Regularization methods in four-point interaction theories: handling magnetic field and chiral imbalance effects

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Regularization of four-point interaction theories

In the set of the two lectures we will study:

- **First Lecture on 27 November: Handling Chiral Imbalance Effects**

- **Second Lecture on 28 November: Handling Magnetic Field Effects**

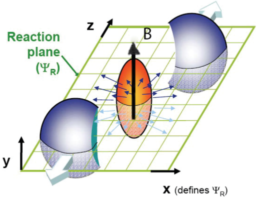
We will apply these effects in the **two-flavor Nambu–Jona-Lasinio model**[1][2], however, several of these procedures can be applied in other effective models or theories.

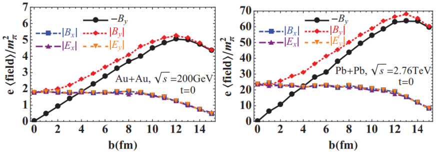
[1] Y. Nambu and G,.J. Lasinio. Phys. Rev. 122, 345 – Published 1 April, 1961.
[2] Y. Nambu and G,.J. Lasinio. Phys. Rev. 124, 246 – Published 1 October, 1961

Content of the first lecture

Magnetic Fields in QCD

1. **High Temperatures (noncentral HIC):** $eB \sim 10^{19}$ G at BNL (RHIC) and $eB \sim 10^{20}$ G at LHC (ALICE) .
2. **High Densities (magnetars):** $eB \sim 10^{17}$ G.
3. **Primordial universe and electroweak phase transition.**





Top right fig.: Incera, Ferrer Eur.Phys.J.A 52 (2016) 8, 266. **Bottom right fig.**: Huang, X.-G. 2016 Rep. Prog. Phys. 79 076302.

Lattice QCD in strong magnetic fields

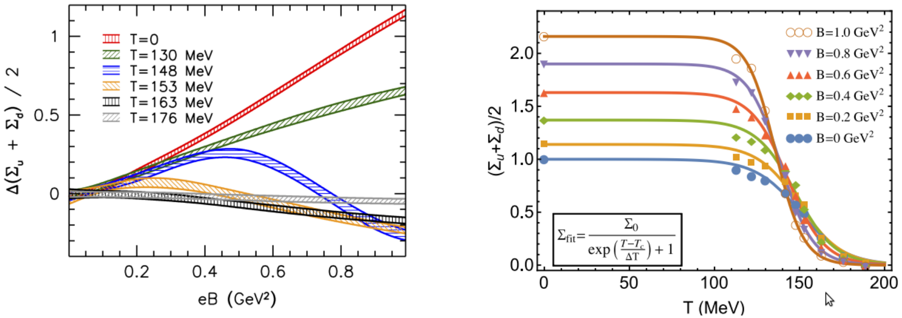


Figure: *

Right plot: V. Miransky, I. Shovykovy. Phys.Rept. 576 (2015) 1-209.
Left plot: G. Bali, F. Bruckmann, G. Endrödi, Z. Fodor, S. Katz, A. Schäffer. Phys.Rev.D 86 (2012) 071502

What LQCD tell us about very strong magnetic fields?

M. S. D'Elia, et al.Phys. Rev. D 105, 034511 (2022)

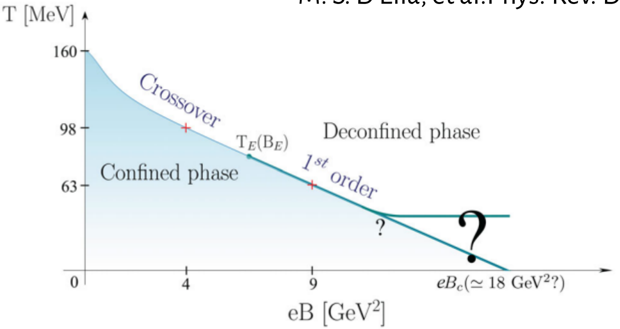


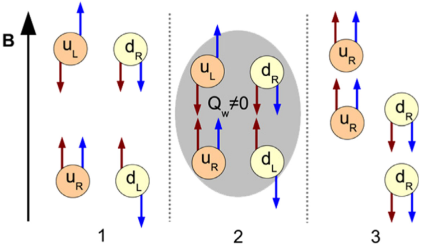
Figure: QCD phase diagram. The end point is located in the range $(4\mathrm{GeV}^2, 65\mathrm{MeV}) < (eB_E, T_E) < (9\mathrm{GeV}^2, 95\mathrm{MeV})$.

Chiral Magnetic Effect - Physical Motivations

$$\boxed{\vec{J} = \frac{e\mu_5}{2\pi^2}\int d^3x\vec{B}}$$

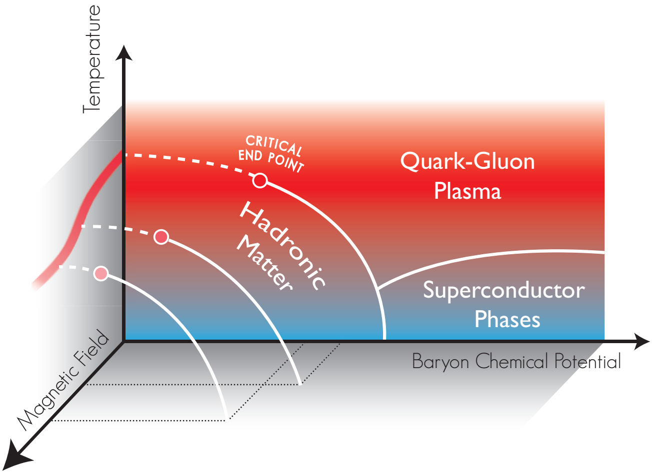
- **The chiral imbalance is the difference between right- and left-handed quarks.**
- **Event-by-event C and CP violating processes in heavy-ion collisions;** (Adler-Jackiw anomaly $\rightarrow$ chiral imbalanced medium)[2];
- **chiral anomaly transport model[3]:** $\sim \mu_5 = 2.1T + \sqrt{eB} > 300\text{MeV}$

- **Chiral Magnetic Effect[1]:**



[1] K. Fukushima, D. Kharzeev, H. Warringa. Phys.Rev.D 78 (2008) 074033
[2] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. 53, 43 (1981).
[2] H. Aoyama, H. Goldberg, and Z. Ryzak, Phys. Rev. Lett. 60, 1902 (1988).
[3] Zilin Yuan, Anping Huang, Wen-Hao Zhou, Guo-Liang Ma and Mei Huang. Phys. Rev. C 109, L031903
Figure: D. Kharzeev, L. Mclerran and H. Warringa. Nucl.Phys.A 803 (2008) 227-253

QCD Phase Diagram



P. Costa, M. Ferreira, C. Providência. PoS Hadron2017 (2018) 161

Nambu–Jona-Lasinio model SU(2)

The lagrangian of the two-flavor NJL model given by

$$\mathcal{L} = \overline{\psi}\left[i\gamma^{\mu}\partial_{\mu} - \tilde{m}\right]\psi + G\left[(\overline{\psi}\psi)^2 + (\overline{\psi}i\gamma_5\vec{\tau}\psi)^2\right], \tag{1}$$

- **Basic definitions:** the current quark masses matrix $\tilde{m} = \mathrm{diag}(m_u, m_d)$ in which $m_u = m_d = m_0$; τ are the Pauli matrices; ψ is the spinor representing the quark fields $\psi = (\psi_u \quad \psi_d)^T$; G is the coupling constant and μ is the quark chemical potential.
- **This is a non-renormalizable model:** the coupling constant G has dimension of MeV^{-2}.
- **This model has three parameters to set:** the cutoff Λ, the current quark mass m_0 and the coupling G.
  The parameters are chosen in order to obtain the vacuum values of the pion decay constant, $f_\pi \approx 93$ MeV; the pion mass $m_\pi \approx 135$ MeV and the quark condensate $\langle \bar{u}u \rangle \approx (-250)^{\frac{1}{3}}$ MeV.

S.P. Klevansky. Rev.Mod.Phys. 64 (1992) 649-708

Nambu–Jona-Lasinio SU(2): some details

To find the quark dispersion relation we start by looking to the functional generator:

$$Z = \mathcal{N} \int D\bar{\psi} D\psi \exp\left[i \int d^4 x \mathcal{L}(\bar{\psi}, \psi) \right] \Rightarrow \text{H.S.} \Rightarrow \begin{cases} \sigma = -2G\bar{\psi}\psi \to -2G\langle \bar{\psi}\psi \rangle \\ \vec{\pi} = -2G\bar{\psi} i \gamma_5 \vec{\tau} \psi \end{cases}$$

where we performed the Hubbard-Stratanovich transformation (H.S.) with $\vec{\pi} = 0$ due to isospin symmetry. The partition function becomes

$$Z = \exp\left[-i \frac{\sigma^2}{4G} \int d^4 x \right] \exp\left[N_c \int d^4 x \int \frac{d^4 p}{(2\pi)^4} \operatorname{Tr}_{f,D} \ln\left(\not{p} - m_c - \sigma \right) \right]. \tag{2}$$

with $N_c = 3$. Evaluating the trace over the flavor and Dirac indices using $\operatorname{Tr} \ln \mathcal{O} \equiv \ln \operatorname{Det} \mathcal{O}$

$$\omega_s(k) = \sqrt{k^2 + M^2} \quad \text{where} \quad \left\{ M = m_0 + \sigma \right.$$

The effective quark masses can be obtained by the minimization of the free energy, i.e., $\frac{\partial F}{\partial M} = 0$.

Thermodynamic Potential at $eB = 0$

We will include temperature by means of the Matsubara formalism

$$\int_{\infty}^{\infty} \frac{d^4 p}{(2\pi)^4} \Rightarrow iT \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d^3 p}{(2\pi)^3}, \quad \text{where} \quad \begin{cases} (p_0, \vec{p}) \Rightarrow (i\omega_n, \vec{p}) \\ \omega_n = (2n+1)\pi T \end{cases} \tag{3}$$

The thermodynamics of this model will be evaluated through the partition function,

$$\mathcal{Z} = \exp\left[-\beta \int d^3x \mathcal{F}\right], \quad \beta = \frac{1}{T} \tag{4}$$

where the Landau Free energy is given by $\mathcal{F}$. Then the thermodynamic potential will be given by

$$\mathcal{F}(T, \mu_5) = \frac{(M - m_0)}{4G} - \underbrace{2N_c N_f \int_\Lambda \frac{d^3k}{(2\pi)^3} \omega(k)}_{\text{needs regularization!}} - 2N_c N_f T \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \log\left(1 + e^{-\frac{\omega(k)}{T}}\right) \tag{5}$$

where $\omega(k) = \sqrt{k^2 + M^2}$.

Nambu–Jona-Lasinio model SU(2) with magnetic field

The SU(2) NJL Lagrangian with the inclusion of a constant external magnetic field is given by:

$$\mathcal{L} = \overline{\psi}\left(i\not{D} - \tilde{m}\right)\psi + G\left[(\overline{\psi}\psi)^2 + (\overline{\psi}i\gamma_5\overrightarrow{\tau}\psi)^2\right] - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{6}$$

with the current quark masses matrix $\tilde{m}$ =diag(m_u, m_d) in the isospin symmetry approximation, $m_u = m_d = m$.
The covariant derivative is given by $\partial^\mu \rightarrow D^\mu = (i\partial^\mu - QA^\mu)$; the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; and the charge matrix Q_q =diag$(2/3, -1/3)e$. The gauge adopted are:

- $A_\mu = \delta_{\mu 2} x_1 B$, ($\overrightarrow{B} = B\hat{e}_z$);

Energy solution of an electron in the presence of $B \neq 0$

$$\Psi^{(+)}(t,\mathbf{r}) = \begin{pmatrix} C_1\, v_{n-1}(\xi) \\ C_2\, v_n(\xi) \\ C_3\, v_{n-1}(\xi) \\ C_4\, v_n(\xi) \end{pmatrix} e^{-iEt+ip_y y+ip_z z}$$

$$\xi = (eB)^{1/2}\left(x + \frac{p_y}{eB}\right), \qquad v_n(\xi) = \frac{1}{(\pi^{1/2}2^n n!)^{1/2}} H_n(\xi)\, e^{-\xi^2/2}$$

The position where the oscillator wave functions are centered: $0 \leq \frac{p_y}{eB} \leq L$.

$$\sum_{p_x} \to \sum_{n=0}^{\infty} g_n, \qquad \sum_{p_y} \to \frac{L}{2\pi}\int dp_y = \frac{L^2 eB}{2\pi}, \qquad \sum_{p_z} \to \frac{L}{2\pi}\int_{-\infty}^{\infty} dp_z$$

$$\frac{2}{V}\sum_{p_x,p_y,p_z} \equiv \frac{2}{(2\pi)^3}\int d^3p \to \sum_{n=0}^{\infty} \alpha_n \frac{eB}{(2\pi)^2}\int_{-\infty}^{\infty} dp_z, \qquad \alpha_n = 2 - \delta_{n0}$$

Thermodynamic Potential at $B \neq 0$

We will include temperature by means of the Matsubara formalism

$$\int_{\infty}^{\infty} \frac{d^4 p}{(2\pi)^4} \Rightarrow iT \frac{|q_f|B}{2\pi} \sum_{\nu=-\infty}^{\infty} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_z}{(2\pi)}, \quad \text{where} \quad \begin{cases} p_0 \Rightarrow i\omega_\nu \\ \omega_\nu = (2\nu + 1)\pi T \end{cases} \tag{7}$$

The thermodynamics of this model will be evaluated through the partition function

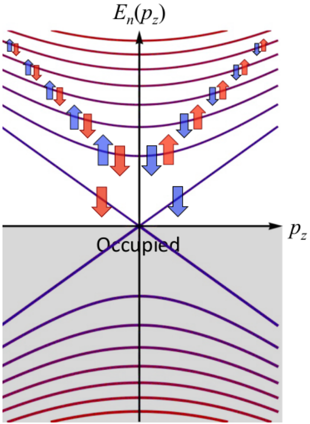
$$\mathcal{Z} = \exp\left[-\beta \int d^3x \mathcal{F}\right], \quad \beta = \frac{1}{T} \tag{8}$$

Then the thermo-magnetic part of the potential will be given by,
$\mathcal{F}^{TM}(eB, T) = \mathcal{F}(eB) + \mathcal{F}(B, T)$, i.e.,

$$\mathcal{F}^{TM}(T, eB) = -\frac{N_c}{2\pi} \sum_{s=\pm 1} \sum_{f=u}^{d} \sum_{n=0}^{\infty} (|q_f|B) \int \frac{dp_z}{(2\pi)} \omega_{n,s}(p_z) - T \frac{N_c}{2\pi} \sum_{s=\pm 1} \sum_{f=u}^{d} \sum_{n=0}^{\infty} (|q_f|B) \int \frac{dp_z}{(2\pi)} \log\left(1 + e^{-\frac{\omega_{n,s}(p_z)}{T}}\right)$$

where $\omega_{n,s}(k) = \sqrt{p_z^2 + (2n - 1 + s)|q_f|B + M_f^2}$.

Landau Level Degeneracy



Let's take $M = 0$ in the energy relation

$$\omega_{n,s}(p_z) = \sqrt{p_z^2 + 2|q_f|Bn'}$$

in which $n' = \left(n + \frac{1}{2} + s\frac{1}{2}\right)$.

- The lowest Landau level is spin polarized, i.e., $\omega^{\pm}_{0,-1} = \pm p_z$
- However, the higher Landau levels are twice degenerated

$$n = n' \quad \text{and} \quad s = -1$$
$$n = n' - 1 \quad \text{and} \quad s = +1$$

Fig: Igor Shovkovy's Lecture on XIV International Workshop on Hadron Physics, Florianópolis, Brazil, 2018.

Thermodynamic Potential at $B \neq 0$

Then we can rewrite the thermo-magnetic part of the potential

$$\mathcal{F}^{TM}(T, eB) = -\frac{N_c}{2\pi}\sum_{f=u}^{d}\sum_{n=0}^{\infty}\alpha_n(|q_f|B)\int\frac{dp_z}{(2\pi)}\omega_n(p_z) - T\frac{N_c}{2\pi}\sum_{f=u}^{d}\sum_{n=0}^{\infty}\alpha_n(|q_f|B)\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}\log\left(1 + e^{-\frac{\omega_n(p_z)}{T}}\right)$$

where $\alpha_n = 2 - \delta_{n,0}$ is the degeneracy factor and $\omega_n(p_z) = \sqrt{p_z^2 + 2n|q_f|B + M^2}$. Here we also observe the **Dimensional Reduction** from 3+1 D$\Rightarrow 1 + 1$ D[3].

$$\underbrace{2N_cN_f\int\frac{d^3p}{(2\pi)^3}\omega(p)}\Rightarrow\frac{N_c}{2\pi}\sum_{f=u}^{d}\sum_{n=0}^{\infty}\alpha_n(|q_f|B)\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}\omega_n(p_z) \tag{9}$$

"The physical reason of this reduction is the fact that the motion of charged particles is restricted in directions perpendicular to the magnetic field[3]".

[3] V.P. Gusynin, V.A. Miransky, I. Shovkovy. Nucl.Phys. B462 (1996) 249-290

The gap equation

The gap equation, $\partial \mathcal{F}/\partial M_f = 0$, gives us the expression to evaluate the effective quark masses

$$\frac{\partial \mathcal{F}}{\partial M} = \frac{M - m_0}{2G} - \frac{N_c}{2\pi} \sum_{f=u}^{d} \sum_{n=0}^{\infty} \alpha_n(|q_f|B) \int_{-\infty}^{\infty} \frac{dp_z}{(2\pi)} \frac{M}{\omega_n(p_z)}$$

$$+ \frac{N_c}{2\pi} \sum_{f=u}^{d} \sum_{n=0}^{\infty} \alpha_n(|q_f|B) \int_{-\infty}^{\infty} \frac{dp_z}{(2\pi)} \frac{M}{\omega_n(p_z)} \frac{2}{1 + \exp\left(\frac{\omega_n(p_z)}{T}\right)} = 0 \tag{10}$$

However, we have divergent contributions in the gap equation and in the free energy

$$\underbrace{\frac{N_c}{2\pi} \sum_{f=u}^{d} \sum_{n=0}^{\infty} \alpha_n(|q_f|B) \int_{-\infty}^{\infty} \frac{dp_z}{(2\pi)} \omega_n(p_z)}_{\text{Divergence in the free energy}} \qquad \underbrace{\frac{N_c}{2\pi} \sum_{f=u}^{d} \sum_{n=0}^{\infty} \alpha_n(|q_f|B) \int_{-\infty}^{\infty} \frac{dp_z}{(2\pi)} \frac{M_f}{\omega_n(p_z)}}_{\text{Divergence in the gap equation}} \tag{11}$$

Form-Factor regularization

The thermodynamical potential regularized with FF is given by

$$\mathcal{F}(eB) = -\frac{N_c}{2}\sum_{f=u,d}(|q_f|B)\sum_{n=0}^{\infty}\sum_{s=\pm1}\int_{-\infty}^{\infty}\frac{dp_3}{2\pi^2}E^f_{n,s}f^f_{n,s}\,, \qquad s=\pm1 \tag{12}$$

where $f^f_{n,s}$ is the Form-Factor function. There are several possibilities

$$f^f_{n,s}(\Lambda, eB) = \frac{1}{1+\exp\left(\frac{((p_z^2+(|q_f|B)(2n+1-ss_f))^{1/2}-\Lambda)}{A}\right)}$$

$$f^f_{n,s}(\Lambda, eB) = \frac{\Lambda^N}{\Lambda^N + (p_z^2 + (|q_f|eB)(2n+1-ss_f))^{N/2}} \tag{13}$$

where Λ is the cutoff, N and A are extra parameters and $s_f = \text{sign}(q_f)$.

S. Avancini, R. Farias, N. Scoccola and **W. Tavares**. Phys.Rev.D 99 (2019) 11, 116002 (2019).

Magnetic Field Independent Regularization

The magnetic field independent regularization is a regularization procedure used to separate divergent contributions from the medium part.

- In the context of QED, this is similar to the subtraction of divergences [4].
- In the context of effective models for QCD, it has been applied long time ago by Klimenko et. al. [5] , Shovkovy et. al.[6], Avancini et. al [7], among others.
- The name MFIR has been applied in the first time in the context of color-superconductivity [8].

When applied to non-renormalizable four-point theories, it enforces the regularization of the medium-independent divergent contribution.

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[4] J. Schwinger. Phys. Rev. 82, 664 (1951)

[5] D. Ebert and K. Klimenko, Nucl. Phys. A 728, 203 (2003)

[6] I. Shovkovy. Lect.Notes Phys. 871 (2013) 13-49

[7] D. Menezes, M. Benghi Pinto, S. Avancini, A. Perez Martinez, and C. Provid^encia, Phys. Rev. C 79, 035807 (2009)

[8] P. Allen, A.G. Grunfeld, N. N. Scoccola. Phys. Rev. D 92, 074041 (2015)

Magnetic Field Independent Regularization

We will start by the divergent contribution in the free energy

$$\begin{aligned}
\mathcal{F}(eB) &= -\frac{N_c}{2\pi}\sum_{f=u}^{d}\sum_{n=0}^{\infty}\alpha_n(|q_f|B)\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}\omega_n(p_z) \\
&= \frac{N_c}{2\pi}\sum_{f=u}^{d}\sum_{n=0}^{\infty}(2-\delta_{n,0})(|q_f|B)\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}\omega_n(p_z) \\
&= \frac{N_c}{2\pi}\sum_{f=u}^{d}(|q_f|B)\left(2\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}\omega_n(p_z)-\sum_{n=0}^{\infty}\delta_{n,0}\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}\omega_n(p_z)\right) \qquad (14)
\end{aligned}$$

Magnetic Field Independent Regularization

continuing...

$$\begin{aligned}
\mathcal{F}(eB) &= \frac{N_c}{2\pi}\sum_{f=u}^{d}(|q_f|B)\left(\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{dp_z}{(\pi)}\omega_n(p_z)-\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}\omega_0(p_z)\right)\\
&= \frac{N_c}{\pi}\sum_{f=u}^{d}(|q_f|B)\left[\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\left(\sum_{n=0}^{\infty}\sqrt{p_z^2+M_f^2+2|q_fB|n}\right)-\frac{1}{2}\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}\sqrt{p_z^2+M_f^2}\right]\\
&= \frac{N_c}{\pi}\sum_{f=u}^{d}(|q_f|B)\left[\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\left(\sqrt{2|q_f|B}\sum_{n=0}^{\infty}\sqrt{\frac{p_z^2+M_f^2}{2|q_f|B}+n}\right)-\frac{1}{2}\int_{-\infty}^{\infty}\frac{dp_z}{(2\pi)}\sqrt{p_z^2+M_f^2}\right]
\end{aligned}$$

Magnetic Field Independent Regularization

Now we will define new variables

$$\begin{cases} x_f = \dfrac{M_f^2}{2|q_f|B} \\ q = \dfrac{p_z}{\sqrt{2|q_f|B}} \\ dq = \dfrac{dp_z}{\sqrt{2|q_f|B}} \end{cases}$$

then, we can rewrite our free energy as

$$\mathcal{F} = \frac{N_c}{\pi}\sum_{f=u}^{d}(|q_f|B)\int_{-\infty}^{\infty}\frac{dq\sqrt{2|q_fB|}}{2\pi}\left[\left(\sqrt{2|q_f|B}\sum_{n=0}^{\infty}\sqrt{q^2+x_f+n}\right)-\frac{1}{2}\sqrt{2|q_f|B}\sqrt{q^2+x_f}\right].$$

Magnetic Field Independent Regularization

continuing...

$$\mathcal{F} = \frac{2N_c}{\pi} \sum_{f=u}^{d} (|q_f|B)^2 \left[\int_{-\infty}^{\infty} \frac{dq}{2\pi} \left(\sum_{n=0}^{\infty} \sqrt{q^2 + x_f + n} \right) - \frac{1}{2} \int_{-\infty}^{\infty} \frac{dq}{(2\pi)} \sqrt{q^2 + x_f} \right] \tag{15}$$

Now we will make use of the dimensional regularization by using

$$\int_{-\infty}^{\infty} \frac{d^d k}{(2\pi)^d} [q^2 + x^2]^{-A} = \frac{\Gamma[A - d/2]}{(4\pi)^{d/2} \Gamma[A]} \frac{1}{(x^2)^{A-d/2}} \tag{16}$$

which can be studied in more details in "*Pierre Ramond. Field theory: a modern primer. Front.Phys. 74 (1989) 1-329*". We will consider $A = -1/2$ and $d = 1 - \epsilon$, where we will take $\epsilon \to 0$ at the end of the calculation.

Magnetic Field Independent Regularization

continuing...

$$\mathcal{F} = \frac{2N_c}{\pi}\sum_{f=u}^{d}\frac{(|q_f|B)^2}{(4\pi)^{\frac{1-\epsilon}{2}}}\frac{\Gamma[-1+\epsilon/2]}{\Gamma\left[-\frac{1}{2}\right]}\left[\left(\sum_{n=0}^{\infty}\frac{1}{(n+x_f)^{-1+\frac{\epsilon}{2}}} - \frac{1}{2x_f^{-1+\frac{\epsilon}{2}}}\right)\right] \tag{17}$$

Making use of $\Gamma[-1/2] = -2\sqrt{\pi}$ and the definition of the Hurwitz zeta function

$$\sum_{n=0}^{\infty}(n+a)^{-s} = \zeta(s,a) \tag{17}$$

which will change our result by

$$\mathcal{F} = \frac{2N_c}{\pi}\sum_{f}(|q_f|B)^2(4\pi)^{-(1-\epsilon)/2}\frac{\Gamma[-1+\epsilon/2]}{-2\sqrt{\pi}}\left[\zeta(-1+\epsilon/2, x_f) - \frac{1}{2}x_f^{-1+\epsilon/2}\right] \tag{18}$$

Magnetic Field Independent Regularization

continuing...

$$\mathcal{F} = -\frac{2N_c}{\pi\sqrt{\pi}}\sum_f (|q_f|B)^2(4\pi)^{-(1-\epsilon)/2}\Gamma[-1+\epsilon/2]\left[\zeta(-1+\epsilon/2,x_f)-\frac{1}{2}x_f^{1-\epsilon/2}\right] \tag{19}$$

$$= -\frac{N_c}{\sqrt{\pi}\pi^2}\sum_f (|q_f|B)^2(4\pi)^{\epsilon/2}\Gamma[-1+\epsilon/2]\left[\zeta(-1+\epsilon/2,x_f)-\frac{x_f}{2}x_f^{-\epsilon/2}\right] \tag{20}$$

We will expand in ϵ the above functions [9] by using $y^\epsilon = 1+\epsilon\log(y)+\mathcal{O}(\epsilon^2)$ and the following expressions

$$\zeta(-1+\epsilon/2,x_f) = \zeta(-1,x_f)+\tfrac{1}{2}\zeta'(-1,x_f)\epsilon+\mathcal{O}(\epsilon^2) \tag{19}$$

$$\Gamma[-1+\epsilon/2] = -\frac{2}{\epsilon}-1+\gamma_E+\mathcal{O}(\epsilon) \tag{21}$$

[9] **Lewis Ryder. Quantum Field Theory. Jun 6, 1996 by Cambridge University Press**

Magnetic Field Independent Regularization

The relation with the Bernoulli Polynomials with the Hurwitz-Zeta function will be useful in this step, by looking

$$\zeta(-n, x) = -\frac{B_{n+1}(x)}{n+1}, \tag{22}$$

which, in particular,

$$B_2(x) = x^2 - x + 1/6. \Rightarrow \zeta(-1, x) = -\frac{x^2}{2} + \frac{x}{2} + \frac{1}{12}. \tag{23}$$

Then, we can conveniently write

$$\zeta(-1+\epsilon/2, x_f) - \frac{x_f}{2} x_f^{-\epsilon/2} = -\frac{x_f^2}{2} + \frac{x_f}{2} + \frac{1}{12} + \frac{\epsilon}{2}\left[\frac{x_f^2}{2}\ln x_f + \zeta'(-1, x_f)\right] + \mathcal{O}(\epsilon^2) \tag{26}$$

Magnetic Field Independent Regularization

Therefore, we have expanded the term highlighted

$$\mathcal{F} = \frac{N_c}{2\pi^2}\sum_f (|q_f|B)^2(4\pi)^{\epsilon/2}\Gamma[-1+\epsilon/2]\underbrace{\left[\zeta(-1+\epsilon/2, x_f) - \frac{x_f}{2}x_f^{-\epsilon/2}\right]}, \tag{24}$$

and now we will expand

$$(4\pi)^{\epsilon/2}\Gamma[-1+\epsilon/2] = -\frac{2}{\epsilon} - \ln(4\pi) - 1 + \gamma_E + O(\epsilon) \tag{27}$$

which is obtained by the same procedure applied before.

Magnetic Field Independent Regularization

We can now write

$$
\begin{aligned}
(4\pi)^{\epsilon/2}\Gamma[-1+\epsilon/2]\left[\zeta(-1+\epsilon/2,x_f)-\frac{x_f}{2}x_f^{-\epsilon/2}\right] \quad = \quad & \frac{x_f^2}{\epsilon}+\frac{x_f^2}{2}(1-\gamma_E+\ln 4\pi)-\frac{x_f^2}{2}\ln x_f \\
& -\zeta'(-1,x_f)-\frac{1}{6\epsilon}+\frac{1}{12}(\gamma_E-1-\ln 4\pi) \\
& +\mathcal{O}(\epsilon)
\end{aligned} \tag{25}
$$

Now we can insert our result in the expression for the free energy

$$
\begin{aligned}
\mathcal{F}=\frac{N_c}{2\pi^2}\sum_f(|q_f|B)^2\Bigg[&\frac{x_f^2}{\epsilon}+\frac{x_f^2}{2}(1-\gamma_E+\ln 4\pi)-\frac{x_f}{2}\ln x_f-\zeta'(-1,x_f) \\
&-\frac{1}{6\epsilon}+\frac{1}{12}(\gamma_E-1-\ln 4\pi)+O(\epsilon)\Bigg]
\end{aligned} \tag{26}
$$

Magnetic Field Independent Regularization

In order to regularize $\mathcal{F}$ we will subtract the vacuum free energy, i.e.,

$$\mathcal{F}_{vac} = -2N_c \sum_f \int \frac{d^3p}{(2\pi)^3} \sqrt{p^2 + M_f^2}$$

We can redefine $p = q/\sqrt{2|q_f|B}$, in order to obtain

$$\mathcal{F}_{vac} = -8N_c \sum_f (|q_f|B)^2 \int \frac{d^3q}{(2\pi)^3} \sqrt{q^2 + x_f} \tag{32}$$

Magnetic Field Independent Regularization

Using the dimensional regularization expression in the free energy of the vacuum, we identify $A = -1/2$ and $d = 3 - \epsilon$. Then, we get

$$\begin{aligned}\mathcal{F}_{vac} &= -8N_c \sum_f (|q_f|B)^2 \frac{1}{(4\pi)^{(3-\epsilon)/2}} \frac{\Gamma\left(-\frac{1}{2} - \frac{3-\epsilon}{2}\right)}{\Gamma(-1/2)} x_f^{1/2+(3-\epsilon)/2} \\ &= \frac{N_c}{2\pi^2} \sum_f (|q_f|B)^2 x_f^2 (4\pi)^{\epsilon/2} \Gamma\left(-2 + \frac{\epsilon}{2}\right) x_f^{-\epsilon/2}\end{aligned} \tag{27}$$

We can evaluate this equation as before, with the additional expression

$$\Gamma\left[-2 + \frac{\epsilon}{2}\right] = \frac{1}{\epsilon} + \frac{1}{2}\left(\frac{3}{2} - \gamma_E\right) + \mathcal{O}(\epsilon) \tag{28}$$

Magnetic Field Independent Regularization

Then, explicitly applying dimension regularization to the vacuum free energy we will have

$$(4\pi)^{\epsilon/2}\Gamma[-2+\epsilon/2]x_f^{-\epsilon/2} = \frac{1}{\epsilon} + \frac{1}{2}\left(\frac{3}{2} - \gamma_E + \ln 4\pi\right) - \frac{1}{2}\ln x_f + O(\epsilon) \tag{29}$$

Therefore,

$$\mathcal{F}_{vac} = \frac{N_c}{2\pi^2}\sum_f (|q_f|B)^2\left[\frac{x_f^2}{\epsilon} + \frac{x_f^2}{2}\left(\frac{3}{2} - \gamma_E + \ln 4\pi\right) - \frac{x_f^2}{2}\ln x_f + O(\epsilon)\right] \tag{30}$$

Comparing with the magnetic field Free energy

$$\begin{aligned}\mathcal{F} = \frac{N_c}{2\pi^2}\sum_f (|q_f|B)^2\Big[&\frac{x_f^2}{\epsilon} + \frac{x_f^2}{2}(1 - \gamma_E + \ln 4\pi) - \frac{x_f}{2}\ln x_f - \zeta'(-1, x_f)\\ &- \frac{1}{6\epsilon} + \frac{1}{12}(\gamma_E - 1 - \ln 4\pi) + O(\epsilon)\Big]\end{aligned} \tag{31}$$

Magnetic Field Independent Regularization

The idea of the MFIR is to employ,

$$\begin{aligned}\mathcal{F}(eB) &= \mathcal{F}(eB) + (\mathcal{F}_{vac} - \mathcal{F}_{vac}) \\ &= (\mathcal{F}(eB) - \mathcal{F}_{vac}) + \mathcal{F}_{vac} \\ &= \bar{\mathcal{F}}(eB) + \mathcal{F}_{vac} \end{aligned} \tag{32}$$

where I have defined $\bar{\mathcal{F}} = \mathcal{F}(eB) - \mathcal{F}_{vac}$. At the end of the day we regularize just $\mathcal{F}_{vac}$.
So, our finite free energy $\bar{\mathcal{F}}$ can be written as

$$\begin{aligned}\bar{\mathcal{F}} = -\frac{N_c}{2\pi^2}\sum_f (|q_f|B)^2 \Big[\zeta'(-1, x_f) - \frac{1}{2}(x_f^2 - x_f)\ln x_f + \frac{x_f^2}{4} \\ + \frac{1}{6\epsilon} - \frac{1}{12}(\gamma_E - 1 - \ln 4\pi) + \mathcal{O}(\epsilon)\Big]\end{aligned} \tag{33}$$

Magnetic Field Independent Regularization

However $\bar{\mathcal{F}}$ is not yet finite, since we still have a term

$$\mathcal{F}^{ren} = \frac{N_c}{2\pi^2} \sum_f (|q_f|B)^2 \frac{1}{6\epsilon} \tag{34}$$

This contribution is **mass-independent**, and do not contribute to the gap equation $\partial \mathcal{F}/\partial M = 0$.
Therefore we can drop this term[10].
Our final result for $\bar{\mathcal{F}}$ is

$$\bar{\mathcal{F}} = -\frac{N_c}{2\pi^2} \sum_f (|q_f|B)^2 \left[\zeta'(-1, x_f) - \frac{1}{2}(x_f^2 - x_f) \ln x_f + \frac{x_f^2}{4} \right] \tag{35}$$

[10] D. Ebert and K. Klimenko, Nucl. Phys. A 728, 203 (2003).

Magnetic Field Independent Regularization

Therefore, our regularized free energy is given by

$$\mathcal{F} = \mathcal{F}_{vac} + \bar{\mathcal{F}}(eB) + \mathcal{F}(eB, T) \tag{36}$$

where

$$\mathcal{F}_{vac} = -2N_c \sum_f \int_\Lambda \frac{d^3p}{(2\pi)^3} \sqrt{p^2 + M_f^2}$$

$$\bar{\mathcal{F}}(eB) = -\frac{N_c}{2\pi^2} \sum_f (|q_f|B)^2 \left[\zeta'(-1, x_f) - \frac{1}{2}(x_f^2 - x_f) \ln x_f + \frac{x_f^2}{4} \right]$$

$$\mathcal{F}(eB, T) = -T \frac{N_c}{2\pi} \sum_{f=u}^{d} \sum_{n=0}^{\infty} \alpha_n(|q_f|B) \int_{-\infty}^{\infty} \frac{dp_z}{(2\pi)} \log\left(1 + e^{-\frac{\omega_n(p_z)}{T}}\right)$$

Magnetic Field Independent Regularization

The regularization of the vacuum contribution can be, in principle, chosen by any usual method, for instance

$$\mathcal{F}_{vac} = -\frac{2N_c}{2\pi^2}\sum_f \int_0^{\Lambda} \frac{dpp^2}{(2\pi)^3}\sqrt{p^2+M_f^2}, \quad \text{3D sharp cutoff}$$
$$\mathcal{F}_{vac} = -\frac{2N_c}{2\pi^2}\sum_f \int_0^{\infty} \frac{dpp^2}{(2\pi)^3}\sqrt{p^2+M_f^2}U(p,\Lambda) \quad \text{3D soft-cutoff or Form-Factor Regularization}$$
$$\mathcal{F}_{vac} = -2N_c\sum_f \sum_i C_i \int \frac{d^3p}{(2\pi)^3}\sqrt{p^2+M_f^2+\alpha_i} \quad \text{Pauli-Villars Regularization} \tag{37}$$

...and proper-time, 4D sharp-Cutoff among others. For detailed analysis, see Ref.(S.P. Klevansky, Rev. Mod. Phys. 64, 649 (1992).)

The gap equation in the MFIR procedure

The only non-trivial evaluation in $\partial\mathcal{F}/\partial M_f = 0$ is the derivative of the zeta function

$$\frac{\partial}{\partial x_f}\zeta(z, x_f) = -z\,\zeta(z+1, x_f), \tag{38}$$

$$\frac{\partial}{\partial x_f}\zeta'(-1, x_f) = \frac{\partial}{\partial z}\frac{\partial}{\partial x_f}\zeta(z, x_f)\Big|_{z=-1} = -\frac{\partial}{\partial z}\,z\,\zeta(z+1, x_f)\Big|_{z=-1}, \tag{39}$$

where, we can use the following expressions

$$\zeta'(0, x_f) = \ln\Gamma(x_f) - \tfrac{1}{2}\ln(2\pi), \qquad \mathrm{Re}(x_f) > 0, \tag{40}$$

$$\zeta(0, x_f) = \tfrac{1}{2} - x_f = -B_1(x_f), \tag{41}$$

Therefore

$$\frac{\partial}{\partial x_f}\zeta'(-1, x_f) = -\zeta(0, x_f) + \zeta'(0, x_f) = -\left(\tfrac{1}{2} - x_f\right) + \left(\ln\Gamma(x_f) - \tfrac{1}{2}\ln(2\pi)\right). \tag{42}$$

The gap equation in the MFIR procedure

The gap equation will be, at the end of the evaluation

$$0 = \frac{\partial \mathcal{F}_{vac}}{\partial M_f} + \frac{\partial \bar{\mathcal{F}}(eB)}{\partial M_f} + \frac{\mathcal{F}(eB,T)}{\partial M_f} \tag{43}$$

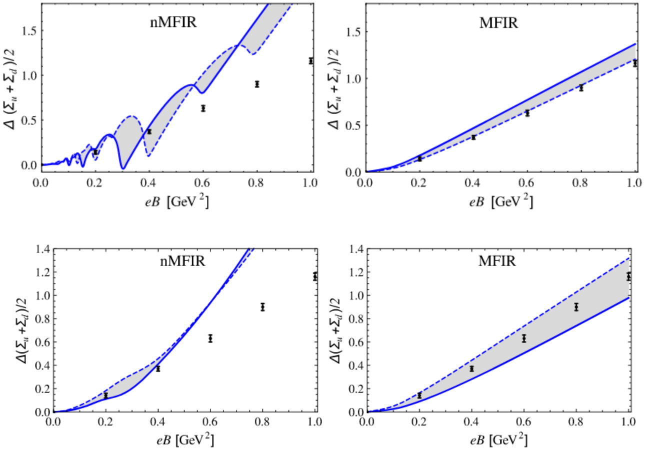
where

$$\frac{\partial \mathcal{F}_{vac}}{\partial M_f} = -2N_c \sum_f \int_\Lambda \frac{d^3p}{(2\pi)^3} \frac{M_f}{\sqrt{p^2+M_f^2}}$$

$$\frac{\partial \bar{\mathcal{F}}(eB)}{\partial M_f} = -\sum_q \frac{MN_c(|Q_q|B)}{2\pi^2} \left(\ln\Gamma(x_f) - \frac{1}{2}\ln(2\pi) + x_f - \frac{1}{2}(2x_f-1)\ln x_f \right).$$

$$\frac{\partial \mathcal{F}(eB,T)}{\partial M_f} = -\frac{N_c}{2\pi} \sum_{f=u}^{d} \sum_{n=0}^{\infty} \alpha_n(|q_f|B) \int_{-\infty}^{\infty} \frac{dp_z}{(2\pi)} \frac{2M_f}{1+e^{\frac{\omega_n(p_z)}{T}}}$$

Regularization at $B \neq 0$ (MFIR) vs (nMFIR)



Avancini, Farias, Scoccola, **Tavares**, Phys. Rev. D 99, 116002 (2019)

Vacuum Magnetic Regularization

Let's look again to the divergence we have eliminated in the MFIR procedure

$$\mathcal{F}^{ren} = \frac{N_c}{2\pi^2} \sum_f (|q_f|B)^2 \frac{1}{6\epsilon} \tag{44}$$

This contribution is **mass-independent**, and do not contribute to the gap equation $\partial\mathcal{F}/\partial M = 0$. Therefore we can drop this term[11]. However, we can subtract a sum a term that cancels out this divergence

$$\begin{aligned} \mathcal{F}_{\text{field}} &= \frac{1}{3}\sum_f \frac{(|q_f|B)^2}{4\pi^2} \int_0^{\Lambda} \frac{dp}{(2\pi)^2} \frac{p}{p^2+M^2}, \quad \text{then we apply dimensional regularization...} \\ &= \frac{1}{2\pi^2}\sum_f \frac{(|q_f|B)^2}{6\epsilon} - \frac{1}{3}\sum_f \frac{(|q_f|B)^2}{8\pi^2} \log x_f \end{aligned} \tag{45}$$

[11] D. Ebert and K. Klimenko, Nucl. Phys. A 728, 203 (2003).

Vacuum Magnetic Regularization

Therefore, we will have to employ in this procedure

$$\begin{aligned} \mathcal{F} &\Rightarrow (\mathcal{F} - \mathcal{F}_{field}) + \mathcal{F}_{field} \\ &= \mathcal{F}^{VMR} + \mathcal{F}_{field} \end{aligned} \tag{46}$$

where the remaining term, $\mathcal{F}_{field}$ is regularized by some usual method, for instance, 3D sharp cutoff

$$\mathcal{F}_{field} = \frac{1}{3}\frac{(|q_f|B)^2}{4\pi^2}\int_0^\Lambda dp\,\frac{p}{p^2+M^2} = \frac{1}{3}\frac{(|q_f|B)^2}{4\pi^2}\frac{1}{2}\ln\left(p^2+M^2\right)\Big|_0^\Lambda = -\frac{1}{3}\frac{(|q_f|B)^2}{8\pi^2}\ln\frac{M^2}{\Lambda^2+M^2}\,. \tag{47}$$

in order to avoid violation of the Goldstone Theorem[12] we will expand the logarithmic term

$$\mathcal{F}_{field} = -\frac{1}{3}\frac{(|q_f|B)^2}{8\pi^2}\ln\frac{M^2}{\Lambda^2}\left(1+\frac{M^2}{\Lambda^2}\right)^{-1} \approx \frac{1}{3}\frac{(|q_f|B)^2}{8\pi^2}\ln\frac{\Lambda^2}{M^2} + \mathcal{O}\left(\frac{\Lambda^2}{M^2}\right) \tag{48}$$

[12] Avancini et al. Phys.Rev.D 103 (2021) 5, 056009

Vacuum Magnetic Regularization

The thermodynamical potential in the VMR scheme at $T = 0$ is given by

$$\mathcal{F} = \frac{(M-m)^2}{4G} + \mathcal{F}_{vac} + \mathcal{F}_{field} + \mathcal{F}^{VMR}(eB), \tag{49}$$

Assuming the definitions : $B_f \equiv |q_f B|$ and $x_f = \frac{M_f^2}{2B_f}$, we have

$$\begin{aligned}
\mathcal{F}(eB) &= -\sum_{f=u,d} \frac{N_c(|q_f|B)^2}{2\pi^2} \left[\zeta'(-1, x_f) - \frac{1}{2}(x_f^2 - x_f)\log x_f + \frac{x_f^2}{4} - \frac{1}{12}(1 + \log x_f) \right] \\
\mathcal{F}_{field} &= -\sum_{f=u,d} \frac{N_c(|q_f|B)^2}{24\pi^2} \ln \frac{M_f^2}{\Lambda^2}.
\end{aligned} \tag{50}$$

and $\mathcal{F}_{vac}$ is the vacuum contribution which depends on the regularization scheme.

Avancini et al. Phys.Rev.D 103 (2021) 5, 056009

Gap equation in VMR and MFIR

The thermodynamical potential in the VMR scheme is given by

$$\mathcal{F} = \frac{(M-m)^2}{4G} + \mathcal{F}_{vac} + \mathcal{F}_{field} + \mathcal{F}(eB), \tag{51}$$

The gap equation is given by $\partial\partial\mathcal{F}M = 0$. Therefore, we have

$$\frac{\partial\mathcal{F}^{MFIR}}{\partial M} \equiv \frac{\partial\mathcal{F}^{VMR}}{\partial M} \to \langle\bar{\psi}\psi\rangle^{MFIR} \equiv \langle\bar{\psi}\psi\rangle^{VMR} \tag{52}$$

which means

$$M = m - 2G\langle\bar{\psi}\psi\rangle \tag{53}$$

Avancini et al. Phys.Rev.D 103 (2021) 5, 056009

Hadron Resonance Gas model + magnetic fields

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QCD equation of state at nonzero magnetic fields in the Hadron Resonance Gas model

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Hadron Resonance Gas model + magnetic fields

B Renormalized free energies

Here we give the renormalized free energy, eq. (3.19), for spins $s = 0$, $1/2$ and 1, without the pure magnetic energy $B_r^2/2$,

$$\Delta f^{\text{vac,r}}(0) = \frac{1}{8\pi^2}(qB)^2 \left[\zeta'(-1, x+1/2) + \frac{x^2}{4} - \frac{x^2}{2}\log(x) + \frac{\log(x)+1}{24}\right],$$

$$\boxed{\Delta f^{\text{vac,r}}(1/2) = \frac{-1}{4\pi^2}(qB)^2 \left[\zeta'(-1, x) + \frac{x}{2}\log(x) + \frac{x^2}{4} - \frac{x^2}{2}\log(x) - \frac{\log(x)+1}{12}\right],}$$

$$\begin{aligned}\Delta f^{\text{vac,r}}(1) = \frac{3}{8\pi^2}(qB)^2 \Bigg[&\zeta'(-1, x-1/2) + \frac{1}{3}(x+1/2)\log(x+1/2) \\ &+ \frac{2}{3}(x-1/2)\log(x-1/2) + \frac{x^2}{4} - \frac{x^2}{2}\log(x) - 7\,\frac{\log(x)+1}{24}\Bigg].\end{aligned} \tag{B.1}$$

where we made use of the identity of eq. (A.8).

In the VMR, we have the same fermionic contribution as obtained in the HRG.

SU(2) NJL model with eB and quark AMM

The SU(2) NJL Lagrangian with quark AMM and a constant magnetic field is given by:

$$\mathcal{L} = \overline{\psi}\left(i\not{D} - \tilde{m} + \frac{1}{2}\hat{a}F^{\mu\nu}\sigma_{\mu\nu}\right)\psi + G\left[(\overline{\psi}\psi)^2 + (\overline{\psi}i\gamma_5\vec{\tau}\psi)^2\right] - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{54}$$

with the current quark masses matrix $\tilde{m}$ =diag(m_u, m_d) in the isospin symmetry approximation, $m_u = m_d = m$.

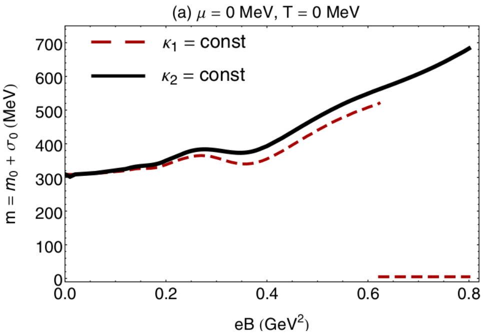
The covariant derivative is given by $\partial^\mu \to D^\mu = (i\partial^\mu - Q_q A^\mu)$; the electromagnetic field tensor is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the charge matrix Q_q =diag$(2/3, -1/3)e$. The gauge adopted is $A_\mu = \delta_{\mu 2} x_1 B$, ($\vec{B} = B\hat{e}_z$).

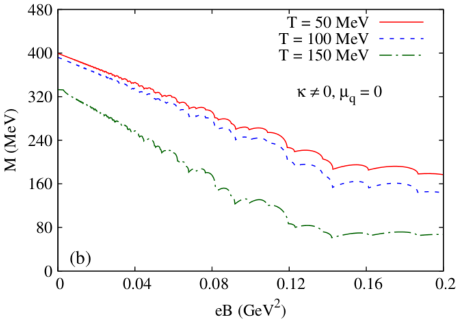
The AMM factor is $\hat{a}$=diag(a_u, a_d) with $a_f = q_f \alpha_f \mu_B$. In the one-loop level approximation, the previous quantities are given by

$$\alpha_f = \frac{\alpha_e q_f^2}{2\pi}, \quad \alpha_e = \frac{1}{137}, \quad \mu_B = \frac{e}{2M}. \tag{55}$$

Some results of quark AMM in the NJL model

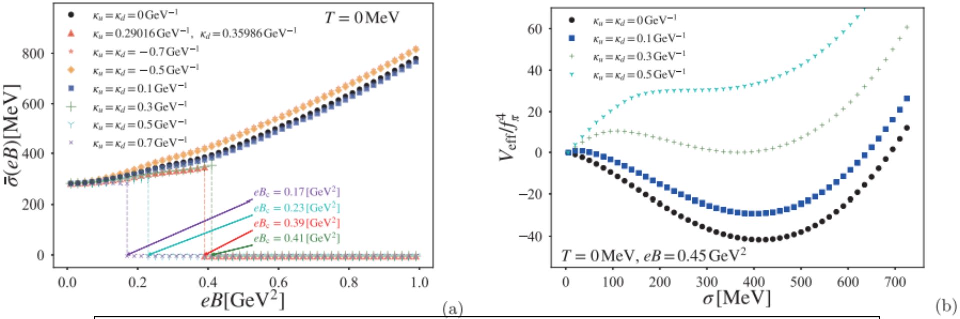
Chaudhuri, et al. Phys.Rev.D 99 (2019) 11, 116025





Sh. Fayazbakhsh, et al. Phys.Rev.D 90 (2014) 10, 105030

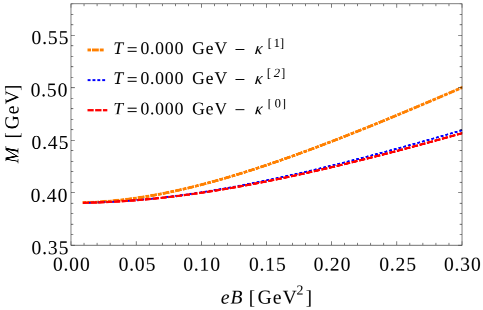
Some results of quark AMM in the NJL model

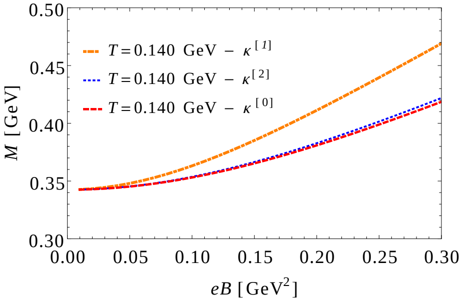


Mamiya Kawaguchi and Mei Huang. Chin.Phys.C 47 (2023) 6, 064103

nVMR vs VMR results: $eB \neq 0$ with AMM$\neq 0$

Farias, R.L.S, Tavares, W.R. et al. Eur. Phys. J. C (2022) 82:674





Regularization at $T \neq 0$ (TRPT)

TRPT : Thermomagnetic Regulated Proper-Time (with Λ);
SPT : Standard proper-Time (without Λ);

$$\Omega(B,T,M) = \frac{(M-m)^2}{4G} + \omega(B,M) + \omega(B,T,M),$$

$$\omega(B,T,M) = \frac{N_f}{8\pi^2}\sum_{f=u}^{d}(|q_f|B)^2\int_{\frac{|q_fB|}{\Lambda^2}}^{\infty}\frac{ds}{s^2}e^{-s\frac{M^2}{|q_f|B}}\coth(s)\tilde{\theta}_3\left(\frac{1}{2},\frac{i|q_fB|}{4\pi sT^2}\right) \to \text{TRPT},$$

$$\omega(B,T,M) = \frac{N_f}{8\pi^2}\sum_{f=u}^{d}(|q_f|B)^2\int_{0}^{\infty}\frac{ds}{s^2}e^{-s\frac{M^2}{|q_f|B}}\coth(s)\tilde{\theta}_3\left(\frac{1}{2},\frac{i|q_fB|}{4\pi sT^2}\right) \to \text{SPT},$$

$$\tilde{\theta}_3\left(\frac{1}{2},\frac{i|q_fB|}{4\pi sT^2}\right) = 2\left\{\sum_{n=1}^{\infty}e^{-|q_fB|n^2/(4sT)}(-1)^n\left[2\cos\left(n\cos^{-1}\frac{3\Phi-1}{2}\right)\right]\right\} \tag{56}$$

Sidney S. Avancini et al., Phys. Rev. D 103, 056009, (2021).

Results

The parameter set is: $\Lambda = 675$ MeV and $m = 3.5$ MeV. The coupling was chosen in order to reproduce IMC in the quark condensate

eB [GeV^2]	VMR and MFIR	SPT and TRPT
0.0	5.83200	5.83200
0.2	5.05349	5.19413
0.4	3.74477	4.05506
0.6	2.69719	3.05269

Table: $G(B)\Lambda^2$ values for the four different regularization prescriptions.

G. Endrödi and G. Markó, JHEP08, 036 (2019)

$$\langle \overline{\psi}_f \psi_f \rangle = -\frac{\partial P}{\partial m_c} \tag{57}$$

$$P = -\Omega, \tag{58}$$

$$\Delta P = P(T, B) - P(0, B), \tag{59}$$

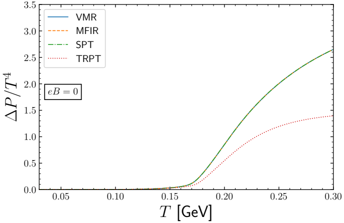
$$C_v = T\frac{\partial \mathcal{S}}{\partial T} , \tag{60}$$

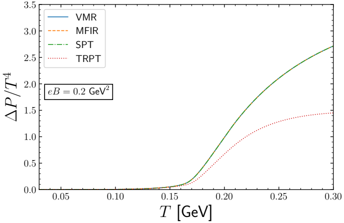
$$C_s^2 = \frac{\partial P}{\partial \mathcal{E}} = \frac{\mathcal{S}}{C_v} \tag{61}$$

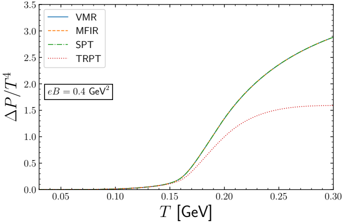
$$\mathcal{M} = \partial P / \partial B. \tag{62}$$

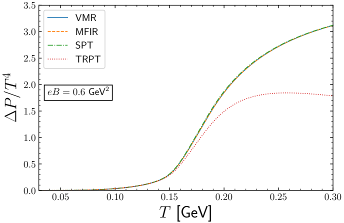
Sidney S. Avancini et al., Phys. Rev. D 103, 056009, (2021).

Reduced Pressure









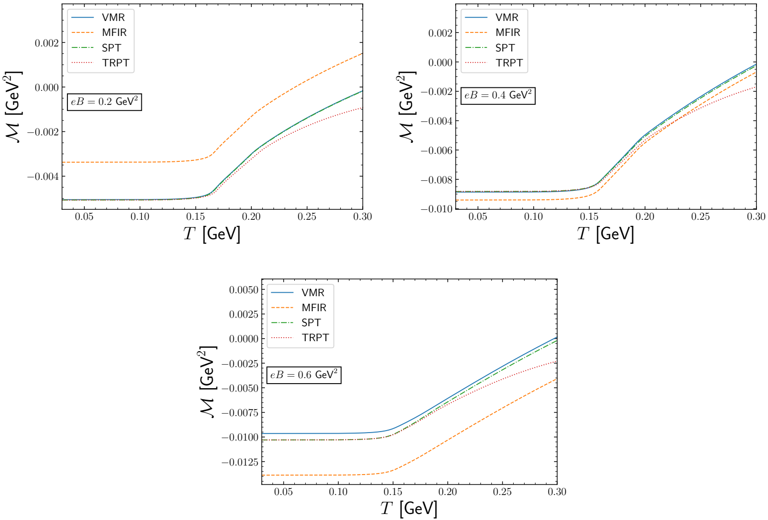
At $eB = 0 \rightarrow$ Pedro Costa et al., Symmetry 2 (2010) 1338-1374.

At $eB = 0 \rightarrow$ Pedro Costa et al., Phys.Rev.D 81 (2010) 016007.

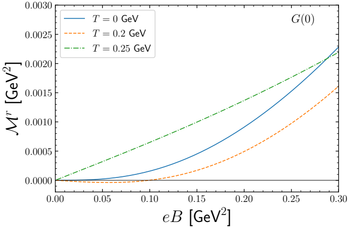
Sidney S. Avancini et al., Phys. Rev. D 103, 056009, (2021).

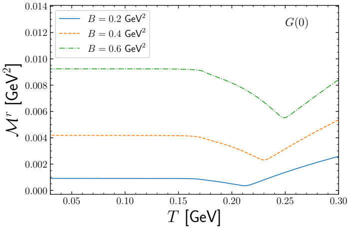
At $eB = 0$ this has been explored in the first two references

Magnetization vs T



Renormalized Magnetization





$$\mathcal{M}^r = \mathcal{M} - (eB)^2 \lim_{eB\to 0} \frac{\mathcal{M}\cdot eB}{(eB)^2}, \qquad (63)$$

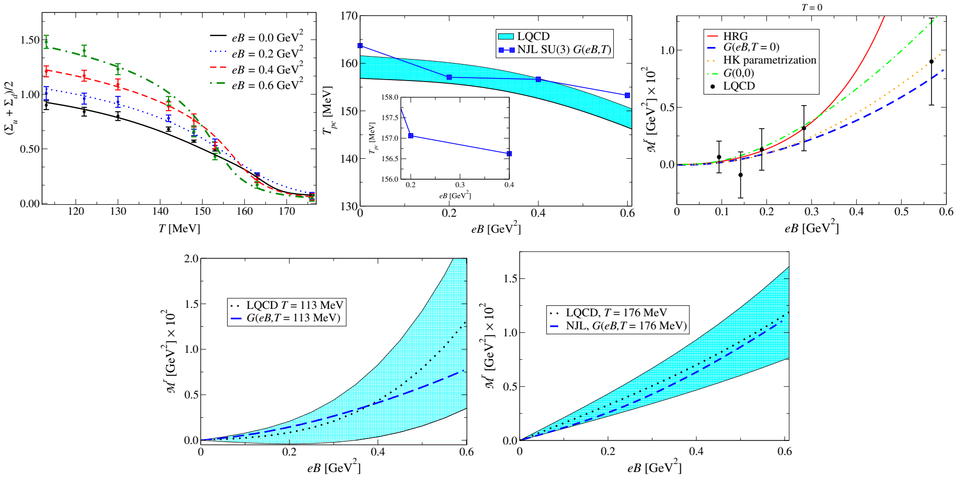
G.S. Bali et al.JHEP 04 (2013) 130

Obs: A constant G was adopted in this work.

The $\mathcal{M}^r$ with $G(B,T)$ is obtained in the SU(3) NJL model:**Tavares, W.R., Farias, R.L.S. et al. Eur. Phys. J. A 57, 278 (2021).**
For more discussions about $G(B,T)$ see: **Farias, R.L.S., Timóteo, V.S., Avancini, S.S. et al. Eur. Phys. J. A 53, 101 (2017).**

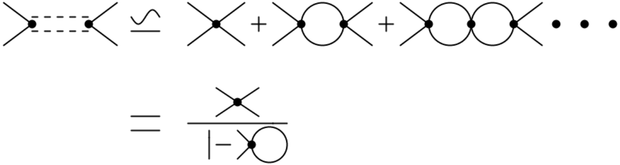
Sidney S. Avancini et al., Phys. Rev. D 103, 056009, (2021).

Renormalized Magnetization in NJL SU(3)+G(B,T)



Tavares W., Farias R. Avancini S. Timóteo V., Pinto M. Krein G..Eur.Phys.J.A 57 (2021) 9, 278

Mesons in NJL with $eB \neq 0$

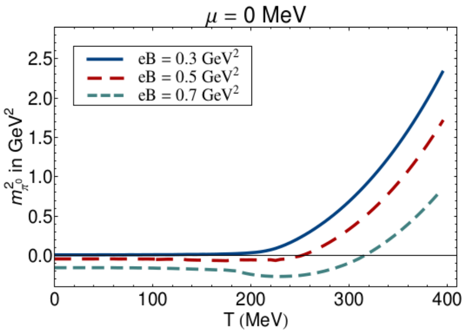


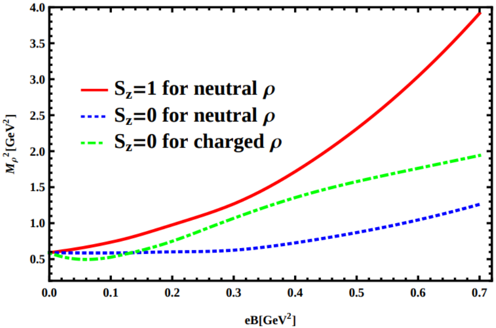
$$\frac{1}{i}\Pi_{\pi^0}(q^2) = -\int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}_{f,c,D}[\gamma^5\tau^3\tilde{S}(k)\gamma^5\tau^3\tilde{S}(k+q)],$$

$$\frac{1}{i}\Pi_{\sigma}(q^2) = \int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}_{f,c,D}[\tilde{S}(k)\tilde{S}(k+q)],$$

$$\frac{1}{i}\Pi_{\rho}^{\mu\nu,33}(q^2) = \int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}_{f,c,D}[\gamma^\mu\tau^3\tilde{S}(k)\gamma^\nu\tau^3\tilde{S}(k+q)],$$

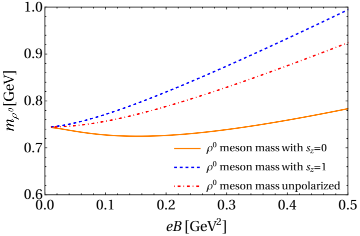
Mesons in NJL with $eB \neq 0$ – nMFIR procedures

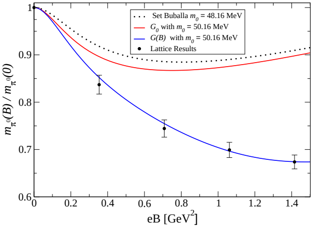


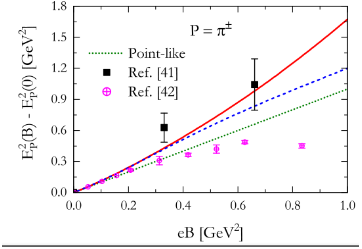


Right:Hao Liu, Lang Yu, Mei Huang. Phys. Rev. D 91, 014017 (2015).
Left:Sh. Fayazbakhsh, S. Sadeghian, and N. Sadooghi .Phys. Rev. D 86, 085042 (2012).

Mesons in NJL with $eB \neq 0$ – MFIR procedures







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Summary

Magnetic Fields in QCD

- **Magnetic Fields:** We expect very strong magnetic fields in noncentral HIC, magnetars and the primordial universe.

Regularization

- **nonMFIR procedures:** Is a set of regularization prescriptions that does not separate the vacuum from the magnetic field. Usually, makes use of Form-Factors.

- **MFIR procedure:** Is a regularization prescription, originally performed in QED defined as a subtraction of divergences. For effective model purposes, the separation of vacuum and magnetic field contributions is performed, and one regularizes just the remaining divergent contribution.

- **VMR procedure:** Is a regularization prescription based on MFIR, that includes a mass-independent vacuum-magnetic contribution in the free energy.

Conclusions

- **MFIR vesus non-MFIR:** The MFIR procedure avoids that regularization artifacts that changes several physical quantities, e.g., meson masses, magnetization among others.
- **Examples in literature:** This procedure has been applied in the context meson masses, thermodynamics, decay of neutral mesons among others.
- **Future applications:** This procedure can be interesting to separate divergences in renormalizable theories and beyond mean field approaches.

Gracias!

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