Universality of the chiral soliton lattice in the low energy limit of QCD

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Motivation

The Chiral soliton lattice (ChSL)

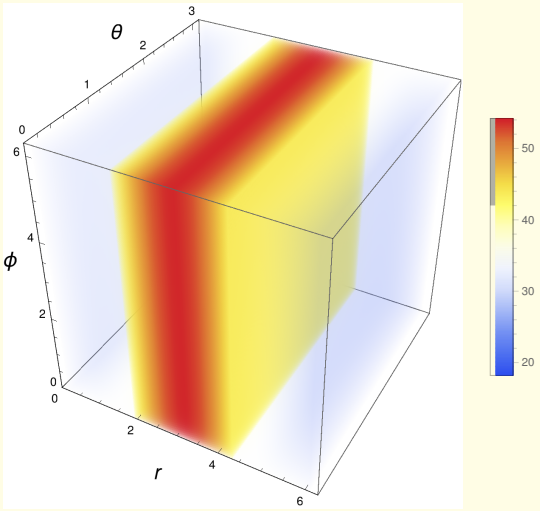
- The ChSL is a lattice of aligned parallel domain walls -two-dimensional topological solitons- separated at equal space intervals. I. Dzyaloshinskii, Sov. Phys. JETP **19** no. 4, (1964) 960–971.
- It emerges in certain materials or physical systems, especially under extreme conditions such as the presence of intense magnetic fields or in high-density particle physics. J. O. Andersen, W. R. Naylor and A. Tranberg, Rev. Mod. Phys. **88**, 025001 (2016).
- In the context of QCD, the ChSL is the ground state of QCD at finite baryon chemical potential in the presence of a critical homogeneous magnetic field. T. Brauner and N. Yamamoto, JHEP **04**, 132 (2017).

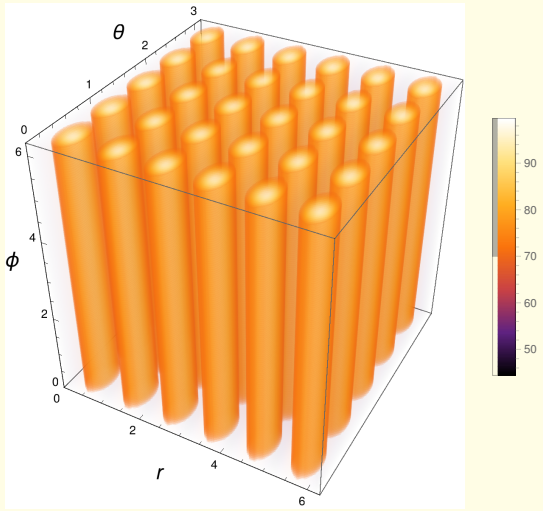
The question: *Is it possible to derive the ChSL, with all its virtues, from a realistic, effective model?*

Motivation

The Skyrme model

- The Skyrme model is a non-linear effective field theory that represents the low energy limit of QCD at leading order in the 't Hooft expansion. *T. H. R. Skyrme, Nucl. Phys. **31**, 556-569 (1962).*
- It allows to describe the static properties of nucleons. G. S. Adkins, C. R. Nappi and E. Witten, Nucl. Phys. B **228**, 552 (1983).
- It has a topological charge that corresponds to the baryon number. E. Witten, Nucl. Phys. B **223**, 422-432 (1983).
- Although it is a non-linear theory with coupled partial differential equations, it admits exact solutions describing baryons at a finite volume. F. Canfora, M. Lagos and A. Vera, Eur. Phys. J. C **80**, no.8, 697 (2020).





The gauged (generalized) Skyrme model

The gauged generalized Skyrme model is described by the action

$$I(U,A_\mu) = \frac{f_\pi^2}{4}\int_{\mathcal{M}} d^4x\sqrt{-g}\Big[\mathrm{Tr}\left(R_\mu R^\mu + \frac{\lambda}{8}G_{\mu\nu}G^{\mu\nu}\right) - m_\pi^2\,\mathrm{Tr}(2\mathbb{I}-U-U^{-1})\Big] - \frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} + I_{\mathrm{corr}}\ ,$$

where $U(x)\in SU(2)$, A_μ is the electromagnetic four-potential, and

$$R_\mu = U^{-1}D_\mu U\ , \qquad G_{\mu\nu} = [R_\mu, R_\nu]\ , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\ .$$

The covariant derivative is defined as $D_\mu U = \nabla_\mu U + A_\mu\,[t_3, U]$, where $t_a = i\sigma_a$.

Here, the coupling f_π is the pions decay constant, m_π is the pions mass, and λ is a positive number fixed experimentally. Moreover, I_{corr} represents the sub-leading corrections that come from the large N_c expansion.

The field equations of the system are

$$D_\mu\left(R^\mu + \frac{\lambda}{4}[R_\nu, G^{\mu\nu}]\right) - \frac{m_\pi^2}{2}(U-U^{-1}) = 0\ ,$$

$$\nabla_\mu F^{\mu\nu} = -\frac{f_\pi^2}{4}\mathrm{Tr}\Big[[t_3,U]\Big(R_\mu + \frac{\lambda}{4}[R^\nu, G_{\mu\nu}]\Big)U^{-1}\Big]\ .$$

The gauged (generalized) Skyrme model

The topological current ρ^μ is written as

$$\rho^\mu = \epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\Big[R_\nu R_\alpha R_\beta - 3\nabla_\nu[A_\alpha t_3(U^{-1}\nabla_\beta U + (\nabla_\beta U)U^{-1})]\Big] \ .$$

The first term is the usual topological density, while the second is the Callan-Witten term. Integrating, we obtain the topological (baryonic) charge

$$n_{\mathrm{B}} = \frac{1}{24\pi^2}\int_\Sigma \rho^0 \ .$$

The baryon chemical potential, μ_B, is added via the Wess-Zumino-Witten term, given by

$$I_{\mathrm{WZW}} = \frac{1}{24\pi^2}\int d^4x\left(\mu_B\rho^0 - \frac{1}{2}A_\mu\rho^\mu\right) \ .$$

The Ansatz

A general element of $SU(2)$ in the exponential representation is written as

$$U^{\pm 1}(x^\mu) = \cos(\alpha)\,\mathbf{1}_2 \pm \sin(\alpha)\,n^i t_i \qquad \text{with} \qquad \begin{cases} n^1 = \sin\Theta\cos\Phi\ , \\ n^2 = \sin\Theta\sin\Phi\ , \\ n^3 = \cos\Theta\ , \end{cases}$$

where α, Θ and Φ are the three degrees of freedom of the $SU(2)$ field.
In this representation, the topological charge density takes the form

$$\rho^0 = 12\sin^2\alpha\sin\Theta\,\mathrm{d}\alpha\,\mathrm{d}\Theta\,\mathrm{d}\Phi + 12\,\mathrm{d}\left[\left(\alpha - \frac{1}{2}\sin(2\alpha)\right)\sin\Theta A\,\mathrm{d}\Theta - \alpha\cos\Theta F\right]\ ,$$

where $A = A_i \mathrm{d}x^i$ and $F = \frac{1}{2}F_{ij}\mathrm{d}x^i\mathrm{d}x^j$.

The Ansatz

The Skyrme field equations are

$$
\begin{gathered}
-\Box\alpha + \sin(\alpha)\cos(\alpha)\left(\nabla_\mu\Theta\nabla^\mu\Theta + \sin^2\Theta\nabla_\mu\Phi\nabla^\mu\Phi\right) \\
+\lambda\begin{pmatrix}
\sin(\alpha)\cos(\alpha)\left((\nabla_\mu\alpha\nabla^\mu\alpha)(\nabla_\nu\Theta\nabla^\nu\Theta) - (\nabla_\mu\alpha\nabla^\mu\Theta)^2\right) \\
+\sin(\alpha)\cos(\alpha)\sin^2(\Theta)\left((\nabla_\mu\alpha\nabla^\mu\alpha)(\nabla_\nu\Phi\nabla^\nu\Phi) - (\nabla_\mu\alpha\nabla^\mu\Phi)^2\right) \\
+2\sin^3(\alpha)\cos(\alpha)\sin^2(\Theta)\left((\nabla_\mu\Theta\nabla^\mu\Theta)(\nabla_\nu\Phi\nabla^\nu\Phi) - (\nabla_\mu\Theta\nabla^\mu\Phi)^2\right) \\
-\nabla_\mu\left(\sin^2(\alpha)(\nabla_\nu\Theta\nabla^\nu\Theta)\nabla^\mu\alpha\right) + \nabla_\mu\left(\sin^2(\alpha)(\nabla_\nu\alpha\nabla^\nu\Theta)\nabla^\mu\Theta\right) \\
-\nabla_\mu\left(\sin^2(\alpha)\sin^2(\Theta)(\nabla_\nu\Phi\nabla^\nu\Phi)\nabla^\mu\alpha\right) + \nabla_\mu\left(\sin^2(\alpha)\sin^2(\Theta)(\nabla_\nu\alpha\nabla^\nu\Phi)\nabla^\mu\Phi\right)
\end{pmatrix} = 0\,,
\end{gathered}
$$

$$
\begin{gathered}
-\sin^2(\alpha)\Box\Theta - 2\sin(\alpha)\cos(\alpha)\nabla_\mu\alpha\nabla^\mu\Theta + \sin^2(\alpha)\sin(\Theta)\cos(\Theta)\nabla_\mu\Phi\nabla^\mu\Phi \\
+\lambda\begin{pmatrix}
\sin^2(\alpha)\sin(\Theta)\cos(\Theta)\left((\nabla_\mu\alpha\nabla^\mu\alpha)(\nabla_\nu\Phi\nabla^\nu\Phi) - (\nabla_\mu\alpha\nabla^\mu\Phi)^2\right) \\
+\sin^4(\alpha)\sin(\Theta)\cos(\Theta)\left((\nabla_\mu\Theta\nabla^\mu\Theta)(\nabla_\nu\Phi\nabla^\nu\Phi) - (\nabla_\mu\Theta\nabla^\mu\Phi)^2\right) \\
-\nabla_\mu\left(\sin^2(\alpha)(\nabla_\nu\alpha\nabla^\nu\alpha)\nabla^\mu\Theta\right) + \nabla_\mu\left(\sin^2(\alpha)(\nabla_\nu\alpha\nabla^\nu\Theta)\nabla^\mu\alpha\right) \\
-\nabla_\mu\left(\sin^4(\alpha)\sin^2(\Theta)(\nabla_\nu\Phi\nabla^\nu\Phi)\nabla^\mu\Theta\right) + \nabla_\mu\left(\sin^4(\alpha)\sin^2(\Theta)(\nabla_\nu\Theta\nabla^\nu\Phi)\nabla^\mu\Phi\right)
\end{pmatrix} = 0\,,
\end{gathered}
$$

$$
\begin{gathered}
-\sin^2(\alpha)\sin^2(\Theta)\Box\Phi - 2\sin(\alpha)\cos(\alpha)\sin^2(\Theta)\nabla_\mu\alpha\nabla^\mu\Phi - 2\sin^2(\alpha)\sin(\Theta)\cos(\Theta)\nabla_\mu\Theta\nabla^\mu\Phi \\
+\lambda\begin{pmatrix}
-\nabla_\mu\left[\sin^2(\alpha)\sin^2(\Theta)(\nabla_\nu\alpha\nabla^\nu\alpha)\nabla^\mu\Phi\right] + \nabla_\mu\left[\sin^2(\alpha)\sin^2(\Theta)(\nabla_\nu\alpha\nabla^\nu\Phi)\nabla^\mu\alpha\right] \\
-\nabla_\mu\left[\sin^4(\alpha)\sin^2(\Theta)(\nabla_\nu\Theta\nabla^\nu\Theta)\nabla^\mu\Phi\right] + \nabla_\mu\left[\sin^4(\alpha)\sin^2(\Theta)(\nabla_\nu\Theta\nabla^\nu\Phi)\nabla^\mu\Theta\right]
\end{pmatrix} = 0\,.
\end{gathered}
$$

Gauged solitons at finite volume

Fixing any one of the degrees of freedom to a constant implies that the contribution of the usual topological charge vanishes. However, the Callan-Witten term is not necessarily zero under this assumption. In fact, fixing $\Theta = \pi$, the topological charge density becomes

$$\rho^0 = -12F\,\mathrm{d}\alpha \ .$$

Therefore, one can construct topological solitons considering only two non-trivial degrees of freedom! As long as there is a non-zero electromagnetic field strength.

Let us consider a finite volume system described by the metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \ .$$

Gauged solitons at finite volume

A simple choice for the matter field that considerably reduces the field equations and, at the same time, leads to a non-vanishing topological charge is

$$\alpha = \alpha(t, x) \ , \qquad \Theta = \pi \ ,$$

together with the Maxwell potential that leads to a constant magnetic field in the x direction

$$A_\mu = (0, 0, Bz, 0) \ .$$

For this Ansatz, the Maxwell equations are automatically satisfied, while the Skyrme equations are reduced to the sine-Gordon one

$$\Box\alpha - m_\pi^2 \sin(\alpha) = 0 \ , \qquad \Box = -\partial_t^2 + \partial_x^2 \ .$$

The non-trivial topological charge density comes from the Callan-Witten term, resulting in

$$n_{\rm B} = \frac{1}{24\pi^2} \int 12B \, {\rm d}\alpha {\rm d}y {\rm d}z = B(\alpha(x_f) - \alpha(x_i)) \ .$$

Assuming the boundary conditions $\alpha(-l/2) = 0$ and $\alpha(l/2) = n\pi$, with n an integer, we obtain

$$n_{\rm B} = n\pi \, B \ .$$

Gauged solitons at finite volume

In the static case, $\alpha = \alpha(x)$, the sine-Gordon equation can be reduced to a first order equation

$$\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x}\right)^2 + 2m_\pi^2 \cos\alpha = \ell \ ,$$

A general solution for this equation is given by the Jacobi amplitude $\mathrm{Am}(x, m)$, as

$$\alpha(x) = 2\mathrm{Am}\left(\frac{1}{2}\sqrt{\ell - 2m_\pi^2}\left(x + \frac{l}{2}\right), \frac{4m_\pi^2}{2m_\pi^2 - \ell}\right) \ ,$$

where the integration constant, ℓ, is related to the baryon charge as follows

$$4n\,K\left(\frac{4m_\pi^2}{2m_\pi^2 - \ell}\right) = l\sqrt{\ell - 2m_\pi^2} \ .$$

This equation has a finite number of solutions for ℓ as long as $l > 2\pi$.

Emergent chiral soliton lattice

The field configuration presented above, through the description of topological solitons in a finite volume in the context of the generalized Skyrme theory, is nothing less than (a generalization of) the ChSL.

To see this, we can calculate the Hamiltonian density

$$\mathcal{H} = f_\pi^2 \left(\frac{1}{2} (\alpha')^2 + m_\pi^2 (1 - \cos(\alpha)) \right) - \frac{\mu_B \, B}{4\pi^2} \alpha' + \frac{B^2}{2} \ .$$

From the above expression, we can see the following:

- In the absence of the last term, the Hamiltonian corresponds to the free energy density of the ChSL; this is the sine-Gordon theory with the coupling between the baryon chemical potential, the magnetic field, and the gradient of the pion field.

  Using the elementary properties of the Jacobi elliptic functions, one can write the energy in terms of the elliptic modulus. Then, by minimizing the energy, it is possible to derive the critical value for the magnetic field above which the ChSL is formed, $B > B_{\rm ChSL}$;

$$B_{\rm ChSL} = \frac{16\pi m_\pi f_\pi^2}{\mu_B} \ .$$

Emergent chiral soliton lattice

- In the general case when $\vec{B}^2$ is different from zero, the solution can be interpreted as a periodic array of hadronic layers that generates its own magnetic field. This configuration is, of course, much more difficult to analyze; however, its chiral limit, $m_\pi \to 0$, can be studied. In fact, the configuration minimizing the free energy satisfies

$$(\alpha') = \frac{\mu_B B}{4\pi^2 f_\pi^2} \ .$$

  In this case, the free energy density of the system becomes

$$\lim_{m_\pi \to 0} \mathcal{H} = -\frac{1}{2}\left(\frac{\mu_B B}{4\pi^2 f_\pi}\right)^2 + \frac{B^2}{2} \ .$$

  The chiral limit of the system shows a similarity with the behavior of ordinary type-II superconductors. G. W. Evans and A. Schmitt, JHEP **09**, 192 (2022).

Generalized Skyrme terms

The construction performed above also works when higher-order corrections coming from the large N_c expansion are supplemented to the Skyrme action. L. Marleau, Phys. Lett. B **235**, 141 (1990).

The first two sub-leading terms are given by

$$\mathcal{L}_6 = \frac{c_6}{96}\,\mathrm{Tr}\left[G_\mu{}^\nu G_\nu{}^\rho G_\rho{}^\mu\right] \ ,$$
$$\mathcal{L}_8 = -\frac{c_8}{256}\left(\mathrm{Tr}\left[G_\mu{}^\nu G_\nu{}^\rho G_\rho{}^\sigma G_\sigma{}^\mu\right] - \mathrm{Tr}\left[\{G_\mu{}^\nu, G_\rho{}^\sigma\} G_\nu{}^\rho G_\sigma{}^\mu\right]\right) .$$

Taking into account the above new contributions, the Skyrme equations are given by

$$\frac{f_\pi^2}{2}\left(D_\mu\left(R^\mu + \frac{\lambda}{4}[R_\nu, G^{\mu\nu}]\right) - \frac{m_\pi^2}{2}(U - U^{-1})\right) + 3c_6[R_\mu, D_\nu[G^{\rho\nu}, G_\rho{}^\mu]]$$
$$+4c_8\left[R_\mu, D_\nu\left(G^{\nu\rho}G_{\rho\sigma}G^{\sigma\mu} + G^{\mu\rho}G_{\rho\sigma}G^{\nu\sigma} + \{G_{\rho\sigma}, \{G^{\mu\rho}, G^{\nu\sigma}\}\}\right)\right] = 0 .$$

The Maxwell equations will also be affected by including these new terms due to the $U(1)$ connection are present in each of the corrections.

$$J_\mu = -\frac{f_\pi^2}{2}\mathrm{Tr}\left[\hat{O}\left(R_\mu + \frac{\lambda}{4}[R^\nu, G_{\mu\nu}]\right)\right] + \frac{c_6}{32}\mathrm{Tr}\left[\hat{O}\left([R^\alpha, [G_{\mu\nu}, G_\alpha{}^\nu]]\right)\right]$$
$$-\frac{c_8}{64}\mathrm{Tr}\left[\hat{O}\left([R^\alpha, G_\alpha{}^\nu G_\nu{}^\rho G_{\rho\mu} + G_{\rho\mu}G_\nu{}^\rho G_\alpha{}^\nu + \{G^{\nu\rho}, \{G_{\mu\nu}, G_{\rho\alpha}\}\}]\right)\right] ,$$

where we have defined $\hat{O} = U^{-1}[t_3, U]$.

Generalized Skyrme terms

Here, it is important to note two very relevant facts:

- All new contributions to the electromagnetic current are proportional to $\hat{O}$. However, the matrix U depends only on the identity matrix and the generator t_3, namely,

$$U(\alpha) = \cos(\alpha)\mathbf{1}_2 - \sin(\alpha)t_3 \ .$$

  Thus, the tensor $\hat{O}$ is identically zero, and therefore, all extra contributions to the current eventually cancel out.

- Something similar happens with the Skyrme field equations. It follows that R_μ depends only on one of the generators of the group

$$R_\mu = -\partial_\mu \alpha \, t_3 \ .$$

Then, it is clear that the tensor $G_{\mu\nu} = [R_\mu, R_\nu]$ must be $G_{\mu\nu} = 0$.

The solution presented here is not only a solution to the Skyrme model, but also to the generalized Skyrme model. The ChSL is a universal feature of the low energy limit of QCD.

About the coupling with quark matter

Our construction opens the possibility of coupling the ChSL directly to quark matter via the usual Yukawa coupling.

The idea is to consider an interaction term in the Lagrangian of the form,

$$\mathcal{L}_{\text{int}} = \bar{\psi}\left(i\gamma^{\mu}D_{\mu} + gU^{\gamma_5}\right)\psi \ ,$$

where ψ is a spin-isospin spinor, whose mass is negligible compared to the energy scale of the coupling with the U field, and $D_{\mu} = (\partial_{\mu} - ieU^{-1}[t_3, U]A_{\mu})$ is the usual Maxwell-covariant derivative acting on the space of spinors. The matter field is coupled in the minimal parity-invariant manner, as

$$U^{\gamma_5} = \left(\frac{1+\gamma_5}{2}\right)U + \left(\frac{1-\gamma_5}{2}\right)U^{-1} = \cos(\alpha) - i\gamma_5\tau_3\sin(\alpha) \ ,$$

where the ChSL Ansatz dramatically simplifies the expression for the coupling.

Summary

- The CSL can be derived from the generalized Skyrme model, which describes hadrons at low energies.
- The key point in this derivation is the topological charge derived from the Callan-Witten term.

This allows us to explore:

- The coupling of baryons with quarks in this regime.
- Domain walls that generate their own electromagnetic field.