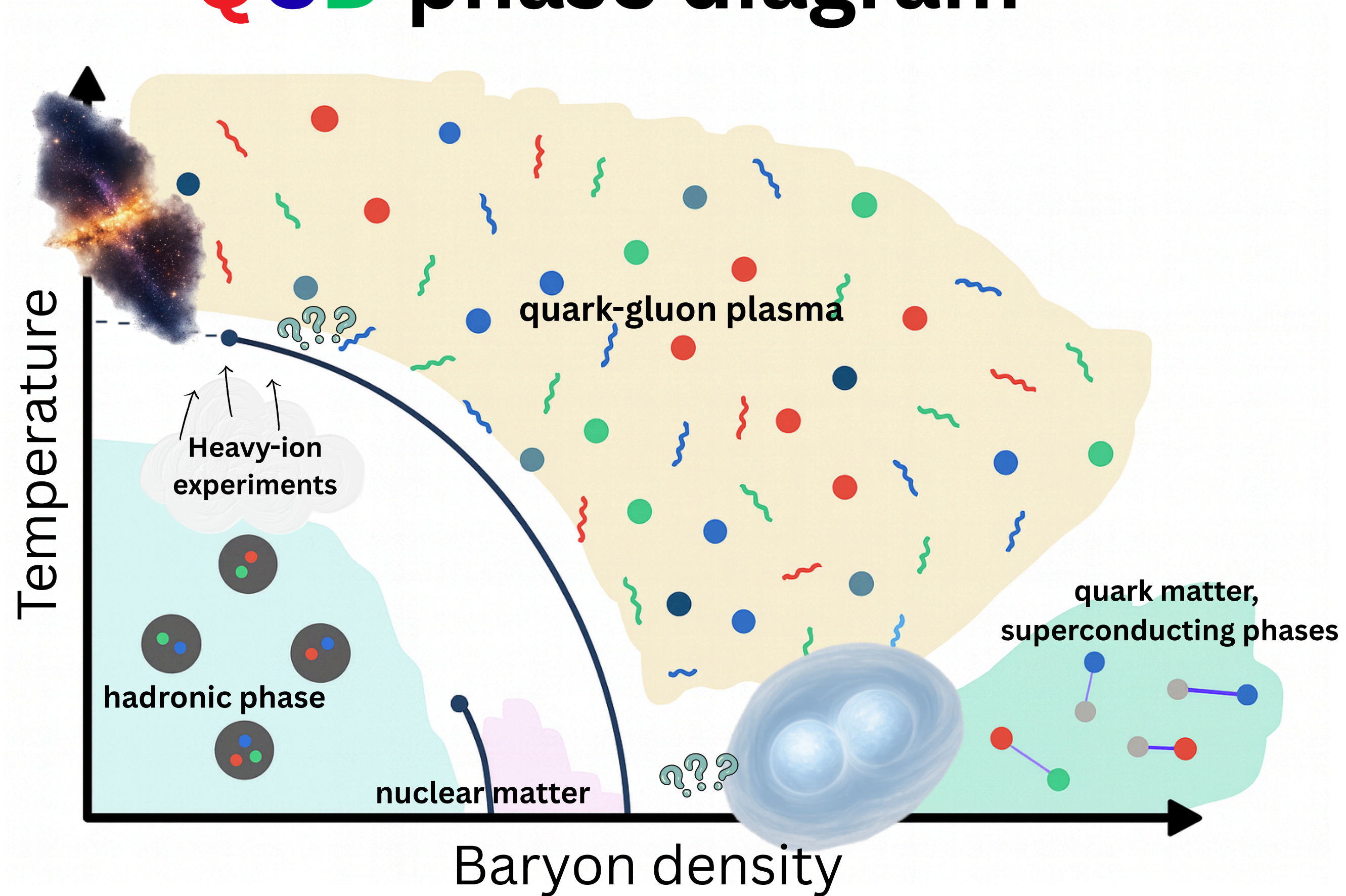
• gluon action

$$S[A] = \int_0^{1/T} d\tau \int_V \frac{1}{2} \text{tr} F_{\mu\nu} F_{\mu\nu}$$

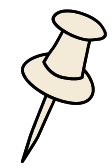
• Dirac operator

$$\mathcal{D}(\mu_j) = \not{D} + m_j + \mu_j \gamma^0$$

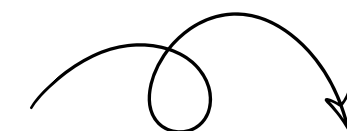
QCD phase diagram



QCD thermodynamics



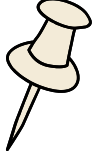

thermodynamics from $p(T, \mu_j) \sim \log Z(T, \mu_j)$



- equation of state
- entropy density
- etc.

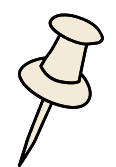
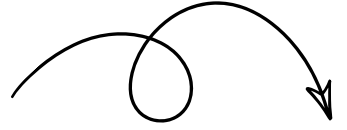
recall $Z = \text{tr} e^{-\frac{1}{T}(\hat{H} - \mu \hat{N})}$

QCD thermodynamics

-  thermodynamics from $p(T, \mu_j) \sim \log Z(T, \mu_j)$ 
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 - etc.
- perturbation theory: $p = \bigcirc + \bigcirc\text{---} + \dots$

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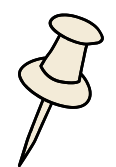
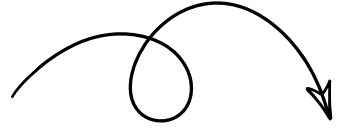
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• lattice gauge theory:

recall $Z = \text{tr} e^{-\frac{1}{T}(\hat{H} - \mu \hat{N})}$

$$n_j = \frac{\langle \hat{N}_j \rangle}{V} = \frac{\partial p}{\partial \mu_j}, \quad \hat{N}_j \equiv \int_V \bar{\psi}_j \gamma^0 \psi_j$$

QCD thermodynamics

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 fermion **Sign Problem**:

$$\gamma^5 \mathcal{D}(\mu_j) \gamma^5 = \mathcal{D}^\dagger(-\mu_j) \quad \Longrightarrow \quad \det \mathcal{D}(\mu_j) \in \mathbb{C}$$

“ γ^5 – hermiticity”

complex Boltzmann weight

~~Monte Carlo importance sampling~~

phase-quenched QCD

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• positive measure: $e^{-S[A]} \prod_{j=1}^{N_f} |\det[\mathcal{D}(\mu_j)]| \geq 0$ (assume $\theta_{\text{QCD}} = 0$)

$$\longrightarrow Z_{\text{PQ}}(T, \mu_j) \geq Z(T, \mu_j)$$

no Sign Problem!

[Cohen, **PRL** '03]

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• QCD pressure bounded from above

$$p_{\text{PQ}}(T, \mu_j) \geq p(T, \mu_j)$$

phase-quenched QCD

$$|\det \mathcal{D}(\mu_j)| = \sqrt{\det \mathcal{D}(\mu_j) \det \mathcal{D}(-\mu_j)}$$

$$[\det \mathcal{D}(\mu_j)]^* = \det \mathcal{D}(-\mu_j)$$

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[lattice: Abbott et al., **PRL** ‘24]

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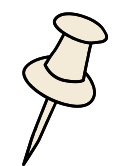
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perturbation theory: $\alpha_s(T, \mu_j) \ll 1$

$$\alpha_s \equiv \frac{g^2}{4\pi}$$

- PQ Feynman rule

$$\sqrt{\det \mathcal{D}(\mu_j)} = \exp \left\{ \frac{1}{2} \text{tr} \log \mathcal{D}(\mu_j) \right\}$$

$$\frac{1}{2} \left(\text{circle}_{+\mu} + \text{circle}_{-\mu} \right)$$

phase-quenched QCD

$$\Delta p \equiv p_{\text{PQ}} - p$$

- we know $\Delta p \geq 0$

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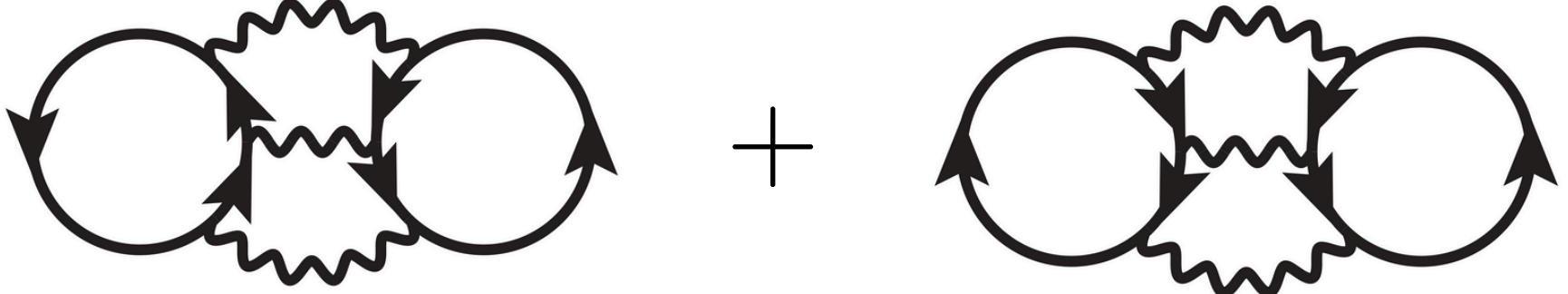
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[Moore and Gorda, **JHEP** '23]
($T = 0$)

$$\Delta p = \text{[diagram 1]} + \text{[diagram 2]} + O(\alpha_s^4 \log \alpha_s)$$


fully finite at $T = 0$

[PN, Paatelainen, Seppänen, **PRD** '24]

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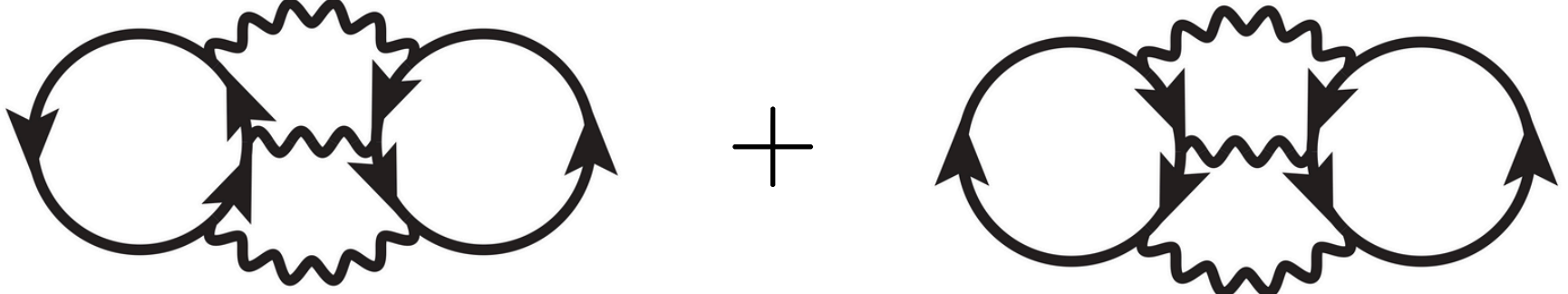
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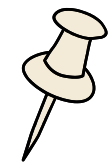
$$\Delta p = \text{[diagram 1]} + \text{[diagram 2]} + O(\alpha_s^4 \log \alpha_s)$$




caveat: potentially large quark-pairing contributions $\Delta p_{\text{pairing}} \sim \mu^2 \Delta_{\text{PQ}}^2$, $\Delta_{\text{PQ}} \sim g^{-5} e^{-\frac{1}{\sqrt{2}} c/g}$

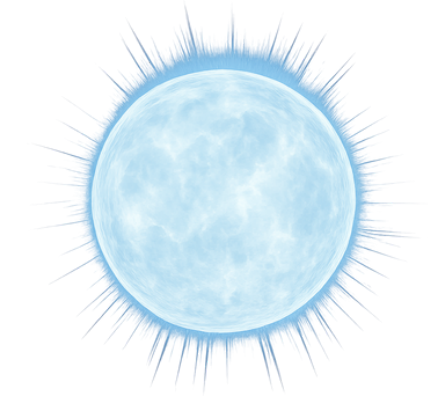
[Fujimoto, **PRD** '23]

evaluating Δp



setup: u , d and s quarks in beta equilibrium at finite T

$$\mu_u = \mu_d = \mu_s, \quad \mu_B = 3\mu_u$$



$$d \leftrightarrow u + e^- + \bar{\nu}_e$$

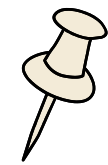
$$s \leftrightarrow u + e^- + \bar{\nu}_e$$

• • •

$$T_c^{\text{PQ}} \sim 40 - 110 \text{ MeV}$$

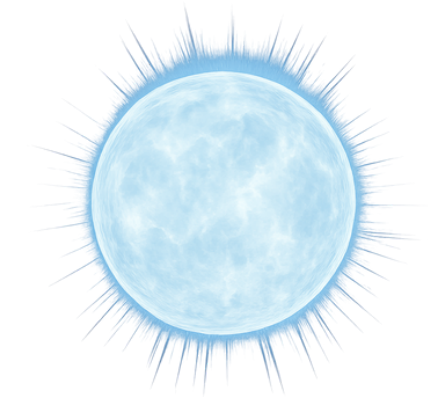
$$(\mu_B \sim 2.5 - 3.0 \text{ GeV})$$

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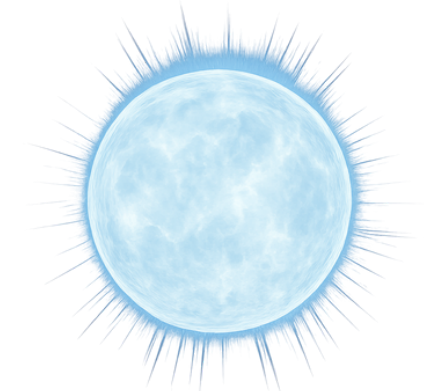
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$$\Delta p = \text{[diagram 1]} + \text{[diagram 2]} + \cancel{\Delta p_{\text{pairing}}}$$

The diagram shows the evaluation of the pressure difference Δp . It is composed of two identical Feynman diagrams added together, followed by a crossed-out term $\Delta p_{\text{pairing}}$. Each Feynman diagram consists of two fermion loops (represented by solid lines with arrows) connected by two wavy lines (representing gluons). The first diagram has the wavy lines connected by a vertical line, and the second diagram has the wavy lines connected by a horizontal line.

evaluating Δp



📌 setup: u , d and s quarks in beta equilibrium at finite T

$$\begin{aligned} d &\leftrightarrow u + e^- + \bar{\nu}_e \\ s &\leftrightarrow u + e^- + \bar{\nu}_e \\ &\quad \bullet \quad \bullet \quad \bullet \end{aligned}$$

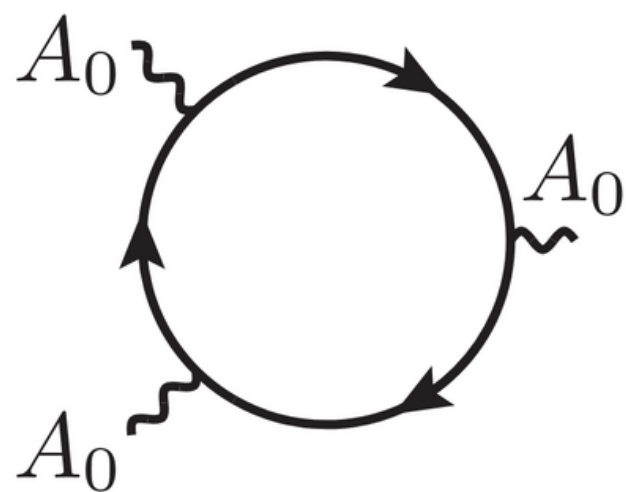
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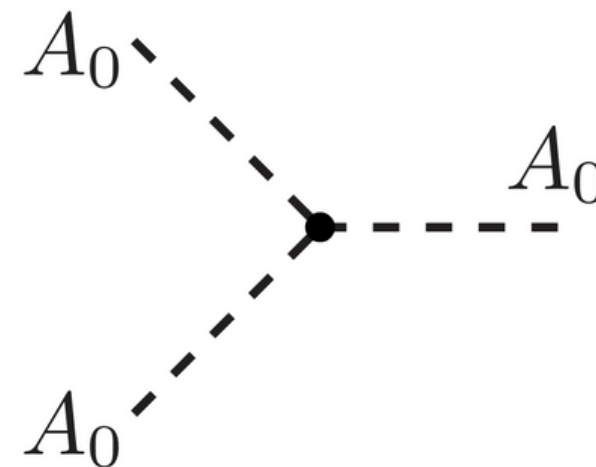
$$(\mu_B \sim 2.5 - 3.0 \text{ GeV})$$

$$\Delta p = \text{[gluon loop diagram]} + \text{[gluon loop diagram]}$$

• static low-momentum gluons at $T > 0$ source infrared divs.



resummation



“electrostatic QCD”

$$\sim i\bar{\gamma} \text{tr} (A_0)^3$$

[Hart et al. ‘03]

evaluating Δp

$$\Delta p = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]}$$

$$= \alpha_s^3 (c_1 + c_2 \log \alpha_s)$$

The diagrams represent Feynman diagrams for the evaluation of Δp . The first two diagrams show two fermion loops connected by a gluon exchange (wavy line). The first diagram has the gluon exchange between the top and bottom lines of the loops, while the second diagram has it between the left and right lines. The third diagram is a fermion loop with a dashed line (representing a ghost or scalar) connecting two points on the loop.

evaluating Δp

$$\Delta p = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]}$$

The diagram shows the evaluation of Δp as a sum of three terms. The first two terms are identical Feynman diagrams: two fermion loops connected by a gluon exchange (represented by a wavy line). The third term is a diagram with a dashed circle and two dots on a horizontal line passing through its center.

$$= \alpha_s^3 (c_1 + c_2 \log \alpha_s)$$

[Catani et al.; **JHEP** '08]

[Capatti et al.; **PRL** '19]

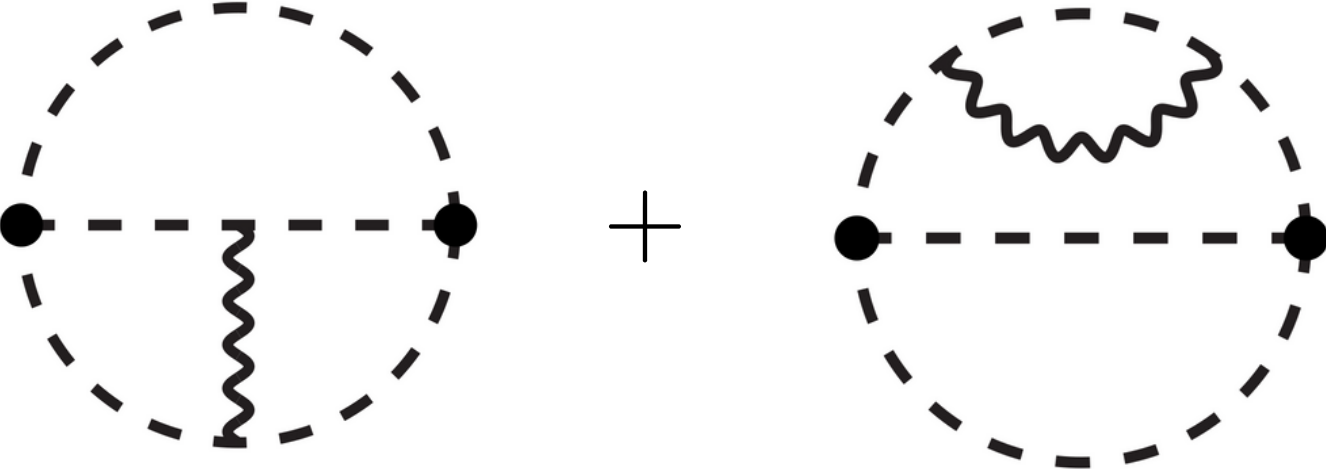
- evaluated with thermal Loop Tree Duality:

numerical Monte Carlo
framework

[**PN**, Paatelainen, Seppänen; **PRD** '24]

[Kärkkäinen, **PN**, Nurmela, Paatelainen,
Seppänen, Vuorinen; **PRL** '25]

evaluating Δp

$$\Delta p \Big|_{O(\alpha_s^{7/2})} =$$


The equation shows the evaluation of Δp at the order $O(\alpha_s^{7/2})$. The result is the sum of two Feynman diagrams. Both diagrams consist of a dashed circle with a horizontal dashed line passing through its center, and two solid black dots at the points where the line intersects the circle. In the first diagram, a vertical wavy line connects the two dots. In the second diagram, a wavy line connects the two dots, and an additional wavy line segment is attached to the top of the circle.

evaluating Δp

$$\Delta p \Big|_{O(\alpha_s^{7/2})} = \text{[Diagram 1]} + \text{[Diagram 2]}$$

- full result to $O(\alpha_s^{7/2})$:

$$\Delta p = \alpha_s^3 (c_1 + c_2 \log \alpha_s) + c_3 \alpha_s^{7/2}$$

- check nonperturbative condition $\Delta p \geq 0$

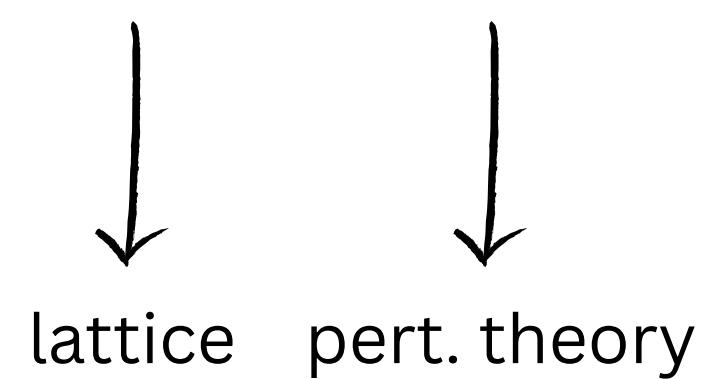
[Gorda, **PN**, Sandbote, Paatelainen,
Seppänen; **2511.09627**]

conclusions

- thermal QCD relevant for compact stars, heavy-ion collisions and cosmology

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- try to get the best of lattice simulations and perturbation theory

$$p(T, \mu_B) = p_{\text{PQ}} - \Delta p$$


The diagram illustrates the decomposition of the pressure $p(T, \mu_B)$ into two parts. Below the term p_{PQ} in the equation, there is a downward-pointing arrow leading to the word "lattice". Similarly, below the term Δp , there is a downward-pointing arrow leading to the words "pert. theory".

conclusions

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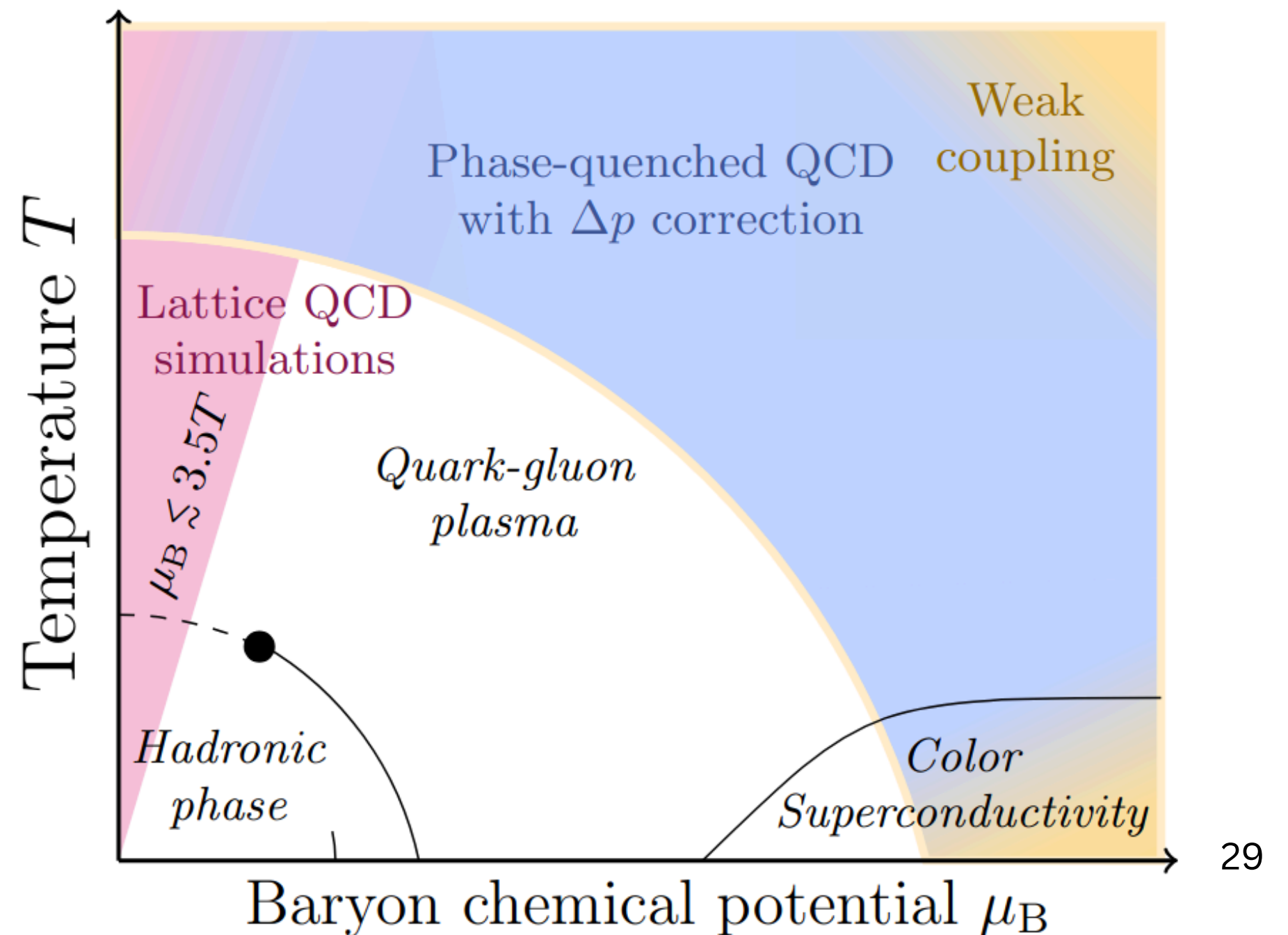
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\downarrow
lattice

\downarrow
pert. theory

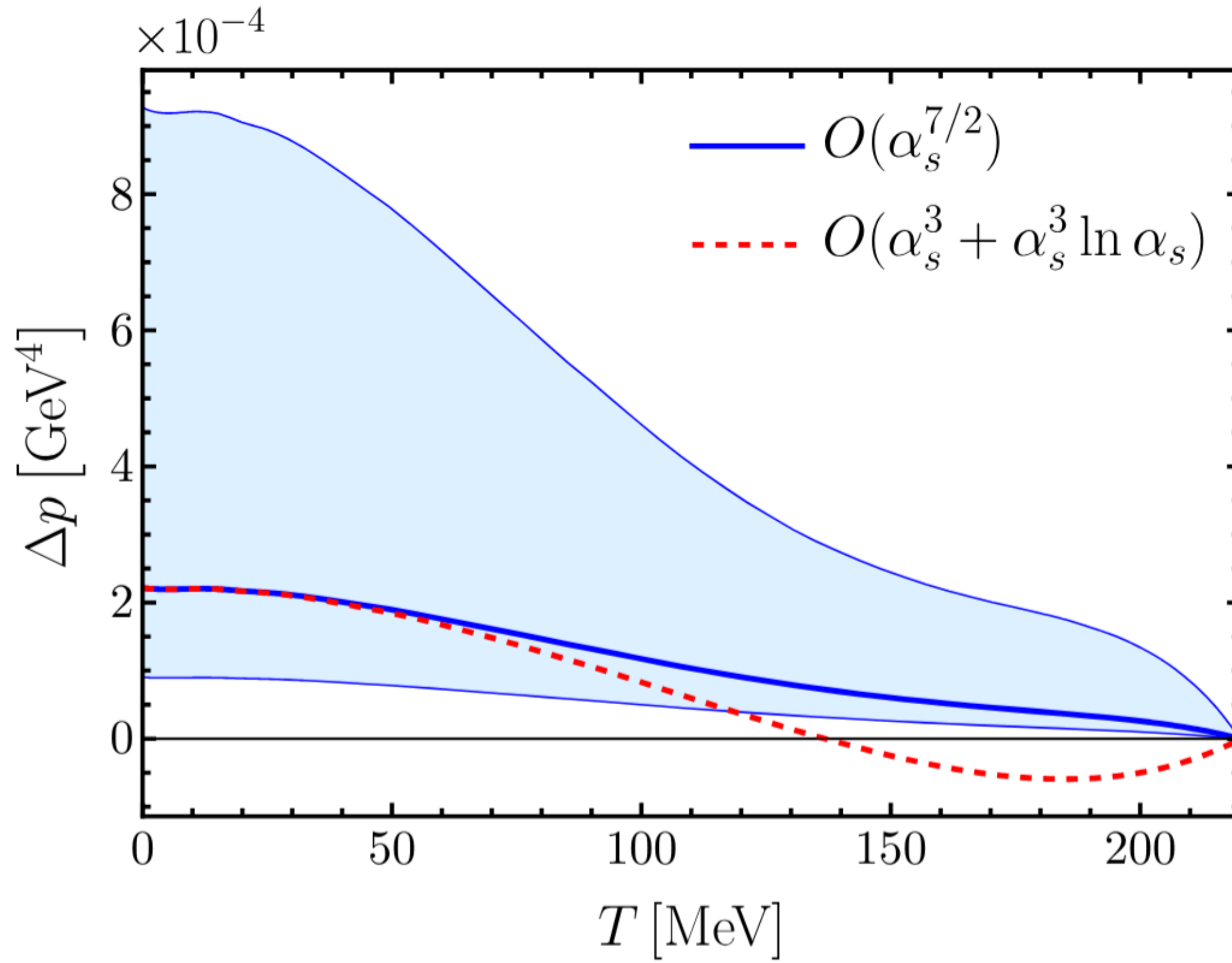
- room for systematic improvements on Δp

$$O(\alpha_s^4) \longrightarrow \text{five loops!}$$

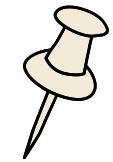


Backup

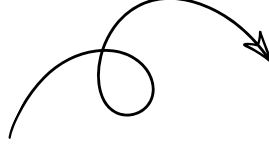
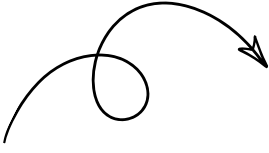
results



thermal Loop Tree Duality



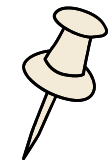
goal: render a Feynman (sum)-integral locally finite and integrable in $D = 4$

- (i) subtract ultraviolet divergences  well-known from vacuum QFT (R-operation")
- (ii) subtract infrared divergences  of thermal origin (use effective theories)
- (iii) integrate energy components via Residue theorem (original "Loop Tree Duality")
- (iv) numerical Monte Carlo integration

$$\Gamma = \underbrace{(\Gamma - \text{CT}_\Gamma)}_{R(\Gamma)} + \underbrace{\text{CT}_\Gamma}_{\substack{\downarrow \\ D = 4 - 2\varepsilon}}$$

defines the renormalization scheme

quark pairing



ground state of QCD at high density is superconducting

(2SC, CFL phases, etc.)

[Alford, Rajagopal, Wilczek
+ others, late '90s]

- condensation of Cooper pairs

$$p_{\text{pairing}}(T \ll \mu) \sim \mu^2 \Delta^2, \quad \Delta \sim g^{-5} e^{-c/g}$$

exponential suppression
 $\mu \gtrsim 1 \text{ GeV}$

[Son, PRD '99]

$$\Delta p = p_{\text{PQ}} - p + \Delta p_{\text{pairing}}$$

- 2-flavor isospin case: pairing **enhancement** $\Delta_{\text{PQ}} \sim g^{-5} e^{-\frac{1}{\sqrt{2}}c/g}$

[Fujimoto, PRD '23]



PQ theory: potentially large pairing effects; however

$$T_c^{\text{PQ}} = 40 - 110 \text{ MeV} \longrightarrow \text{condensate melts}$$

$(\mu_B = 2.5 - 3 \text{ GeV})$

[Gorda, PN, Sandbote, Paatelainen,
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