A renormalon-motivated resummation for low-energy QCD observables

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Introduction

- We consider the case of a low energy spacelike QCD observable $\mathcal{D}(Q^2)$ whose perturbation expansion has in general *noninteger* powers of the coupling $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ (due to nonzero value of the associated anomalous dimension), $\mathcal{D}(Q^2) = a(Q^2)^{\nu_0} + \mathcal{O}(a^{\nu_0+1})$.
- If the renormalon structure of $\mathcal{D}(Q^2)$ is known, then the renormalon structure of an associated auxiliary quantity $\mathcal{D}^{(1)}(Q^2)$ is known and its so called characteristic function $F_{\mathcal{D}^{(1)}}(t)$ can be obtained.
- The quantity $\mathcal{D}(Q^2)$ can then be evaluated as an integral (over t) of the product of $F_{\mathcal{D}^{(1)}}(t)$ and $\widetilde{a}_{\nu_0}(tQ^2)$, where the latter is a generalised logarithmic derivative of order ν_0 of the coupling $a(tQ^2)$.
- The procedure is then extended to the evaluation of timelike observables $\mathcal{F}(\sigma)$, which involves some additional concepts.
- The procedure is then applied to the evaluation of a specific timelike observable $\mathcal{F}(\sigma)$ that has $\nu_0 = 1/3$.
- Instead of the perturbative coupling $a(Q^2)$, a (holomorphic) coupling $\mathcal{A}(Q^2)$ that has no Landau singularities is used as the basis, in order to avoid ambiguities in the evaluation.

Spacelike $\mathcal{D}(Q^2)$

The perturbation expansion of the considered low energy spacelike observable $\mathcal{D}(Q^2)$ is

$$\mathcal{D}(Q^2) = \sum_{n=0}^{\infty} d_n(\nu_0;\kappa) a(\kappa Q^2)^{\nu_0+n} \tag{1}$$

$$= \sum_{n=0}^{\infty} \widetilde{d}_n(\nu_0;\kappa) \widetilde{a}_{\nu_0+n}(\kappa Q^2). \tag{2}$$

Here, $Q^2 \equiv -q^2 > 0$ is spacelike; ν_0 is the power index of the leading term $(0 < \nu_0 \leq 1)$; $\kappa \equiv \mu^2/Q^2 > 0$ is the renormalisation scale parameter.
The (generalised) logarithmic derivatives $\widetilde{a}_{\nu_0+n}$ are related to the powers a^{ν_0+m}

$$\widetilde{a}_{\nu} = \sum_{m=0}^{\infty} k_m(\nu) a^{\nu+m} \qquad (k_0(\nu) = 1), \tag{3}$$

$$a^{\nu} = \sum_{m=0}^{\infty} \widetilde{k}_m(\nu) \widetilde{a}_{\nu+m} \qquad (\widetilde{k}_0(\nu) = 1). \tag{4}$$

Spacelike $\mathcal{D}(Q^2)$

When $\nu = n$ is integer, then:

$$\widetilde{a}_{n+1}(Q^2) \equiv \frac{(-1)^n}{n!\beta_0{}^n} \left(\frac{d}{d\ln Q^2} \right)^n a(Q^2). \tag{5}$$

We have $\widetilde{a}_\nu = a^\nu + \mathcal{O}(a^{\nu+1})$.
Explicit expressions for $k_m(\nu)$ and $\widetilde{k}_m(\nu)$ were obtained by (Kotikov & G.C., JPG (2012)) for $m \le 4$. When a holomorphic coupling $\mathcal{A}(Q^2)$ is used as a basis instead of $a(Q^2)$, the corresponding (exact) $\widetilde{\mathcal{A}}_\nu(Q^2)$ was constructed in (Kotikov & G.C., JPG (2012)) as an integral involving the spectral function $\rho_{\mathcal{A}}(\sigma) = \mathrm{Im}\mathcal{A}(-\sigma - i\varepsilon)$ (see later).
The coefficients $\widetilde{d}_n$ and d_m are then related

$$\widetilde{d}_n(\nu_0, \kappa) = \sum_{s=0}^{n} \widetilde{k}_{n-s}(\nu_0 + s) d_s(\nu_0; \kappa), \tag{6}$$

$$d_n(\nu_0, \kappa) = \sum_{s=0}^{n} k_{n-s}(\nu_0 + s) \widetilde{d}_s(\nu_0; \kappa). \tag{7}$$

Spacelike $\mathcal{D}(Q^2)$

We now construct an associated auxiliary quantity $\mathcal{D}^{(1)}(Q^2)$

$$\mathcal{D}^{(1)}(Q^2) \equiv \sum_{n=0}^{\infty} \widetilde{d}_n(1;\kappa)\widetilde{a}_{n+1}(\kappa Q^2), \tag{8}$$

where

$$\widetilde{d}_n(1;\kappa) \equiv \frac{\Gamma(\nu_0)\Gamma(1+n)}{\Gamma(\nu_0+n)}\widetilde{d}_n(\nu_0;\kappa). \tag{9}$$

$\mathcal{D}^{(1)}(Q^2)$ can be shown to be κ-independent.
We will apply to the evaluation of $\mathcal{D}^{(1)}(Q^2)$ the renormalon-motivated formalism that was developed for the observables with $\nu_0 = 1$ (G.C., 2019 [PRD]):

Spacelike $\mathcal{D}(Q^2)$

The modified Borel transform $\widetilde{B}$ of $\mathcal{D}^{(1)}(Q^2)$ is defined as

$$\widetilde{B}[\mathcal{D}^{(1)}](u;\kappa) \equiv \sum_{n=0}^{\infty} \frac{\widetilde{d}_n(1;\kappa)}{n! {\beta_0}^n} u^n. \tag{10}$$

In practice we need to know the renormalon (singularity) structure of $\widetilde{B}[\mathcal{D}^{(1)}](u)$. Then the quantity $\mathcal{D}^{(1)}(Q^2)$ is evaluated as (G.C., 2019 [PRD])

$$\mathcal{D}^{(1)}(Q^2)_{\rm res.} = \int_0^{\infty} \frac{dt}{t} F_{\mathcal{D}^{(1)}}(t) a(tQ^2), \tag{11}$$

where the characteristic function $F_{\mathcal{D}^{(1)}}(t)$ is the inverse Mellin transformation of the modified Borel $\widetilde{B}[\mathcal{D}^{(1)}](u)$

$$F_{\mathcal{D}^{(1)}}(t) = \frac{1}{2\pi i} \int_{-i\infty}^{+\infty} du \widetilde{B}[\mathcal{D}^{(1)}](u) t^u. \tag{12}$$

Spacelike $\mathcal{D}(Q^2)$

Theor. 1: Then it can be proved that the original spacelike observable $\mathcal{D}(Q^2)$ has the form

$$\mathcal{D}(Q^2)_{\rm res.} = \int_0^\infty \frac{dt}{t} F_{\mathcal{D}^{(1)}}(t) \widetilde{a}_{\nu_0}(tQ^2), \tag{13}$$

In order to prove this, we must perform Taylor expansion of $\widetilde{a}_{\nu_0}(tQ^2)$ around $(\ln)\kappa Q^2$, using recursive relations for $\widetilde{a}_{\nu_0+n}(\kappa Q^2)$'s.
To obtain the needed $F_{\mathcal{D}^{(1)}}(t)$, the (leading) renormalon (singularity) structure of $\widetilde{B}[\mathcal{D}^{(1)}](u)$ [cf. Eq. (12)] should be known. However, in practice, we may know the renormalon structure of $B^{(\nu_0)}[\mathcal{D}](u)$.
The question then: how to get from
$B^{(\nu_0)}[\mathcal{D}](u) \mapsto \widetilde{B}^{(\nu_0)}[\mathcal{D}](u) \mapsto \widetilde{B}[\mathcal{D}^{(1)}](u)$? We recall:

$$B^{(\nu_0)}[\mathcal{D}](u;\kappa) \equiv \sum_{n=0}^{\infty} \frac{d_n(\nu_0;\kappa)}{n! \beta_0{}^n} u^n, \tag{14}$$

$$\widetilde{B}^{(\nu_0)}[\mathcal{D}](u;\kappa) \equiv \sum_{n=0}^{\infty} \frac{\widetilde{d}_n(\nu_0;\kappa)}{n! \beta_0{}^n} u^n, \tag{15}$$

Spacelike $\mathcal{D}(Q^2)$

The following two Theorems can be proved:

Theor. 2: If

$$B^{(\nu_0)}[\mathcal{D}](u;\kappa) = \frac{\mathcal{K}(\kappa)}{(p-u)^{s_0}}\left[1+\mathcal{O}((p-u))\right] \Rightarrow \quad (16)$$

$$\widetilde{B}^{(\nu_0)}[\mathcal{D}](u;\kappa) = \frac{\widetilde{\mathcal{K}}(\kappa)}{(p-u)^{\widetilde{s}_0}}\left[1+\mathcal{O}((p-u))\right], \quad (17)$$

where $\widetilde{s}_0 = s_0 - p\beta_1/\beta_0^2$. We recall the RGE:
$da(\mu^2)/d\ln\mu^2 = -\beta_0 a(\mu^2)^2 - \beta_1 a(\mu^2)^3 - \ldots$.
For UV renormalons, the substitutions $(p-u) \mapsto (p+u)$ and $\widetilde{s}_0 = s_0 + p\beta_1/\beta_0^2$ are made. The theorem is valid even for timelike observable (see later).

Theor. 3: If $\widetilde{B}^{(\nu_0)}[\mathcal{D}](u;\kappa)$ is as in Theor. 2, then

$$\widetilde{B}[\mathcal{D}^{(1)}](u;\kappa) = \frac{\widetilde{\mathcal{K}}^{(1)}(\kappa)}{(p-u)^{\widetilde{s}_0-\nu_0+1}}\left[1+\mathcal{O}((p-u))\right], \quad (18)$$

Spacelike $\mathcal{D}(Q^2)$

When knowing the renormalon structure Eq. (18) of $\widetilde{B}[\mathcal{D}^{(1)}](u)$, the needed characteristic function $F_{\mathcal{D}^{(1)}}(t)$ is obtained then via inverse Mellin Eq. (12).

Theor. 4: If $\widetilde{B}[\mathcal{D}^{(1)}](u)$ has the renormalon form as in Theor. 3, i.e., $\widetilde{B}[\mathcal{D}^{(1)}](u) = \widetilde{\mathcal{K}}^{(1)}/(p-u)^{\widetilde{s}}$ then

$$F_{\mathcal{D}^{(1)}}(t) = \Theta(1-t)\widetilde{\mathcal{K}}^{(1)} \frac{t^p}{\Gamma(\widetilde{s})(-\ln t)^{1-\widetilde{s}}} \qquad \text{with}: \ \widetilde{s} = s_0 - p\frac{\beta_1}{\beta_0^2} - \nu_0 + 1. \tag{19}$$

This is then applied in the resummation Eq. (13) for $\mathcal{D}(Q^2)$.
This then concludes the case of resummation of spacelike observables $\mathcal{D}(Q^2)$.

Timelike $\mathcal{F}(\sigma)$

A typical timelike observable $\mathcal{F}(\sigma)$ is a function of positive squared energy $\sigma > 0$. It has the related spacelike quantity (observable) $\mathcal{D}(Q^2)$. Furthermore, the auxiliary spacelike quantity $\mathcal{D}^{(1)}(Q^2)$ has the corresponding auxiliary timelike quantity $\mathcal{F}^{(1)}(\sigma)$. The two pairs are related via

$$\mathcal{F}(\sigma) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi \, \mathcal{D}(\sigma e^{i\phi}), \tag{20}$$

$$\mathcal{F}^{(1)}(\sigma) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi \, \mathcal{D}^{(1)}(\sigma e^{i\phi}). \tag{21}$$

$$\mathcal{D}(Q^2) = Q^2 \int_0^\infty \frac{d\sigma \mathcal{F}(\sigma)}{(\sigma + Q^2)^2}, \tag{22}$$

$$\mathcal{D}^{(1)}(Q^2) = Q^2 \int_0^\infty \frac{d\sigma \mathcal{F}^{(1)}(\sigma)}{(\sigma + Q^2)^2} \tag{23}$$

Timelike $\mathcal{F}(\sigma)$

If the expansion of $\mathcal{D}(Q^2)$ in powers of $a(\kappa Q^2)$ has powers $a(\kappa Q^2)^{\nu_0+n}$ [cf.Eqs. (1)-(2)], then the expansion of $\mathcal{F}(\sigma)$ has powers $a(\kappa\sigma)^{\nu_0+n}$.

$$\mathcal{F}(\sigma) = \sum_{n=0}^{\infty} f_n(\nu_0;\kappa) a(\kappa\sigma)^{\nu_0+n} \tag{24}$$

$$= \sum_{n=0}^{\infty} \widetilde{f}_n(\nu_0;\kappa) \widetilde{a}_{\nu_0+n}(\kappa\sigma), \tag{25}$$

and for $\mathcal{F}^{(1)}(\sigma)$ (that is obtained via Eq. (21)):

$$\mathcal{F}^{(1)}(\sigma) = \sum_{n=0}^{\infty} \widetilde{f}_n(1;\kappa) \widetilde{a}_{1+n}(\kappa\sigma). \tag{26}$$

We recall that the coefficients $\widetilde{d}_n(1;\kappa)$ of $\mathcal{D}^{(1)}$ were defined via a simple rescaling of the coefficients $\widetilde{d}_n(\nu_0;\kappa)$ of $\mathcal{D}$:

$$\widetilde{d}_n(1;\kappa) \equiv \frac{\Gamma(\nu_0)\Gamma(1+n)}{\Gamma(\nu_0+n)} \widetilde{d}_n(\nu_0;\kappa).$$

Timelike $\mathcal{F}(\sigma)$

When applying the resummation of $\mathcal{D}(Q^2)$ Eq. (13) to the integration form (20) and exchanging the order of integration, we obtain the resummed form for $\mathcal{F}(\sigma)$:

$$\mathcal{F}(\sigma)_{\rm res.} = \int_0^\infty \frac{dt}{t} F_{\mathcal{D}^{(1)}}(t) \widetilde{\mathfrak{h}}_{\nu_0}(t\sigma), \tag{27}$$

where $\widetilde{\mathfrak{h}}_\nu(\sigma)$ is the timelike analog of the spacelike (generalised logarithmic derivative) $\widetilde{a}_\nu(Q^2)$, defined and investigated in (Kotikov and G.C., 2012 [JPG])

$$\widetilde{\mathfrak{h}}_\nu(\kappa\sigma) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \; \widetilde{a}_\nu(\kappa\sigma e^{i\phi}). \tag{28}$$

Timelike $\mathcal{F}(\sigma)$

We can then show the following relation between modified Borel of $\mathcal{D}^{(1)}$ and $\mathcal{F}^{(1)}$:

Theor. 5:

$$\widetilde{B}[\mathcal{D}^{(1)}](u;\kappa)\frac{\sin(\pi u)}{\pi u} = \widetilde{B}[\mathcal{F}^{(1)}](u;\kappa), \tag{29}$$

and the similar version for the modified Borel transforms of $\mathcal{D}$ and $\mathcal{F}$:

Theor. 6: If

$$\begin{aligned} \widetilde{B}^{(\nu_0)}[\mathcal{D}](u;\kappa) &= \frac{\widetilde{\mathcal{K}}(\kappa)}{(p-u)^{\widetilde{s}_0}} \Rightarrow \\ \widetilde{B}^{(\nu_0)}[\mathcal{F}](u;\kappa) &= \frac{\sin(\pi p)}{\pi p}\frac{\widetilde{\mathcal{K}}(\kappa)}{(p-u)^{\widetilde{s}_0}}\left[1+\mathcal{O}\left((p-u)\right)\right]. \end{aligned} \tag{30}$$

We recall that $\widetilde{B}^{(\nu_0)}[\mathcal{D}](u;\kappa) = \sum \widetilde{d}_n(\nu_0;\kappa)u^n/(n!\beta_0{}^n)$ and $\widetilde{B}^{(\nu_0)}[\mathcal{F}](u;\kappa) = \sum \widetilde{f}_n(\nu_0;\kappa)u^n/(n!\beta_0{}^n)$.

Timelike $\mathcal{F}(\sigma)$

One of the consequences of Theor. 5 and 6 the following **Lemma:**

$$\widetilde{f}_n(1;\kappa) = \frac{\Gamma(\nu_0)\Gamma(1+n)}{\Gamma(\nu_0+n)} \widetilde{f}_n(\nu_0;\kappa)\left(1+\mathcal{O}(1/n)\right), \qquad (31)$$

i.e., the relation between $\widetilde{f}_n(1;\kappa)$ and $\widetilde{f}_n(\nu_0;\kappa)$ is in the leading order in $1/n$ equal to that between the coefficients $\widetilde{d}_n(1;\kappa)$ and $\widetilde{d}_n(\nu_0;\kappa)$.
By **Lemma** Eq. (31), **Theor. 3** can be applied also to $\widetilde{B}^{(\nu_0)}[\mathcal{F}](u)$:

If $\widetilde{B}^{(\nu_0)}[\mathcal{F}](u) \sim 1/(p-u)^{\widetilde{s}_0}$, then $\widetilde{B}[\mathcal{F}^{(1)}](u) \sim 1/(p-u)^{\widetilde{s}_0-\nu_0+1}$.
From here, $\widetilde{B}[\mathcal{D}^{(1)}](u)$ is obtained by **Theor. 5**.

We note that **Theor. 2** is valid also for timelike observables, i.e., we can replace there $B^{(\nu_0)}[\mathcal{D}](u) \mapsto B^{(\nu_0)}[\mathcal{F}](u)$ and $\widetilde{B}^{(\nu_0)}[\mathcal{D}](u) \mapsto \widetilde{B}^{(\nu_0)}[\mathcal{F}](u)$.

So: $B^{(\nu_0)}[\mathcal{F}](u) \mapsto \widetilde{B}^{(\nu_0)}[\mathcal{F}](u) \mapsto \widetilde{B}[\mathcal{F}^{(1)}](u) \mapsto \widetilde{B}[\mathcal{D}^{(1)}](u)$.

In order to apply the described resummations formally consistently, i.e., without ambiguities due to Landau singularities of pQCD coupling $a(tQ^2)$ (at $0 < tQ^2 \lesssim 1 \ \mathrm{GeV}^2$), we should use a spacelike coupling $a(Q^2) \mapsto \mathcal{A}(Q^2)$ that is free from Landau singularities. In principle, we can use any such coupling that is defined via its spectral (discontinuity) function $\rho_{\mathcal{A}}(\sigma) \equiv \mathrm{Im}\mathcal{A}(-\sigma - i\varepsilon)$. For example, one choice of such spectral function can be (3δAQCD, $0 < {M_1}^2 < {M_2}^2 < {M_3}^2 < {M_0}^2$)

$$\rho_{\mathcal{A}}(\sigma) = \pi \sum_{j=1}^{3} \mathcal{F}_j \delta(\sigma - {M_j}^2) + \Theta(\sigma - {M_0}^2)\rho_a(\sigma), \tag{32}$$

where: $\mathcal{A}(Q^2) = (1/\pi) \int_{M_1^2}^{\infty} d\sigma \rho_{\mathcal{A}}(\sigma)/(\sigma + Q^2)$.

The generalised logarithmic derivative $\widetilde{\mathcal{A}}_{\nu}(Q^2)$ (the analog of $\widetilde{a}_{\nu}(Q^2)$) is then (Kotikov and G.C., 2021 [JPG])

$$\widetilde{\mathcal{A}}_{\nu}(Q^2) = \frac{1}{\pi} \frac{(-1)}{{\beta_0}^{\nu-1}\Gamma(\nu)} \int_0^{\infty} \frac{d\sigma}{\sigma} \rho_{\mathcal{A}}(\sigma) \mathrm{Li}_{-\nu+1}\left(-\frac{\sigma}{Q^2}\right) \quad (\nu > 0) \tag{33}$$

The timelike coupling $\widetilde{\mathfrak{H}}_{\nu}$ (the analog of $\widetilde{\mathfrak{h}}_{\nu}$) is then

$$\widetilde{\mathfrak{H}}_{\nu}(\kappa\sigma) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, \widetilde{\mathcal{A}}_{\nu}(\kappa\sigma e^{i\phi}). \quad (34)$$

$$= -\frac{\sin(\pi\nu)}{\pi^2(\nu-1)\beta_0{}^{\nu-1}}\int_0^{\infty} \frac{dw}{w^{\nu-1}}\rho_{\mathcal{A}}(\sigma e^{w}) \quad (0<\nu<2), \quad (35)$$

The resummations of the quantities $\mathcal{D}^{(1)}$, $\mathcal{D}$ and $\mathcal{F}$ in this formulation are

$$\mathcal{D}^{(1)}(Q^2)_{\rm res.} = \int_0^{\infty} \frac{dt}{t} F_{\mathcal{D}^{(1)}}(t)\mathcal{A}(te^{-\widetilde{K}_e}Q^2), \quad (36)$$

$$\mathcal{D}(Q^2)_{\rm res.} = \int_0^{\infty} \frac{dt}{t} F_{\mathcal{D}^{(1)}}(t)\widetilde{\mathcal{A}}_{\nu_0}(te^{-\widetilde{K}_e}Q^2), \quad (37)$$

$$\mathcal{F}(\sigma)_{\rm res.} = \int_0^{\infty} \frac{dt}{t} F_{\mathcal{D}^{(1)}}(t)\widetilde{\mathfrak{H}}_{\nu_0}(te^{-\widetilde{K}_e}\sigma). \quad (38)$$

Holomorphic QCD

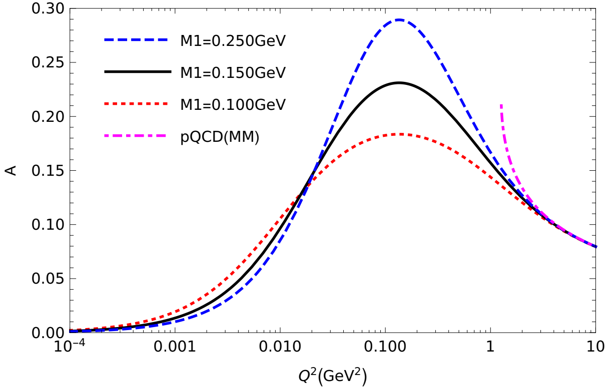

Figure: The specific 3δ-parametrised choice of $\rho_{\mathcal{A}}(\sigma)$ gives: $\mathcal{A}(Q^2)$ for positive Q^2, in 3δAQCD, with $n_f = 3$, $\alpha_s^{\overline{\mathrm{MS}}}(M_Z^2) = 0.1180$, and three different values of the IR-threshold scale parameter M_1 (C. Ayala, G.C. et al., 2017 [JPG]); cf. also (Pelaez et al., 2017 [PRD]). Based on lattice calculations (Bogolubsky et al., 2009 [PLB]). The underlying pQCD coupling is included, all in (Lambert) MiniMOM scheme (LMM).

The specific 3δ-parametrised choice of $\rho_{\mathcal{A}}(\sigma)$ then gives

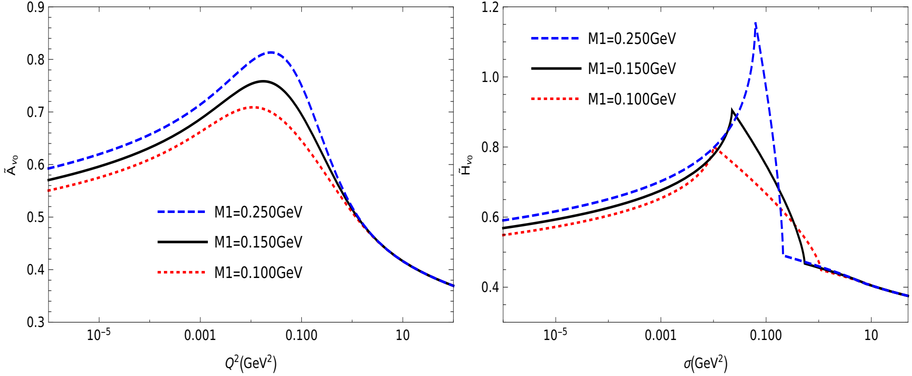

Figure: The spacelike running coupling $\widetilde{\mathcal{A}}_{\nu_0}(Q^2)$ for positive Q^2 (left-hand figure), and the timelike running coupling $\widetilde{\mathfrak{H}}_{\nu_0}(\sigma)$ (right-hand figure), in 3δAQCD, with $\nu_0 = 1/3$ and $n_f = 3$, $\alpha_s^{\overline{\rm MS}}(M_Z^2) = 0.1180$, and three different values of the IR-threshold scale parameter M_1.

Implementation: an example

We consider $\mathcal{F}(\sigma) \propto \hat{C}(m)$ of (Grozin and Neubert, 1997 [NPB]), which is proportional to the factor of the Wilson coefficient of the chromomagnetic operator in the heavy-quark effective theory (HQET) for hadronic bound states containing one heavy quark (c or b). Strictly speaking, $\mathcal{F}(\sigma) = \pi^{-\nu_0} \hat{C}(m)$, where $\sigma = m$ is the (pole) mass of the heavy quark. In this case, the first four expansion coefficients $f_j(\nu_0; \kappa)$ ($j = 0, \ldots, 3$) are known, and $\nu_0 = 1/3, 9/25$ (for $n_f = 3, 4$) Further, the (leading) renormalon structure of the Borel $B^{(\nu_0)}[\mathcal{F}](u)$ is known (Grozin and Neubert, 1997 [NPB])

$$B^{(\nu_0)}[\mathcal{F}](u;1) = \left\{ \frac{S_+}{\left(\frac{1}{2} - u\right)^{+\nu_0 + \beta_1/(2\beta_0^2)}} + \frac{S_0}{\left(\frac{1}{2} - u\right)^{+\beta_1/(2\beta_0^2)}} + \frac{S_-}{\left(\frac{1}{2} - u\right)^{-\nu_0 + \beta_1/(2\beta_0^2)}} \right\} \left[1 + \mathcal{O}\left(1/2 - u\right)\right]. \quad (39)$$

Application of **Theor. 2** then gives

$$\widetilde{B}^{(\nu_0)}[\mathcal{F}](u;1) = \left\{ \frac{\widetilde{S}_+}{\left(\frac{1}{2}-u\right)^{\nu_0}} + \widetilde{S}_0 \ln\left(\frac{1}{2}-u\right) + \frac{\widetilde{S}_-}{\left(\frac{1}{2}-u\right)^{-\nu_0}} \right\} \left[1+\mathcal{O}\left(1/2-u\right)\right]. \tag{40}$$

Since **Lemma** is valid (relating $\widetilde{f}_n(1)$ with $\widetilde{f}_n(\nu_0)$), **Theor. 3** can be applied, giving

$$\widetilde{B}[\mathcal{F}^{(1)}](u;\kappa) = \left\{ \frac{\widetilde{S}_+^{(1)}}{\left(\frac{1}{2}-u\right)^{1}} + \frac{\widetilde{S}_0^{(1)}}{\left(\frac{1}{2}-u\right)^{-\nu_0+1}} + \frac{\widetilde{S}_-^{(1)}}{\left(\frac{1}{2}-u\right)^{-2\nu_0+1}} \right\} \left[1+\mathcal{O}\left(1/2-u\right)\right]. \tag{41}$$

Implementation: an example

The use of **Theor. 5** then gives

$$\widetilde{B}[\mathcal{D}^{(1)}](u;\kappa) = \left\{ \frac{\widetilde{K}_{+}^{(1)}(\kappa)}{\left(\frac{1}{2}-u\right)^{1}} + \frac{\widetilde{K}_{+0}^{(1)}(\kappa)}{\left(\frac{1}{2}-u\right)^{-\nu_0+1}} + \frac{\widetilde{K}_{-}^{(1)}(\kappa)}{\left(\frac{1}{2}-u\right)^{-2\nu_0+1}} \right\} \left[1+\mathcal{O}\left(1/2-u\right)\right], \tag{42}$$

where $\widetilde{K}_q^{(1)} = (\pi/2)\widetilde{S}_q^{(1)}$. We have the κ-dependence

$$\widetilde{K}_q^{(1)}(\kappa) = \exp\left(\ln(\kappa)u\right)\widetilde{K}_q^{(1)} \quad (q=+,0,-). \tag{43}$$

Therefore, we include in our ansatz for $\widetilde{B}[\mathcal{D}^{(1)}](u;\kappa)$ this factor. This then gives us our ansatz ("the point of departure"):

$$\widetilde{B}[\mathcal{D}^{(1)}](u) = \exp(\widetilde{K}_e^{(1)}u) \left\{ \frac{\widetilde{K}_{+}^{(1)}}{\left(\frac{1}{2}-u\right)^{1}} + \frac{\widetilde{K}_{0}^{(1)}}{\left(\frac{1}{2}-u\right)^{-\nu_0+1}} + \frac{\widetilde{K}_{-}^{(1)}}{\left(\frac{1}{2}-u\right)^{-2\nu_0+1}} \right\}. \tag{44}$$

The four parameters here are determined by the knowledge of the first four coefficients $f_n(\nu_0;\kappa)$ $(n=0,\ldots,3;\ \kappa=1)$.

Implementation: an example

The relevant characteristic function for the resummations of $\mathcal{D}^{(1)}$, $\mathcal{D}$ and $\mathcal{F}$ is then, according to Theorem 4

$$F_{\mathcal{D}^{(1)}}(t) = \Theta(1-t)t^{1/2}\left\{ \widetilde{K}_{+}^{(1)} + \frac{\widetilde{K}_{0}^{(1)}}{\Gamma(-\nu_0+1)(-\ln t)^{\nu_0}} + \frac{\widetilde{K}_{-}^{(1)}}{\Gamma(-2\nu_0+1)(-\ln t)^{2\nu_0}} \right\}. \tag{45}$$

This then gives us the final result for the evaluation (resummation) of $\mathcal{F}(\sigma)$:

$$\mathcal{F}(\sigma)_{\rm res.} = \int_0^1 \frac{dt}{t} F_{\mathcal{D}^{(1)}}(t) \widetilde{\mathfrak{H}}_{\nu_0}(t e^{-\widetilde{K}_e}\sigma), \tag{46}$$

Implementation: an example

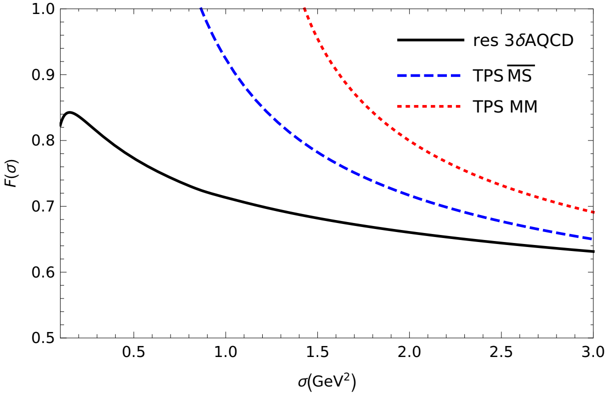

Figure: The renormalon-resummed $\mathcal{F}(\sigma)$, as a function of the squared timelike momentum (squared mass) σ, in 3δAQCD, for $n_f = 3$, $M_1 = 0.150$ GeV and $\alpha_s^{\overline{\rm MS}}(M_Z^2) = 0.1180$. For comparison, we include also the corresponding pQCD TPS in the $\overline{\rm MS}$ and LMM schemes, with three terms included ($N = 3$).

Implementation: an example

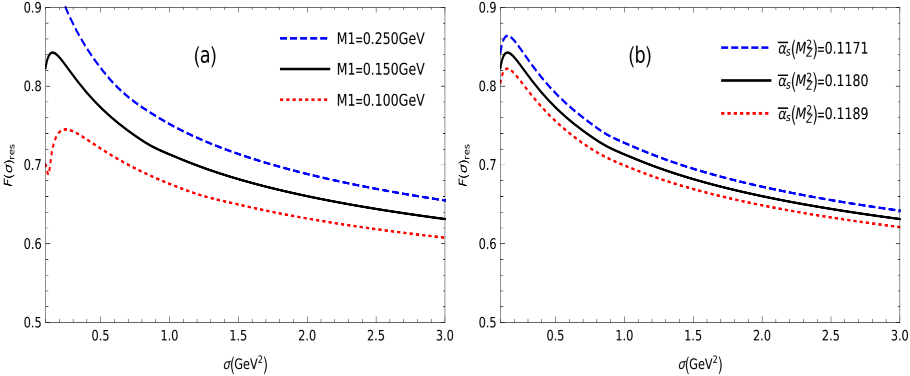

Figure: The resummed values of $\mathcal{F}(\sigma)$ as in the previous Figure, but now when (a) the IR-threshold scale M_1 is varied and $\alpha_s^{\overline{\rm MS}}(M_Z^2) = 0.1180$; (b) when $\alpha_s^{\overline{\rm MS}}(M_z^2)$ value is varied and $M_1 = 0.150$ GeV.

Implementation: an example

Results:

$$\mathcal{F}(m_c^2)_{\rm res.} = 0.6365^{-0.0058}_{+0.0060}(m_c)^{+0.0108}_{-0.0105}(\alpha_s)^{+0.0243}_{-0.0245}(M_1) = 0.6365 \pm 0.0273. \quad (47)$$

$$\mathcal{F}(m_b^2)_{\rm res.} = 0.4792 \mp 0.0010(m_b)^{+0.0055}_{-0.0053}(\alpha_s)^{+0.0061}_{-0.0065}(M_1) = 0.4792^{+0.0084}_{-0.0083}. \quad (48)$$

pQCD TPS approach gives:

$$\mathcal{F}(m_c^2)^{\rm TPS[3]} = 0.6604^{-0.0116}_{+0.0130}(m_c)^{+0.0120}_{-0.0115}(\alpha_s) \pm 0.0854({\rm TPS}) = 0.6604^{+0.0872}_{-0.0869}, \quad (49)$$

$$\mathcal{F}(m_b^2)^{\rm TPS[4]} = 0.4807 \mp 0.0014(m_b) \pm 0.0049(\alpha_s) \pm 0.0184({\rm TPS}) = 0.4807 \pm 0.0191. \quad (50)$$

Implementation: an example

The ratio $\mathcal{F}(m_b^2)/\mathcal{F}(m_c^2)$ is related to the following ratio of mass splitting between the ground-state pseudoscalar and vector mesons, in the bottom and charm quark systems:

$$\frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} = \pi^{\nu_0(4)-\nu_0(3)} \frac{\mathcal{F}(m_b^2)}{\mathcal{F}(m_c^2)} \left[1 + \Lambda_{\rm eff} \left(\frac{1}{m_c} - \frac{1}{m_b}\right) + \ldots\right]. \quad (51)$$

Here, hadronic parameter $\Lambda_{\rm eff}$ in the subleading terms is a combination of the hadronic parameters. The ratio of mass splitting is 0.8776, using the data PDG2024. We now use in this relation the results (47)-(50) and we extract the value of this hadronic parameter

$$\begin{aligned} \textbf{res} : \Lambda_{\rm eff} &= \left(0.335^{-0.006}_{+0.005}(m_c) \mp 0.004(m_b)^{+0.075}_{-0.074}(M_1 \& \alpha_s)\right) \ {\rm GeV} \\ &= (0.335 \pm 0.075) \ {\rm GeV}, \quad (52) \\ \textbf{TPS} : \Lambda_{\rm eff} &= \left(0.435^{-0.027}_{+0.028}(m_c) \pm 0.006(m_b)^{+0.265}_{-0.285}({\rm TPS} \& \alpha_s)\right) \ {\rm GeV} \\ &= \left(0.435^{+0.266}_{-0.286}\right) \ {\rm GeV}. \quad (53) \end{aligned}$$

Conclusions

- We presented a renormalon-motivated evaluation (resummation) for general spacelike QCD observables $\mathcal{D}(Q^2)$ whose perturbation expansion in powers of couplings is $\mathcal{D}(Q^2) - a(\kappa Q^2)^{\nu_0} + \mathcal{O}(a^{\nu_0+1})$ with ν_0 in general noninteger. This approach is an extension of a previous approach that had be constructed for the case of ν_0 integer ($\nu_0 = 1$).
- The resummation involved a characteristic function $F_{\mathcal{D}^{(1)}}(t)$ that can be obtained if the renormalon structure of $\mathcal{D}(Q^2)$ is known.
- In order to have the correct holomorphic behaviour of the evaluated (resummed) $\mathcal{D}(Q^2)_{\rm res.}$, it was necessary to replace in the integration the pQCD generalised logarithmic derivative $\widetilde{a}_{\nu_0}(tQ^2)$ by a holomorphic analog $\widetilde{\mathcal{A}}_{\nu_0}(tQ^2)$.
- We then extended this approch to the evaluation of timelike QCD observables $\mathcal{F}(\sigma)$ (with ν_0 in general noninteger).
- We applied this approach to a timelike low energy QCD quantity, the scheme invariant factor of the Wilson coefficient of the chromomagnetic factor of the heavy-quark effective Lagrangian, and compared the obtained results with those of a naive pQCD evaluation.