Parameters Evolution of the Pomeron Using the Functional Renormalisation Group

Luis Cancino Arancibia[1] Dr. Carlos Contreras Hidalgo[2]

[1]Universidad San Sebastián
[2]Universidad Técnica Federico Santa María

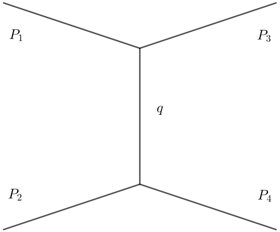
December 4, 2025

Matrix S

To understand the transitions between one particle state (at time $t = t_A$ and 4-momentum p_A) and another (at $t = t_B$ and 4-momentum p_B), we need an object that describes this time evolution, namely the S-matrix. If both states coincide, S must be equal to the identity matrix, so it is commonly defined as

$$S = 1 + iT, \quad \text{donde} \quad \langle b_{fin}; t_B | T | a_{in}; t_A \rangle = (2\pi)^4 \delta^4 (p_A - p_B) A(a \to b), \tag{1}$$

where T is the transition matrix that encodes the deviations from the trivial case, and A is the probability amplitude for the process.



Before the advent of quantum chromodynamics (QCD), processes were studied imposing certain properties that the S-matrix must satisfy:
-S must be Lorentz invariant
-S must be analytic $\Rightarrow A_{a+\bar{c}\to\bar{b}+d}(s,t,u) = A_{a+b\to c+d}(t,s,u)$
-S must be unitary $\Rightarrow$ the optical theorem. A special case of this theorem is that of an elastic, forward process (i.e. $\theta = t = 0$), which is given by

$$2Im[A(s,0)] = -i(2\pi)^4 \sum_X \int d \prod_X \delta^4(p_{in} - p_X)|A(in \to X)| \approx \Phi \cdot \sigma_t, \tag{2}$$

where $\sigma_t = \sum_X \sigma(in \to X)$ is the total sum of all possible cross sections for the initial state, and Φ is the incident flux.

- Sommerfeld rewrote partial-wave expansion series as complex integrals with poles at integer values of l associated with $\sin(\pi l)$

$$A(s,t) \approx \oint dl \frac{(2l+1)a(l,t)}{\sin(\pi l)} P\left(l, 1+\frac{2s}{t}\right), \tag{3}$$

where P are the Legendre polynomials, a the partial-wave amplitude, and $1+\frac{2s}{t} \equiv \theta$ is the scattering angle generated by the collision.

- Integrating over the complex plane and taking the limit $s \gg |t|$, the amplitude of these particles, which we call *reggeons* in the t-channel, takes the form

$$A(s,t) \approx \beta(t)(\eta + e^{-i\pi\alpha(t)})s^{\alpha(t)} \approx s^{\alpha(t)}, \tag{4}$$

where $\alpha(t)$ is the Regge trajectory.

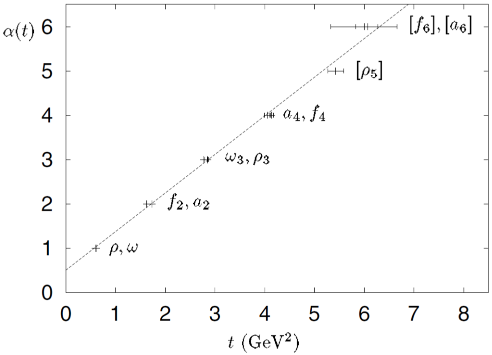
Using the optical theorem, we obtain for the *cross section* and its derivative

$$\sigma_t \approx s^{\alpha(0)-1} \quad ; \quad \frac{d\sigma}{dt} \approx s^{2\alpha(t)-2}. \tag{5}$$

If we expand the Regge trajectory around $t = 0$, for small values of t we have

$$\alpha_R(t) = \alpha_R(0) + \alpha'_R t, \tag{6}$$

where $\alpha_R(0)$ is the intercept and α'_R the slope of the trajectory of a *reggeon* R.



Soft-Pomeron

- In 1973, Jaroszkiewicz and Landshoff proposed a model for $p-p$ collisions at high energies and small t, in which the *pomeron* is exchanged from the *quark* sea. In this work, they compute that the differential cross section must be

$$\frac{d\sigma}{dt} = \frac{(3\beta F_p(t))^4}{4\pi \sin^2(0.5\pi\alpha_{SP}(t))}\left(\frac{s}{m_p}\right)^{2\alpha_{SP}(t)-2}, \tag{7}$$

  where m_p is the proton mass, β the coupling between *pomeron* and *quarks*, $F_p(t)$ the proton form factor, and $\alpha_{SP}(t)$ the Regge trajectory of the *Soft-Pomeron*. They compare the model with data from the *Intersecting Storage Rings* (*ISR*) at *CERN*, obtaining for the slope of the *Soft-Pomeron* the value

$$\alpha'_{SP} = 0.25 \pm 0.02 \ GeV^{-2}, \tag{8}$$

  and this value of the slope is used by subsequent models and data for the *Soft-Pomeron* case.

- In 1992, Donnachie and Landshoff performed a phenomenological analysis comparing experimental data with their own Regge model, where the total cross section can be written as

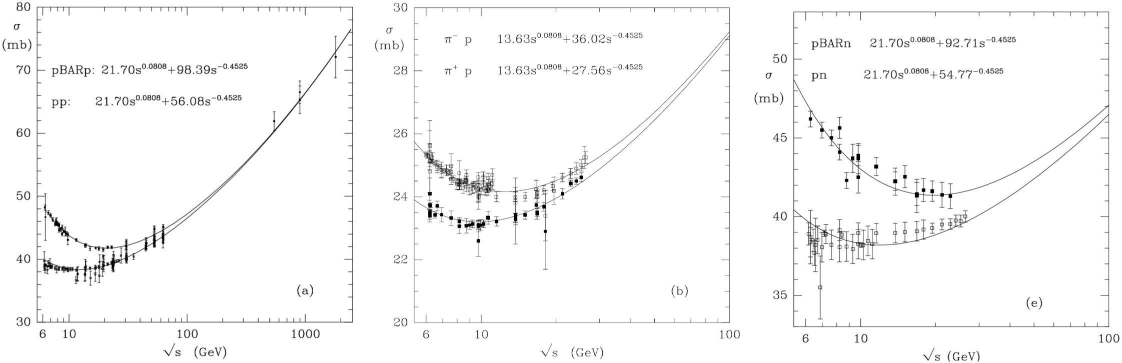
$$\sigma = Xs^{\varepsilon} + Ys^{-\eta}, \tag{9}$$

  where $\varepsilon = \alpha_{SP}(0) - 1$ (as in (5)) and η has the same physical interpretation, but associated with reggeized mesons.

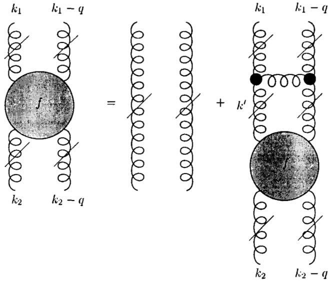
Pomerons

- Since the 1970s, different theoretical models including Reggeons have been compared with experimental data; the table summarizes the parameters of the Pomerons.

	Intercept $(Z^{-1}\mu_R)$	**Slope** (α'_R)
SOFT POMERON	0.0808 – 0.114	~ 0.25
HARD POMERON	0.3 – 0.5	~ 0.1



- Using QCD, the pomeron is a bound state of gluons, described by the *BFKL* evolution equation, which allows one to compute the *pomeron* contribution as a sum over states corresponding to gluon ladders



- The intercept appears as eigenvalues of the *BFKL* kernel. In the limit $t \to 0$, the intercept is

$$\omega_n(\nu) = -\frac{2N_c\alpha_s}{\pi} Re[\gamma((|n|+1)/2 - i\nu) - \gamma(1)], \tag{10}$$

where $\gamma(x)$ is the digamma function.

- The function has a maximum given by the first order (i.e. $n = \nu = 0$). This is related to the *pomeron* intercept, so approximately the intercept is

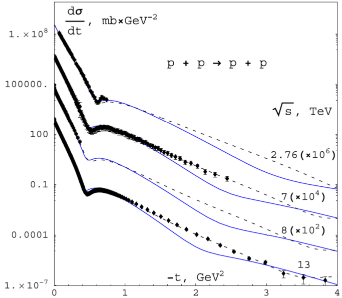
$$\omega_0(0) = -\frac{2N_c\alpha_s}{\pi} Re[\gamma(1/2) - \gamma(1)] = -\frac{2N_c\alpha_s}{\pi}(-ln(4)) = \alpha_s \cdot 2.6476... , \quad (11)$$

where they find that if $\alpha_s \approx 0.2$ the intercept should be $\omega_0(0) \approx 0.5$.

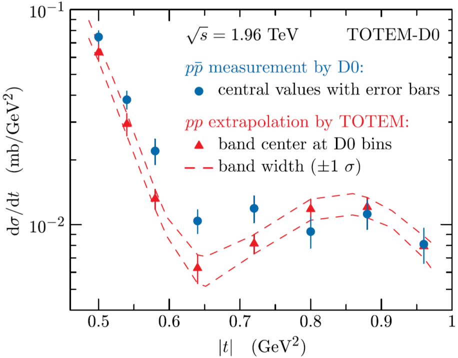
- On the other hand, in 1986 Lipatov found a lower bound for the intercept (at first order) of the *bare pomeron*. This bound is

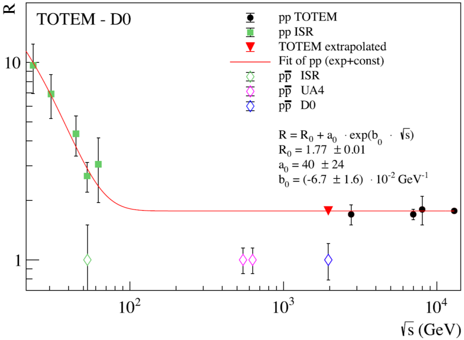
$$\alpha_{HP}(0) - 1 \gtrsim 0.3, \quad (12)$$

where $\alpha_{HP}(0)$ is the intercept of the *Hard-Pomeron*.



- In 2021, the collaboration between the DØ and TOTEM experiments published the discovery of the exchange of a gluonic bound state, colorless and C-odd (i.e. it couples differently to pp than to $p\bar{p}$ collisions and therefore contributes with opposite sign to one of the amplitudes).





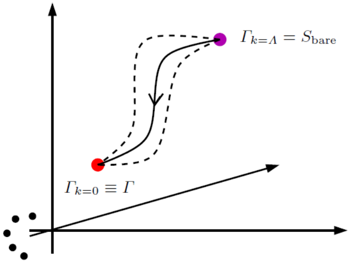
- After this slide, t will no longer be the Mandelstam variable, but we will define it as a function depending on the momentum and the scale, $t \equiv \ln(k/k_0)$.

Renormalization group

- We have seen that in different energy regions different *pomerons* dominate with different observables. To study how the observables evolve, we need to use the renormalization group, which tells us how the parameters vary through the β functions

$$\beta_i = \Lambda \frac{\partial \lambda_i}{\partial \Lambda}. \tag{13}$$

- The IR limit (i.e. $k \to 0$), which gives us the full effective action $\Gamma = \Gamma_{k=0}$, shows two extreme cases of the theory that we would like to connect. As shown in Fig. 12, if we start at the scale $k = \Lambda$, we can find in theory space (i.e. in the space of the parameters of the action) trajectories that connect these points.



- In order to investigate the IR region of the strong interaction, we need non-perturbative methods. For this reason we use the functional renormalization group, since this method allows the study of non-perturbative sectors.
- The β functions are obtained by solving the Wetterich equation

$$\partial_t \Gamma_k = \frac{1}{2} Tr[\Big(\Gamma_k^{(2)} + R_k\Big)^{-1} \partial_t R_k]. \tag{14}$$

where

$$\Gamma_k^{(2)} = \Gamma_{k,0}^{(2)} - V_k \quad ; \quad Tr[\ \dots\ ] \equiv \int\int \frac{dq^D}{(2\pi)^D}\frac{d\omega}{(2\pi)} Tr[\ \dots\ ].$$

- We perform an expansion of the form $(A-B)^{-1} = -A^{-1}(1 + BA^{-1} + B^2A^{-2} + \dots)$ around the interaction matrix

$$(\Gamma_{k,0}^{(2)} + R_k - V_k)^{-1} = -G_{k,0}(1 + V_k G_{k,0} + V_k G_{k,0} V_k G_{k,0} + \dots). \tag{15}$$

- The free Green function $G_{k,0}$ is

$$G_{k,0} = \frac{1}{\Gamma_{k,0}^{(2)} + R_k}, \tag{16}$$

where R_k is the regulator matrix.

We start with M different *reggeons* represented by scalar fields ψ_i, and the effective action has the form

$$\Gamma_k[\psi_i^\dagger, \psi_i] = \int d^D x d\tau [Z_i(\frac{1}{2}\psi_i^\dagger \hat{\partial}_\tau \psi_i - \alpha_i' \psi_i^\dagger \nabla^2 \psi_i) + m_{i,l}(\psi_i \psi_l^\dagger + \psi_i^\dagger \psi_l) - \mu_i \psi_i^\dagger \psi_i - V_k[\psi_i^\dagger, \psi_i; \lambda_j] \ ], \qquad (17)$$

where τ is the *rapidity*; D is the transverse space dimension (in this thesis we use $D = 2$); $l \neq i \equiv 1, .., M$; $j \equiv 1, ..., N$, where N is the number of couplings appearing in the potential; Z_i is the wave function renormalization; α_i' is the slope of the Regge trajectory of the corresponding *reggeons*; λ_j are the couplings associated with the potential; and μ_i and m are denoted as the 'mass' parameters of the fields, which are related to the intercepts of the Regge trajectories via $Z^{-1}\mu_i = \alpha_i(0) - 1$.

In this work we use the Litim regulator, which suppresses modes larger than the scale k, achieved via the Heaviside function $\Theta(x)$. The explicit form of the regulator that we use is

$$R_1 = Z_1\alpha_1'(k^2 - q^2)\Theta(k^2 - q^2) \tag{18}$$

$$R_2 = Z_2\alpha_2'(k^2 - q^2)\Theta(k^2 - q^2) = rZ_2\alpha_1'(k^2 - q^2)\Theta(k^2 - q^2), \tag{19}$$

and its derivatives

$$\begin{aligned}\dot{R}_1 &= -Z_1\alpha_1'(k^2 - q^2)\Theta(k^2 - q^2)\,(\zeta_1 + \eta_1) + 2k^2 Z_1\alpha_1'\Theta(k^2 - q^2) \\ &= 2k^2 Z_1\alpha_1'\Theta\left(k^2 - q^2\right)\left(1 - \frac{1}{2}\left(1 - \frac{q^2}{k^2}\right)(\zeta_1 + \eta_1)\right)\end{aligned} \tag{20}$$

$$\dot{R}_2 = 2k^2 rZ_2\alpha_1'\Theta\left(k^2 - q^2\right)\left(1 - \frac{1}{2}\left(1 - \frac{q^2}{k^2}\right)(\zeta_2 + \eta_2)\right), \tag{21}$$

Potential

The potential, depending only on dimensionless parameters, is

$$\begin{aligned} V[\tilde{\psi}, \tilde{\psi}^\dagger, \tilde{\chi}, \tilde{\chi}^\dagger] =& \alpha_1' k^{2D}[i\tilde{\lambda}_1\tilde{\psi}^\dagger(\tilde{\psi}^\dagger + \tilde{\psi})\tilde{\psi} + i\tilde{\lambda}_2\tilde{\chi}^\dagger(\tilde{\chi}^\dagger + \tilde{\chi})\tilde{\chi} + i\tilde{\lambda}_3\tilde{\psi}^\dagger(\tilde{\chi}^\dagger + \tilde{\chi})\tilde{\psi} + \\ & i\tilde{\lambda}_4\tilde{\chi}^\dagger(\tilde{\psi}^\dagger + \tilde{\psi})\tilde{\chi} + i\tilde{\lambda}_5((\tilde{\chi}^\dagger)^2\tilde{\psi} + \tilde{\chi}^2\tilde{\psi}^\dagger) + i\tilde{\lambda}_6((\tilde{\psi}^\dagger)^2\tilde{\chi} + \tilde{\psi}^2\tilde{\chi}^\dagger)] \end{aligned} \tag{22}$$

with which we construct the interaction matrix, whose elements are the second derivatives of the potential with respect to the fields.

$$V_k = \begin{pmatrix} \partial_{\tilde{\psi}\tilde{\psi}}V & \partial_{\tilde{\psi}\tilde{\psi}^\dagger}V & \partial_{\tilde{\psi}\tilde{\chi}}V & \partial_{\tilde{\psi}\tilde{\chi}^\dagger}V \\ \partial_{\tilde{\psi}^\dagger\tilde{\psi}}V & \partial_{\tilde{\psi}^\dagger\tilde{\psi}^\dagger}V & \partial_{\tilde{\psi}^\dagger\tilde{\chi}}V & \partial_{\tilde{\psi}^\dagger\tilde{\chi}^\dagger}V \\ \partial_{\tilde{\chi}\tilde{\psi}}V & \partial_{\tilde{\chi}\tilde{\psi}^\dagger}V & \partial_{\tilde{\chi}\tilde{\chi}}V & \partial_{\tilde{\chi}\tilde{\chi}^\dagger}V \\ \partial_{\tilde{\chi}^\dagger\tilde{\psi}}V & \partial_{\tilde{\chi}^\dagger\tilde{\psi}^\dagger}V & \partial_{\tilde{\chi}^\dagger\tilde{\chi}}V & \partial_{\tilde{\chi}^\dagger\tilde{\chi}^\dagger}V \end{pmatrix}. \tag{23}$$

For the free case (without interactions), we perform a Fourier transform of the action, obtaining

$$\Gamma_{k,0}[\bar{\psi}^\dagger,\bar{\psi},\bar{\chi}^\dagger,\bar{\chi}] = \int d^D q d\omega [\bar{\psi}^\dagger Z_1(i\omega+\alpha_1' q^2)\bar{\psi} + \bar{\chi}^\dagger Z_2(i\omega+\alpha_2' q^2)\bar{\chi} + m(\bar{\psi}\bar{\chi}^\dagger + \bar{\psi}^\dagger\bar{\chi}) - \mu_1\bar{\psi}^\dagger\bar{\psi} - \mu_2\bar{\chi}^\dagger\bar{\chi}\ ],$$

where $\bar{\psi}^\dagger,\bar{\psi},\bar{\chi}^\dagger,\bar{\chi}$ are the Fourier transforms of their respective fields, and we reorder the terms in the action to obtain the second functional derivative $(\Gamma_0^{(2)})_{kl} \equiv \frac{\delta\Gamma_0}{\delta\Psi_k\delta\Psi_l^T}$

$$\Gamma_{k,0} = \frac{1}{2}\int d^D q d\omega\ \Psi^T\Gamma_{k,0}^{(2)}\Psi, \qquad \text{where} \qquad \Psi^T = \begin{pmatrix}\bar{\psi} & \bar{\psi}^\dagger & \bar{\chi} & \bar{\chi}^\dagger\end{pmatrix}$$

$$\Gamma_{k,0}^{(2)} = k^D\begin{pmatrix} 0 & (-i\omega+\alpha_1' q^2 - k^2\alpha_1'\tilde{\mu}_1) & 0 & k^2\alpha_1'\tilde{m} \\ (i\omega+\alpha_1' q^2 - k^2\alpha_1'\tilde{\mu}_1) & 0 & k^2\alpha_1'\tilde{m} & 0 \\ 0 & k^2\alpha_1'\tilde{m} & 0 & (-i\omega+\alpha_2' q^2 - k^2\alpha_1'\tilde{\mu}_2) \\ k^2\alpha_1'\tilde{m} & 0 & (i\omega+\alpha_2' q^2 - k^2\alpha_1'\tilde{\mu}_2) & 0 \end{pmatrix}$$

Integration over q

- Only $\dot{R}$ contains a dependence on q in the case $q^2 < k^2$, and all the terms in the integral will be multiplied by one of the elements of R (i.e. we will only have integrals of the type $\int dq^D(1+q^2)...$). Therefore, after performing the integration, we can redefine this matrix so that it contains the factors arising from the q integration

$$\dot{R}_k = k^{2D}\alpha_1' \begin{pmatrix} 0 & Z_1 N_D A_D(\zeta_1,\eta_1) & 0 & 0 \\ Z_1 N_D A_D(\zeta_1,\eta_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & rZ_2 N_D A_D(\zeta_2,\eta_2) \\ 0 & 0 & rZ_2 N_D A_D(\zeta_2,\eta_2) & 0 \end{pmatrix},$$

where $N_D = \frac{1}{(2\sqrt{\pi})^D \Gamma(D/2)}$ and $A_D(\zeta_x,\eta_x) = \frac{2}{D}(1 - \frac{\zeta_x+\eta_x}{D+2})$.

- With the q integration done, the right-hand side of the Wetterich equation becomes

$$\frac{1}{2}Tr[(\Gamma_k^{(2)} + R_k)^{-1}\dot{R}_k] = -ik^D\alpha_1' \int \frac{dz'}{2\pi} Tr[\dot{R}_k G_{k,0}(1 + V_k G_{k,0} + V_k G_{k,0} V_k G_{k,0} + ...)] \tag{24}$$

- We find that the poles in the z integral can be defined as

$$z = x \pm y, \tag{25}$$

where $x \equiv (1 + r - \mu_1 - \mu_2)/2$ and $y \equiv \sqrt{(1 - r - \mu_2 + \mu_1)^2 + (2m)^2}/2$.

- Out of the 4 poles, only two contribute because they lie inside the integration contour. The integrals to be computed are

$$-i \int \frac{dz'}{2\pi} \frac{(z + r - \mu_2)^a(-z + r - \mu_2)^b(z + 1 - \mu_1)^c(-z + 1 - \mu * 1)^d}{(-z + x + y)^r(-z + x - y)^r(z + x + y)^s(z + x - y)^s} = \sum Res(r, s, a, b, c, d),$$

where the function $Res(r, s, a, b, c, d)$ is defined as

$$\sum Res(r, s, a, b, c, d) = \lim_{z \to -(x+y)} \frac{1}{s!} \frac{d^{s-1}}{dz^{s-1}} \left(\frac{(z + r - \mu_2)^a(-z + r - \mu_2)^b(z + 1 - \mu_1)^c(-z + 1 - \mu_1)^d}{(-z + x + y)^r(-z + x - y)^r(z + x - y)^s} \right) +$$

$$\lim_{z \to -(x-y)} \frac{1}{s!} \frac{d^{s-1}}{dz^{s-1}} \left(\frac{(z + r - \mu_2)^a(-z + r - \mu_2)^b(z + 1 - \mu_1)^c(-z + 1 - \mu_1)^d}{(-z + x + y)^r(-z + x - y)^r(z + x + y)^s} \right).$$

Solution

- The left-hand side of the Wetterich equation is

$$\begin{aligned}\partial_t \Gamma_k =& \dot{m}(\psi\chi^\dagger + \psi^\dagger\chi) - \dot{\mu}_1\psi^\dagger\psi - \mu_2\chi^\dagger\chi + i\dot{\lambda}_1\psi^\dagger(\psi^\dagger + \psi)\psi + \\ & i\dot{\lambda}_2\chi^\dagger(\chi^\dagger + \chi)\chi + i\dot{\lambda}_3\psi^\dagger(\chi^\dagger + \chi)\psi + i\dot{\lambda}_4\chi^\dagger(\psi^\dagger + \psi)\chi + \\ & i\dot{\lambda}_5((\chi^\dagger)^2\psi + \chi^2\psi^\dagger) + i\dot{\lambda}_6((\psi^\dagger)^2\chi + \psi^2\chi^\dagger)],\end{aligned} \tag{26}$$

- The right-hand side of the Wetterich equation, up to third order in the matrix V, is

$$\frac{1}{2} Tr[(\Gamma_k^{(2)} + R_k)^{-1}\dot{R}_k] = \alpha_1' k^D \left(Tr[\dot{R}GVGVG] + Tr[\dot{R}GVGVGVG] \right), \tag{27}$$

where we perform the following replacement, derived in the previous subsection,

$$\frac{(z + r - \mu_2)^a(-z + r - \mu_2)^b(z + 1 - \mu_1)^c(-z + 1 - \mu_1)^d}{(-z + x + y)^s(-z + x - y)^s(z + x + y)^r(z + x - y)^r} \to \Sigma Res[r, s, a, b, c, d]. \tag{28}$$

After collecting and filtering our results, we obtain the β functions.

$$\dot{\mu}_1 =(-2+\xi_1+\eta_1)\mu_1+\sum_{i,j}\lambda_i\lambda_j f_{1,ij} \quad ; \quad \dot{\mu}_2 = (-2+\xi_1+\eta_2)\mu_2+\sum_{i,j}\lambda_i\lambda_j f_{2,ij}$$

$$\dot{m} =(-2+\xi_1+\frac{1}{2}(\eta_1+\eta_2))m+\sum_{i,j}\lambda_i\lambda_j f_{m,ij}$$

$$\dot{\lambda}_1 =(-2+\frac{D}{2}+\xi_1+\frac{3}{2}\eta_1)\lambda_1+\sum_{i,j,k}\lambda_i\lambda_j\lambda_k f_{1,ijk} \ ; \ \dot{\lambda}_2 = (-2+\frac{D}{2}+\xi_1+\frac{3}{2}\eta_2)\lambda_2+\sum_{i,j,k}\lambda_i\lambda_j\lambda_k f_{2,ijk}$$

$$\dot{\lambda}_3 =(-2+\frac{D}{2}+\xi_1+\eta_1+\frac{1}{2}\eta_2)\lambda_3+\sum_{i,j,k}\lambda_i\lambda_j\lambda_k f_{3,ijk} \ ; \ \dot{\lambda}_4 = (-2+\frac{D}{2}+\xi_1+\eta_2+\frac{1}{2}\eta_1)\lambda_4+\sum_{i,j,k}\lambda_i\lambda_j\lambda_k f_{4,ijk}$$

$$\dot{\lambda}_5 =(-2+\frac{D}{2}+\xi_1+\eta_2+\frac{1}{2}\eta_1)\lambda_5+\sum_{i,j,k}\lambda_i\lambda_j\lambda_k f_{5,ijk} \ ; \ \dot{\lambda}_6 = (-2+\frac{D}{2}+\xi_1+\eta_1+\frac{1}{2}\eta_2)\lambda_6+\sum_{i,j,k}\lambda_i\lambda_j\lambda_k f_{6,ijk}.$$

Here $f_{k,ij} \equiv f_{k,ij}(\mu_1, \mu_2, m, r, N_D, A_D(\zeta_1, \eta_1), A_D(\zeta_2, \eta_2))$ denotes a certain function with poles at $(1+r-\mu_2-\mu_1)$ and $((1-\mu_1)(r-\mu_2)-m^2)$.

$$
\begin{aligned}
\dot{\mu}_1 = \Big(&\lambda_3^2 \left(r(-1+\mu_1)^2\left(m^2+(r-\mu_2)^2\right)+\left(m^2+(-1+\mu_1)^2\right)(r-\mu_2)^2\right)+\\
&\lambda_1^2\left(m^4+m^2 r\left(m^2+(r-\mu_2)^2\right)-m^2(r-\mu_2)(2+r-2\mu_1-\mu_2)+(r-\mu_2)^2(-1-r+\mu_1+\mu_2)^2\right)-\\
&2m\lambda_1\left(-r\lambda_3(-1+\mu_1)\left(m^2+(r-\mu_2)^2\right)+\left(-m^2+(r-\mu_2)(2+r-2\mu_1-\mu_2)\right)(r\lambda_3-m\lambda_5-\lambda_3\mu_2)+m r\lambda_5\left(m^2-(-1+\mu_1)(-1-2r+\mu_1+2\mu_2)\right)\right)-\\
&2m\lambda_3\lambda_5\left(\left(m^2+(-1+\mu_1)^2\right)(r-\mu_2)+r(1-\mu_1)\left(-m^2+(-1+\mu_1)(-1-2r+\mu_1+2\mu_2)\right)\right)+\\
&\lambda_5^2\left(m^2\left(m^2+(-1+\mu_1)^2\right)+r\left(m^4+(-1+\mu_1)^2(-1-r+\mu_1+\mu_2)^2-m^2(-1+\mu_1)(-1-2r+\mu_1+2\mu_2)\right)\right)+\\
&2\pi\mu_1(-1-r+\mu_1+\mu_2)^2\left(m^2+r(-1+\mu_1)+\mu_2-\mu_1\mu_2\right)^2(-2+\eta_1+\xi_1)\Big)\Big/\left(2\pi\left(m^2+(-1+\mu_1)(r-\mu_2)\right)^2(-1-r+\mu_1+\mu_2)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mu}_2 = &\\
\Big(&-2m\lambda_4\lambda_6\left(r(1-\mu_1)\left(m^2+(r-\mu_2)^2\right)+\left(-m^2+(r-\mu_2)(2+r-2\mu_1-\mu_2)\right)(r-\mu_2)\right)+\lambda_4^2\left(r(-1+\mu_1)^2\left(m^2+(r-\mu_2)^2\right)+\left(m^2+(-1+\mu_1)^2\right)(r-\mu_2)^2\right)+\\
&\lambda_6^2\left(m^4+m^2 r\left(m^2+(r-\mu_2)^2\right)-m^2(r-\mu_2)(2+r-2\mu_1-\mu_2)+(r-\mu_2)^2(-1-r+\mu_1+\mu_2)^2\right)-\\
&2m\lambda_2\left(m\lambda_6\left(m^2-(r-\mu_2)(2+r-2\mu_1-\mu_2)\right)+\lambda_4\left(m^2+(-1+\mu_1)^2\right)(r-\mu_2)-r(m\lambda_6+\lambda_4(-1+\mu_1))\left(-m^2+(-1+\mu_1)(-1-2r+\mu_1+2\mu_2)\right)\right)+\\
&\lambda_2^2\left(m^2\left(m^2+(-1+\mu_1)^2\right)+r\left(m^4+(-1+\mu_1)^2(-1-r+\mu_1+\mu_2)^2-m^2(-1+\mu_1)(-1-2r+\mu_1+2\mu_2)\right)\right)+\\
&2\pi\mu_2(-1-r+\mu_1+\mu_2)^2\left(m^2+r(-1+\mu_1)+\mu_2-\mu_1\mu_2\right)^2(-2+\eta_2+\xi_1)\Big)\Big/\left(2\pi\left(m^2+(-1+\mu_1)(r-\mu_2)\right)^2(-1-r+\mu_1+\mu_2)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{m} = \Big(&-\lambda_3\left(m^2+(-1+\mu_1)(r-\mu_2)\right)^2\\
&\left(r(m\lambda_6+\lambda_4(-1+\mu_1))(1-\mu_1)\left(m^2+(r-\mu_2)^2\right)-\left(m\lambda_6\left(m^2-(r-\mu_2)(2+r-2\mu_1-\mu_2)\right)+\lambda_4\left(m^2+(-1+\mu_1)^2\right)(r-\mu_2)\right)(r-\mu_2)\right)+\\
&\lambda_5\lambda_6\left(m^3+m(-1+\mu_1)(r-\mu_2)\right)^2\left(-m^2-m^2 r+(r-\mu_2)(2+r-2\mu_1-\mu_2)+r(-1+\mu_1)(-1-2r+\mu_1+2\mu_2)\right)-\\
&m\lambda_2\lambda_3\left(m^2+(-1+\mu_1)(r-\mu_2)\right)^2\left(\left(m^2+(-1+\mu_1)^2\right)(r-\mu_2)+r(1-\mu_1)\left(-m^2+(-1+\mu_1)(-1-2r+\mu_1+2\mu_2)\right)\right)-\\
&m\lambda_4\lambda_5\left(m^2+(-1+\mu_1)(r-\mu_2)\right)^2\left(\left(m^2+(-1+\mu_1)^2\right)(r-\mu_2)+r(1-\mu_1)\left(-m^2+(-1+\mu_1)(-1-2r+\mu_1+2\mu_2)\right)\right)+\lambda_2\lambda_5\left(m^2+(-1+\mu_1)(r-\mu_2)\right)^2\\
&\left(m^2\left(m^2+(-1+\mu_1)^2\right)+r\left(m^4+(-1+\mu_1)^2(-1-r+\mu_1+\mu_2)^2-m^2(-1+\mu_1)(-1-2r+\mu_1+2\mu_2)\right)\right)+\lambda_1\left(m^2+(-1+\mu_1)(r-\mu_2)\right)^2\\
&\left(m\left(-m^2+(r-\mu_2)(2+r-2\mu_1-\mu_2)\right)(m\lambda_2+\lambda_4(-r+\mu_2))+\lambda_6\left(m^4-m^2(r-\mu_2)(2+r-2\mu_1-\mu_2)+(r-\mu_2)^2(-1-r+\mu_1+\mu_2)^2\right)+\right.\\
&\left.m r\left(m^3(-\lambda_2+\lambda_6)+m^2\lambda_4(-1+\mu_1)+m\lambda_6(r-\mu_2)^2+\lambda_4(-1+\mu_1)(r-\mu_2)^2+m\lambda_2(-1+\mu_1)(-1-2r+\mu_1+2\mu_2)\right)\right)+\\
&m\pi(-1-r+\mu_1+\mu_2)^2\left(m^2+r(-1+\mu_1)+\mu_2-\mu_1\mu_2\right)^4\left(-4-\eta_1+\eta_2+2(\eta_1+\xi_1)\right)\Big)\Big/\left(2\pi\left(m^2+(-1+\mu_1)(r-\mu_2)\right)^4(-1-r+\mu_1+\mu_2)^2\right)
\end{aligned}
$$

$$\dot{\lambda}_1 = (-2\,\lambda_1^3\,(m^2 + r\,(-1+\mu_1) + \mu_2 - \mu_1\,\mu_2)^3\,(-2\,m^6 - 2\,m^6\,r - 3\,m^4\,r\,(-1+\mu_1) - 4\,m^4\,r^2\,(-1+\mu_1) + 2\,m^4\,r\,(r-\mu_2) - 3\,m^4\,(-1+\mu_1)\,(r-\mu_2) -$$

$$6\,m^2\,(-1+\mu_1)^2\,(r-\mu_2)^2 + m^2\,r^2\,(r-\mu_2)^3 + 6\,m^2\,(-1+\mu_1)\,(r-\mu_2)^3 - 3\,m^2\,r\,(-1+\mu_1)\,(r-\mu_2)^3 - 2\,m^2\,(r-\mu_2)^4 +$$

$$m^2\,r\,(r-\mu_2)^3\,(1+r-\mu_1-\mu_2) + 2\,(r-\mu_2)^3\,(1+r-\mu_1-\mu_2)^3 + 3\,m^4\,(-1+\mu_1)\,\mu_2 + 4\,m^4\,r\,(-1+\mu_1)\,\mu_2 - 2\,m^4\,(r-\mu_2)\,\mu_2 - m^2\,r\,(r-\mu_2)^3\,\mu_2) +$$

$$\lambda_4\,\lambda_5^2\,(-m^2 + r - r\,\mu_1 + (-1+\mu_1)\,\mu_2)^3\,(-2\,m^6 - 2\,m^6\,r - 2\,m^4\,r\,(-1+\mu_1) - m^2\,(-1+\mu_1)^3 - 2\,m^2\,r\,(-1+\mu_1)^4 + 2\,m^4\,r\,(-1+\mu_1)\,\mu_1 +$$

$$m^2\,(-1+\mu_1)^3\,\mu_1 + 4\,m^4\,(r-\mu_2) + 3\,m^4\,r\,(r-\mu_2) - 3\,m^4\,r\,(-1+\mu_1)\,(r-\mu_2) - 3\,m^2\,(-1+\mu_1)^3\,(r-\mu_2) + 6\,m^2\,r\,(-1+\mu_1)^3\,(r-\mu_2) - 4\,m^4\,\mu_1\,(r-\mu_2) -$$

$$3\,m^4\,r\,\mu_1\,(r-\mu_2) - 6\,m^2\,r\,(-1+\mu_1)^2\,(r-\mu_2)^2 + m^2\,(-1+\mu_1)^3\,(-1-r+\mu_1+\mu_2) + 2\,r\,(-1+\mu_1)^3\,(-1-r+\mu_1+\mu_2)^3) + m\,\lambda_3^3\,(m^2 + (-1+\mu_1)\,(r-\mu_2))^3$$

$$(1+r-\mu_1-\mu_2)\,((r-\mu_2)^2\,(2+m^2+r-(4+r)\,\mu_1 + 2\,\mu_1^2 + (-1+\mu_1)\,\mu_2) + r\,(-1+\mu_1)^2\,(m^2 + 2\,r^2 + (-1+\mu_1)\,\mu_2 + 2\,\mu_2^2 - r\,(-1+\mu_1+4\,\mu_2))) -$$

$$2\,m\,\lambda_5^2\,\lambda_6\,(1-\mu_1)\,(m^2 + (-1+\mu_1)\,(r-\mu_2))^3\,(1+r-\mu_1-\mu_2)$$

$$(m^2\,(-1+\mu_1) - 3\,r^2\,(-1+\mu_1)^2 + (1+2\,m^2-2\,\mu_1+\mu_1^2)\,\mu_2 + r\,(-3-m^2+2\,\mu_1^3+3\,\mu_2+\mu_1^2\,(-7+3\,\mu_2) - \mu_1\,(-8+m^2+6\,\mu_2))) -$$

$$\lambda_3\,\lambda_5\,(m^2 + (-1+\mu_1)\,(r-\mu_2))^3\,(-2\,\lambda_6\,(1+r-\mu_1-\mu_2)\,(2\,r^3\,(-1+\mu_1)^3 - r^2\,(-1+\mu_1)\,(-2-6\,m^2+\mu_1^3+\mu_1\,(5+3\,m^2-8\,\mu_2) + 4\,\mu_1^2\,(-1+\mu_2) + 4\,\mu_2) +$$

$$\mu_2\,(m^2\,(3+m^2-6\,\mu_1+3\,\mu_1^2) + (-1+\mu_1)\,(1+3\,m^2-2\,\mu_1+\mu_1^2)\,\mu_2) +$$

$$r\,(-m^2\,(6+(-15+m^2)\,\mu_1 + 12\,\mu_1^2 - 3\,\mu_1^3) + (3-4\,\mu_1+\mu_1^2)\,(1+3\,m^2-2\,\mu_1+\mu_1^2)\,\mu_2 + 2\,(-1+\mu_1)^3\,\mu_2^2)) + m\,\lambda_5\,(1-\mu_1)\,(1+r-\mu_1-\mu_2)$$

$$(m^2\,(-1+\mu_1) - 3\,r^2\,(-1+\mu_1)^2 + (1+2\,m^2-2\,\mu_1+\mu_1^2)\,\mu_2 + r\,(-3-m^2+2\,\mu_1^3+3\,\mu_2+\mu_1^2\,(-7+3\,\mu_2) - \mu_1\,(-8+m^2+6\,\mu_2))) +$$

$$2\,m\,\lambda_4\,(-2\,r\,(1-\mu_1)\,(1-m^2+m^4+3\,r-3\,m^2\,r+3\,r^2 + (-4-9\,r-6\,r^2+m^2\,(2+3\,r))\,\mu_1 + (6-m^2+9\,r+3\,r^2)\,\mu_1^2 -$$

$$(4+3\,r)\,\mu_1^3 + \mu_1^4 + 3\,(-1+\mu_1)\,(1-m^2-2\,r\,(-1+\mu_1) - 2\,\mu_1+\mu_1^2)\,\mu_2 + 3\,(-1+\mu_1)^2\,\mu_2^2) +$$

$$(1+m^2-2\,\mu_1+\mu_1^2)\,(m^2\,(-1+\mu_1) + 3\,r^2\,(-1+\mu_1) + (1-m^2-2\,\mu_1+\mu_1^2)\,\mu_2 + 3\,(-1+\mu_1)\,\mu_2^2 + r\,(-1+m^2-\mu_1^2+\mu_1\,(2-6\,\mu_2)+6\,\mu_2)))) -$$

$$\lambda_3^2\,(m^2 + (-1+\mu_1)\,(r-\mu_2))^3\,(-\lambda_5\,(1+r-\mu_1-\mu_2)\,(2\,r^3\,(-1+\mu_1)^3 - r^2\,(-1+\mu_1)\,(-2-6\,m^2+\mu_1^3+\mu_1\,(5+3\,m^2-8\,\mu_2) + 4\,\mu_1^2\,(-1+\mu_2) + 4\,\mu_2) +$$

$$\mu_2\,(m^2\,(3+m^2-6\,\mu_1+3\,\mu_1^2) + (-1+\mu_1)\,(1+3\,m^2-2\,\mu_1+\mu_1^2)\,\mu_2) +$$

$$r\,(-m^2\,(6+(-15+m^2)\,\mu_1 + 12\,\mu_1^2 - 3\,\mu_1^3) + (3-4\,\mu_1+\mu_1^2)\,(1+3\,m^2-2\,\mu_1+\mu_1^2)\,\mu_2 + 2\,(-1+\mu_1)^3\,\mu_2^2)) -$$

$$2\,m\,\lambda_6\,(1+r-\mu_1-\mu_2)\,((r-\mu_2)^2\,(2+m^2+r-(4+r)\,\mu_1 + 2\,\mu_1^2 + (-1+\mu_1)\,\mu_2) + r\,(-1+\mu_1)^2\,(m^2 + 2\,r^2 + (-1+\mu_1)\,\mu_2 + 2\,\mu_2^2 - r\,(-1+\mu_1+4\,\mu_2))) +$$

$$2\,\lambda_4\,(1-\mu_1)\,((1+m^2-2\,\mu_1+\mu_1^2)\,(r-\mu_2)\,(m^2+r^2-2\,r\,\mu_2+\mu_2^2) -$$

$$r\,(1-\mu_1)\,(r^3\,(-1+\mu_1) + m^2\,(-1+m^2+2\,\mu_1-\mu_1^2) - 3\,m^2\,(-1+\mu_1)\,\mu_2 - 3\,r^2\,(-1+\mu_1)\,\mu_2 - (-1+\mu_1)\,\mu_2^3 + 3\,r\,(-1+\mu_1)\,(m^2+\mu_2^2)))) +$$

$$m\,\lambda_1^2\,(m^2 + (-1+\mu_1)\,(r-\mu_2))^3\,(2\,\lambda_6\,(r-\mu_2)\,(1+r-\mu_1-\mu_2)\,(-r^3\,(-3+\mu_1) + \mu_2\,(m^2 - 3\,(-1+\mu_1)\,\mu_2 - 2\,\mu_2^2) +$$

$$r\,(m^2\,(1-2\,\mu_1) + (-6-m^2+6\,\mu_1)\,\mu_2 - (-7+\mu_1)\,\mu_2^2) + r^2\,(3+m^2-8\,\mu_2+\mu_1\,(-3+2\,\mu_2))) -$$

$$2\,m\,\lambda_4\,((m^2+r^2-2\,r\,\mu_2+\mu_2^2)\,(-m^2+r^2+2\,(-1+\mu_1)\,\mu_2+\mu_2^2-2\,r\,(-1+\mu_1+\mu_2)) -$$

$$r\,(r^3\,(-1+\mu_1) + m^2\,(-1+m^2+2\,\mu_1-\mu_1^2) - 3\,m^2\,(-1+\mu_1)\,\mu_2 - 3\,r^2\,(-1+\mu_1)\,\mu_2 - (-1+\mu_1)\,\mu_2^3 + 3\,r\,(-1+\mu_1)\,(m^2+\mu_2^2))) -$$

$$8\,m\,\lambda_5\,(m^4 - r^4\,(-2+\mu_1) + r^3\,(6+3\,\mu_1^2+3\,\mu_1\,(-3+\mu_2) - 7\,\mu_2) - 3\,m^2\,(-1+\mu_1)\,\mu_2 + (3-m^2-6\,\mu_1+3\,\mu_1^2)\,\mu_2^2 + 3\,(-1+\mu_1)\,\mu_2^3 +$$

$$\mu_2^4 + r\,(m^2\,(-3+m^2+3\,\mu_1) + (-7+4\,m^2+15\,\mu_1-2\,m^2\,\mu_1-9\,\mu_1^2+\mu_1^3)\,\mu_2 + 3\,(4-5\,\mu_1+\mu_1^2)\,\mu_2^2 + (-5+\mu_1)\,\mu_2^3) -$$

$$r^2\,(-4+3\,m^2+\mu_1^3+6\,\mu_1^2\,(-1+\mu_2) + 15\,\mu_2 - 9\,\mu_2^2 + \mu_1\,(9-2\,m^2-21\,\mu_2+3\,\mu_2^2)))) + \lambda_3$$

$$(-5\,r^5\,(-3+\mu_1) + r^2\,(m^2\,(-11+4\,m^2-3\,\mu_1+14\,\mu_1^2) + (-81+21\,m^2+6\,(27+m^2)\,\mu_1 - 81\,\mu_1^2)\,\mu_2 + 3\,(71+5\,m^2-84\,\mu_1+13\,\mu_1^2)\,\mu_2^2 + 20\,(-6+\mu_1)\,\mu_2^3) -$$

$$\mu_2\,(8\,m^4 - 25\,m^2\,(-1+\mu_1)\,\mu_2 + 9\,(3-m^2-6\,\mu_1+3\,\mu_1^2)\,\mu_2^2 + 29\,(-1+\mu_1)\,\mu_2^3 + 10\,\mu_2^4) - r\,(-4\,m^4\,(1+\mu_1) + 2\,m^2\,(2\,(-9+m^2) + 11\,\mu_1 + 7\,\mu_1^2)\,\mu_2 +$$

$$3\,(-27+8\,m^2+(54+m^2)\,\mu_1 - 27\,\mu_1^2)\,\mu_2^2 + (129+5\,m^2-142\,\mu_1+13\,\mu_1^2)\,\mu_2^3 + 5\,(-11+\mu_1)\,\mu_2^4) + r^4\,(42+5\,m^2+13\,\mu_1^2 - 70\,\mu_2 +$$

$$5\,\mu_1\,(-11+4\,\mu_2)) - r^3\,(-27+6\,m^2+5\,(31+3\,m^2)\,\mu_2 - 130\,\mu_2^2 + 3\,\mu_1^2\,(-9+13\,\mu_2) + \mu_1\,(54+3\,m^2-194\,\mu_2+30\,\mu_2^2)))) +$$

$$\lambda_1\,(m^2 + (-1+\mu_1)\,(r-\mu_2))^3\,(-4\,m^2\,\lambda_5^2\,(-m^4 + m^2\,r^2 + r\,(1+m^2-2\,\mu_1+\mu_1^2)\,(1-m^2-2\,r\,(-1+\mu_1) + \mu_1^2 + 2\,\mu_1\,(-1+\mu_2) - 2\,\mu_2) +$$

$$(-1+\mu_1)\,(1+3\,m^2-2\,\mu_1+\mu_1^2)\,\mu_2 + m^2\,\mu_2^2 - r\,(-1+3\,(1+m^2)\,\mu_1 - 3\,\mu_1^2 + \mu_1^3 + m^2\,(-3+2\,\mu_2))) + 4\,m\,\lambda_5\,\lambda_6\,(1+r-\mu_1-\mu_2)$$

$$((r-\mu_2)^2\,(2+m^2+r-(4+r)\,\mu_1 + 2\,\mu_1^2 + (-1+\mu_1)\,\mu_2) + r\,(-1+\mu_1)^2\,(m^2 + 2\,r^2 + (-1+\mu_1)\,\mu_2 + 2\,\mu_2^2 - r\,(-1+\mu_1+4\,\mu_2))) + 4\,m^2\,\lambda_4\,\lambda_5$$

$$(-m^4 + r^3\,(-4+7\,\mu_1-3\,\mu_1^2) - (-1+\mu_1)\,(1-2\,m^2-2\,\mu_1+\mu_1^2)\,\mu_2 - 3\,(-1+\mu_1)^2\,\mu_2^2 - (-1+\mu_1)\,\mu_2^3 + 3\,r^2\,(-1+\mu_1)\,(2-m^2+\mu_1^2-3\,\mu_2+\mu_1\,(-3+2\,\mu_2)) -$$

$$r\,(2-3\,m^2+m^4+\mu_1^4+3\,(-3+m^2)\,\mu_2 + 6\,\mu_2^2 + \mu_1^3\,(-5+3\,\mu_2) + \mu_1\,(-7+4\,m^2-3\,(-7+m^2)\,\mu_2 - 9\,\mu_2^2) + \mu_1^2\,(9-m^2-15\,\mu_2+3\,\mu_2^2))) +$$

$$\lambda_3^2\,(r\,(1-\mu_1)\,(m^4\,(-1+\mu_1) + r^4\,(-1+\mu_1) + m^2\,(7+5\,m^2-14\,\mu_1+7\,\mu_1^2)\,\mu_2 + 6\,m^2\,(-1+\mu_1)\,\mu_2^2 + (5+7\,m^2-10\,\mu_1+5\,\mu_1^2)\,\mu_2^3 +$$

$$(-1+\mu_1)\,\mu_2^4 + 3\,r^2\,(2\,m^2\,(-1+\mu_1) + (5+7\,m^2-10\,\mu_1+5\,\mu_1^2)\,\mu_2 + 2\,(-1+\mu_1)\,\mu_2^2) - r\,(m^2\,(7+5\,m^2-14\,\mu_1+7\,\mu_1^2) +$$

$$12\,m^2\,(-1+\mu_1)\,\mu_2 + 3\,(5+7\,m^2-10\,\mu_1+5\,\mu_1^2)\,\mu_2^2 + 4\,(-1+\mu_1)\,\mu_2^3) - r^3\,(5+7\,m^2+5\,\mu_1^2-4\,\mu_2+2\,\mu_1\,(-5+2\,\mu_2))) +$$

$$(r-\mu_2)\,(-m^4\,(-1+\mu_1) + r^4\,(-1+\mu_1) + m^2\,(3-5\,m^2-6\,\mu_1+3\,\mu_1^2)\,\mu_2 + 6\,(-1+\mu_1)\,(1+3\,m^2-2\,\mu_1+\mu_1^2)\,\mu_2^2 + (3+7\,m^2-6\,\mu_1+3\,\mu_1^2)\,\mu_2^3 +$$

$$(-1+\mu_1)\,\mu_2^4 - r\,(m^2\,(3-5\,m^2-6\,\mu_1+3\,\mu_1^2) + 12\,(-1+\mu_1)\,(1+3\,m^2-2\,\mu_1+\mu_1^2)\,\mu_2 + 3\,(3+7\,m^2-6\,\mu_1+3\,\mu_1^2)\,\mu_2^2 + 4\,(-1+\mu_1)\,\mu_2^3) -$$

$$r^3\,(3+7\,m^2+3\,\mu_1^2-4\,\mu_2+\mu_1\,(-6+4\,\mu_2)) + 3\,r^2\,(-2-6\,m^2+2\,\mu_1^3+3\,\mu_1^2\,(-2+\mu_2) + 3\,\mu_2 + 7\,m^2\,\mu_2 - 2\,\mu_2^2 + 2\,\mu_1\,(3+3\,m^2-3\,\mu_2+\mu_2^2)))) -$$

$$2\,\lambda_3\,(-\lambda_6\,(1+r-\mu_1-\mu_2)\,(-r^4\,(2-3\,\mu_1+\mu_1^2) - 3\,r^2\,(m^2\,(2-3\,\mu_1+\mu_1^2) + (-2-5\,m^2+2\,(2+m^2)\,\mu_1 - 2\,\mu_1^2)\,\mu_2 + (3-4\,\mu_1+\mu_1^2)\,\mu_2^2) +$$

$$\mu_2\,(-m^4 + 3\,m^2\,(-1+\mu_1)\,\mu_2 + (2+3\,m^2-4\,\mu_1+2\,\mu_1^2)\,\mu_2^2 + (-1+\mu_1)\,\mu_2^3) + r\,(m^4\,\mu_1 + 3\,m^2\,(3-4\,\mu_1+\mu_1^2)\,\mu_2 -$$

$$3\,(2+4\,m^2-(4+m^2)\,\mu_1+2\,\mu_1^2)\,\mu_2^2 + (5-6\,\mu_1+\mu_1^2)\,\mu_2^3) + r^3\,(-2-6\,m^2+\mu_1\,(4+3\,m^2-10\,\mu_2) + 7\,\mu_2 + \mu_1^2\,(-2+3\,\mu_2))) +$$

$$m\,\lambda_4\,((m^2+r^2-2\,r\,\mu_2+\mu_2^2)\,(-m^2\,(-1+\mu_1) + r^2\,(-1+\mu_1) - r\,(3+m^2+3\,\mu_1^2+2\,\mu_1\,(-3+\mu_2) - 2\,\mu_2) + (3+m^2-6\,\mu_1+3\,\mu_1^2)\,\mu_2 + (-1+\mu_1)\,\mu_2^2) +$$

$$2\,r\,(1-\mu_1)\,(r^3\,(-1+\mu_1) + m^2\,(-1+m^2+2\,\mu_1-\mu_1^2) - 3\,m^2\,(-1+\mu_1)\,\mu_2 - 3\,r^2\,(-1+\mu_1)\,\mu_2 - (-1+\mu_1)\,\mu_2^3 + 3\,r\,(-1+\mu_1)\,(m^2+\mu_2^2))) +$$

$$m\,\lambda_5\,(r^4\,(-3+5\,\mu_1-2\,\mu_1^2) + r^3\,(-12-11\,m^2+9\,\mu_1^3+\mu_1\,(33+6\,m^2-16\,\mu_2) + 6\,\mu_1^2\,(-5+\mu_2) + 10\,\mu_2) + \mu_2\,(-4\,m^4 +$$

$$(-1+\mu_1)\,(6+13\,m^2-12\,\mu_1+6\,\mu_1^2)\,\mu_2 + (3+5\,m^2-6\,\mu_1+3\,\mu_1^2)\,\mu_2^2 + (-1+\mu_1)\,\mu_2^3) + r\,(m^2\,(-3+2\,m^2+(9+2\,m^2)\,\mu_1 - 9\,\mu_1^2 + 3\,\mu_1^3) + (15 +$$

$$27\,m^2 - 2\,m^4 - 4\,(12+7\,m^2)\,\mu_1 + (54+m^2)\,\mu_1^2 - 24\,\mu_1^3 + 3\,\mu_1^4)\,\mu_2 + 3\,(-6-7\,m^2+(15+2\,m^2)\,\mu_1 - 12\,\mu_1^2 + 3\,\mu_1^3)\,\mu_2^2 + 2\,(3-4\,\mu_1+\mu_1^2)\,\mu_2^3) -$$

$$r^2\,(9+14\,m^2-2\,m^4+3\,\mu_1^4+18\,\mu_1^3\,(-1+\mu_2) - 27\,(1+m^2)\,\mu_2 + 12\,\mu_2^2 + \mu_1^2\,(36+m^2-63\,\mu_2+6\,\mu_2^2) - 3\,\mu_1\,(5\,(2+m^2) - 4\,(6+m^2)\,\mu_2 + 6\,\mu_2^2))))) +$$

$$\pi\,\lambda_1\,(1+r-\mu_1-\mu_2)^3\,(m^2 + r\,(-1+\mu_1) + \mu_2 - \mu_1\,\mu_2)^6\,(-2+\eta_1+2\,(\eta_1+\xi_1)))\,/\,(2$$

$$\pi$$

$$(1+r-\mu_1-\mu_2)^3$$

$$(m^2 + r\,(-1+\mu_1) + \mu_2 - \mu_1\,\mu_2)^6)$$

Algorithm to find fixed points

- Since we have 10 nonlinear differential equations, studying the flow in this space becomes complicated. Therefore, we will compute the fixed points (β_i=0) and study the evolution around these points.
- The most standard methods to find fixed points (e.g. Newton–Raphson) require starting close to the point to obtain high precision. In a 10-dimensional space these algorithms fail. For this reason we use the 'neural network' algorithm of Li and Zhen, which has very good convergence to find fixed points for random initial values.
- After this algorithm brings us close to the fixed points, we apply Newton–Raphson to achieve higher precision ($\beta_i < 10^{-16}$).
- This yields a collection of fixed points, among which we are interested in those that are predominantly IR attractors.

- The behaviour near the FPs is described (at first order) through the stability matrix, which is defined from the expansion of the β functions around a FP. The first order of this expansion is

$$\Delta\beta_i = \left|\frac{\partial\beta_i}{\partial\Lambda_j}\right|_{\Lambda=\beta*} \Delta\Lambda_j + \dots \tag{29}$$

where Λ_j – with $i, j = 10$ – are all the parameters on which the β functions depend.

- The matrix associated with the derivatives of the parameters is called the stability matrix

$$M_{ij} \equiv \left|\frac{\partial\beta_i}{\partial\Lambda_j}\right|_{\Lambda=\beta*} \tag{30}$$

- In the IR (UV) case, if the eigenvalue associated with the direction defined by an eigenvector is positive (negative), this direction is attractive; otherwise it is repulsive.

- The IR-attractor fixed point with $\eta_i \neq 0$ that we analyse is

μ_1	μ_2	m	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	r
0.1384	0.1384	-0.0056	1.0118	1.0118	0.8150	0.8150	-0.1967	-0.1967	1.

- which has the following eigenvalues

2.50	-2.23	-2.09	-2.03	1.94	1.41	0.36	-0.11	0.10	0.03

- We also compute the case $\eta_i = 0$, where the fixed point is

μ_1	μ_2	m	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	r
0.1084	0.1084	-0.0042	0.8631	0.8631	0.6967	0.6967	-0.1664	-0.1664	1.

- which has the eigenvalues

4.25	3.37	1.95	-1.89	-1.74	-1.69	0.89	0.62	0.53	-0.03

Evolution around the fixed points

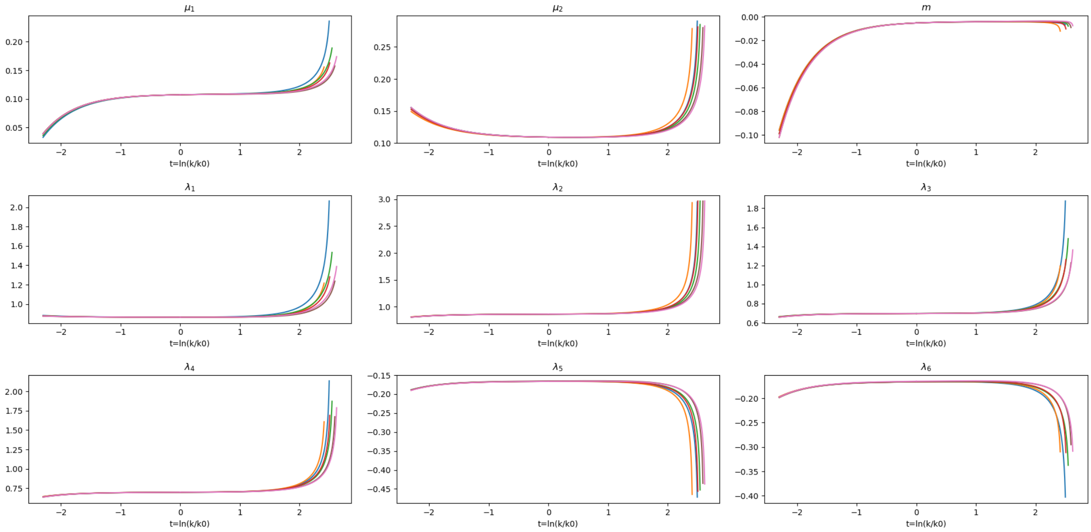
- With the selected points, we focus on the change of the parameters in the subspace of the intercepts (i.e. the subspace of the 'mass' parameters μ_1, μ_2 and m),
- This 3-dimensional subspace has 2^3 regions. Therefore, to sample each region of this subspace, we generate 8 points located at the vertices of a cube, with the FP at the centre of the cube and semi-edge length a.
- Each flow associated with each point exhibits different behaviours for the intercepts and m. We choose the point associated with the following transformation

$$\mu_1 \to \mu_1 - a, \qquad \mu_2 \to \mu_2 + a, \qquad m \to m - a,$$

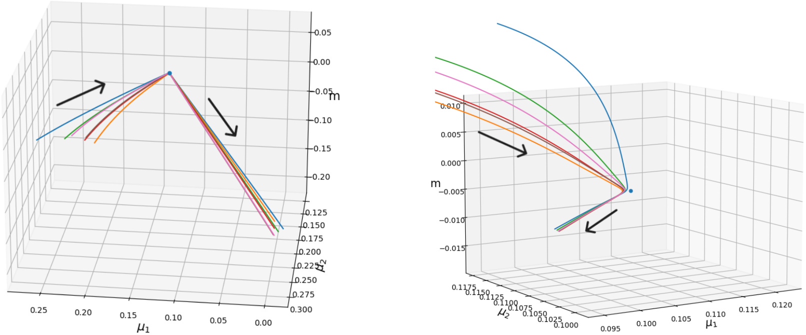
- We then analyse the behaviour of the parameters as each coupling λ is varied. For this purpose, 6 different trajectories are generated, in each of which a is added to a different λ. The colour representing each trajectory is determined by the λ that has been modified.

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6

- The evolution of the parameters for the case $\eta_i = 0$, with $a = 0.001$ and $\Delta t = 0.001$

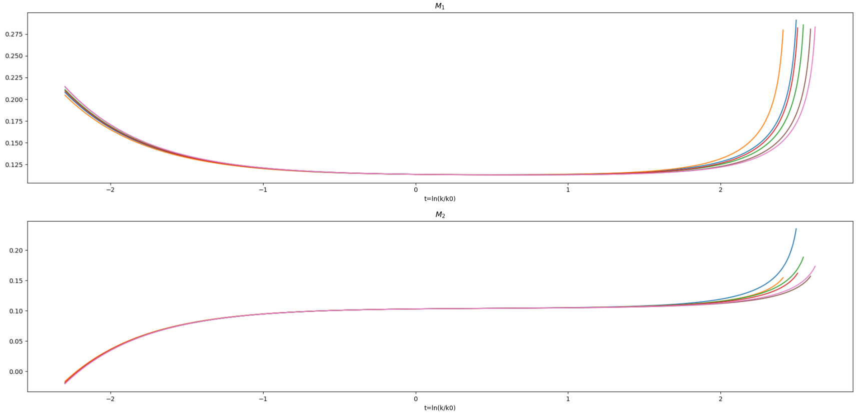


3D projection of the intercept subspace. The arrows indicate the direction in which the system evolves in the IR limit (i.e. negative t). On the left $\eta_i \neq 0$ and on the right $\eta_i = 0$.



- The new diagonalized masses are given by

$$M_1 \equiv \frac{1}{2}(\mu_1 + \mu_2 + \sqrt{(\mu_1 - \mu_2)^2 + 4m^2}) \; M_2 \equiv \frac{1}{2}(\mu_1 + \mu_2 - \sqrt{(\mu_1 - \mu_2)^2 + 4m^2}) \tag{31}$$



Conclusion

- In this work, we developed a method to obtain the β functions by solving the Wetterich equation. Therefore, extending the analysis to higher orders in the potential or to more fields becomes more straightforward.
- We found different flows that exhibit the evolution from a state with decoupled fields in the UV to the fixed point in the IR. These represent two different states of the *pomeron*.
- In future work, we aim to further develop methods to connect two different states in theory space. This can be achieved by finding the connection between a UV fixed point and the IR fixed point studied in this thesis.
- We find that the values of μ_i that fall within the range of phenomenological models are those given by the fixed point with $\eta_i = 0$. Likewise, for the case $\eta_i \neq 0$, at the fixed point all the couplings increase their values, but the properties of the point remain unchanged.
- With this *pomeron* model, we would like to study how it couples to the proton. Thus, we plan to investigate these processes and compute observables that can be compared with experimental values.

Acknowledgements

- This work was carried out thanks to the support of the 'Programa de Incentivo a la Iniciación Científica' (PIIC) of the 'Universidad Técnica Federico Santa María' (UTFSM), and of the Fondecyt Project No. 1191434, in which I participated as a thesis student.
- The author also thank San Sebastián University and Fondecyt Project No. 1220035 for funding the trip and accommodation for this workshop.