Fermions in external fields, the state of the art

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Contents

Magnetic fields everywhere

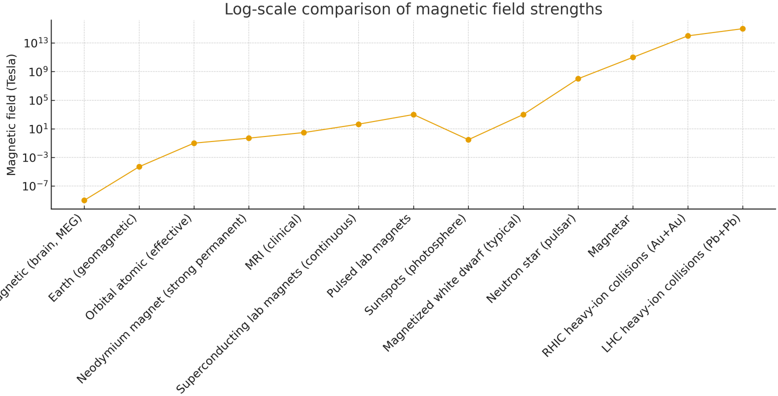


Figure 1: Magnetic fields in the universe.

Fermions in Magnetic fields

Dirac equation

$$(i\not{D} - m)\psi = 0, \qquad D_\mu = \partial_\mu - ieA_\mu.$$

Fermion propagator

In the proper-time representation,

$$\begin{aligned} S_F(x-y) &= (i\not{\partial} + eB\not{F} + m)\int_0^\infty ds\,\frac{eB}{4\pi\sinh(eBs)} \\ &\quad \exp\left[-is\left(m^2 - \frac{(x-y)^2}{4s^2} - \frac{eB}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)\right], \end{aligned}$$

with $\not{F} = \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$, and $F_{\mu\nu}$ is the electromagnetic field strength tensor.

Fermions in Magnetic fields

For a uniform magnetic field in the z-direction,

$$S_F(x,y) = \int_0^\infty ds\, \frac{eB}{(4\pi s)^{3/2}} \frac{e^{-im^2 s}}{\sinh(eBs)} \left(m + i\gamma^\mu \partial_\mu\right) \exp\left[i\mathcal{L}(x,y;s)\right]$$

where the Lagrangian is:

$$\begin{aligned} \mathcal{L}(x,y;s) &= -\frac{eB}{2}\coth(eBs)[(x_1-y_1)^2+(x_2-y_2)^2] \\ &\quad -\frac{(x_3-y_3)^2}{4s}+\frac{(x_0-y_0)^2}{4s} \\ &\quad +\frac{ieB}{2}[(x_1-y_1)(x_2+y_2)-(x_2-y_2)(x_1+y_1)]. \end{aligned}$$

Fermions in Magnetic fields

In the Ritus representation,

$$S_F(x,y) = \int \frac{dp_0 dp_3}{(2\pi)^2} e^{-ip_0(x_0-y_0)+ip_3(x_3-y_3)} \sum_{n,p_y} \mathbb{E}_{n,p_y}(x_\perp) \mathcal{S}_F^n(p) \bar{\mathbb{E}}_{n,p_y}(y_\perp),$$

with $p_\parallel^2 = p_0^2 - p_3^2$, $p_\perp^2 = \frac{1}{2}[(p_1)^2 + (p_2)^2]$ and the Ritus matrices are such that

$$(i \not{D})^2 \mathbb{E}_{n,p_y} = p^2 \mathbb{E}_{n,p_y}.$$

and verify completeness and orthogonality relations.

Fermions in Magnetic fields

For a uniform magnetic field,

$$\mathbb{E}_{n,p_y}(x_\perp) = \frac{1}{\sqrt{L_y}} e^{ip_y y} \phi_n(\xi) u_n,$$

where

$$\phi_n(\xi) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\xi^2/2} H_n(\xi), \qquad \xi = \sqrt{eB}(x - p_y/eB).$$

Here,

$$u_n = \begin{pmatrix} \sqrt{E_n + m}\,\chi_\sigma \\ \sigma^3 p_3 \sqrt{E_n - m}\,\chi_\sigma \end{pmatrix}$$

and $E_n = \sqrt{m^2 + 2neB + p_3^2}$.

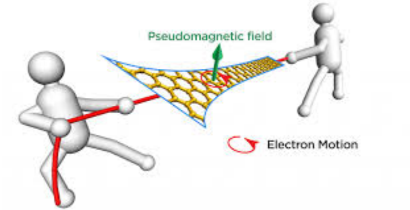
Fermions in Magnetic fields

- $\mathbb{E}_p$ are the states of fermions in the presence of the background field.
- The propagator is diagonal in momentum space
- Generalizations to non-uniform magnetic fields (n-th order SUSY-QM)

Strained graphene

Straintronics

Electric, thermal and mechanical properties of materials can be modified by strain.
In graphene, pseudomagnetic fields can be described in terms of a Dirac equation minimally coupled to an external field,

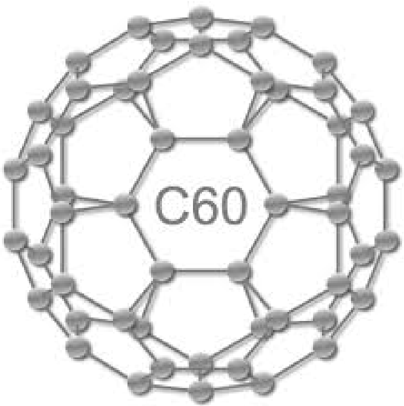


$$(i\not{D} - m)\psi = 0, \qquad D_\mu = \partial_\mu - ieA_\mu .$$

Fullerenes

Fullerenes

- Provide analogs for confinement in QCD-inspired models.
- Their response under strong EM fields allows to probe matter under extreme conditions.
- Their intrinsic curvature provides an analogy of QFT on curved manifolds.
- Exhibit strong couplings parallel to interactions in dense QCD matter.



Dirac equation on a sphere

We begin from the metric of a sphere S^2

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -R^2 & 0 \\ 0 & 0 & -R^2 \sin^2(\theta) \end{pmatrix}$$

and introduce the *dreibein*

$$e_{\mu}^{\hat{\alpha}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R\sin(\theta) \end{pmatrix}.$$

The Dirac equation in curved spacetime is

$$\left(e_{\hat{\alpha}}^{\mu}\gamma^{\hat{\alpha}} v_F p_\mu + m v_F^2\right)\Psi = 0, \tag{1}$$

with $p_\mu = -i\hbar\left(\partial_\mu + \Omega_\mu\right)$ where Ω_μ is the spin connection.

Dirac equation on a sphere

We represent these matrices in terms of the Pauli matrices as

$$\gamma^0 = \sigma^z, \qquad \gamma^1 = i\sigma^y, \qquad \gamma^2 = -i\sigma^x. \tag{2}$$

The spin connection

$$\Omega_\mu \equiv \frac{1}{2}\omega_{\hat{\alpha}\hat{\beta}\mu}\Sigma^{\hat{\alpha}\hat{\beta}}, \qquad \Sigma^{\hat{\alpha}\hat{\beta}} \equiv \frac{1}{4}\left[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\right], \quad \omega^{\hat{\alpha}}_{\hat{\beta}\mu} \equiv -e^{\nu}_{\hat{\beta}} D_\mu \left[e^{\hat{\alpha}}_{\nu}\right], \tag{3}$$

where the covariant derivative $D_\mu \left[e^{\hat{\alpha}}_{\nu}\right] \equiv \partial_\mu e^{\hat{\alpha}}_{\nu} - \Gamma^{\lambda}_{\mu\nu} e^{\hat{\alpha}}_{\lambda}$ with $\Gamma^{\lambda}_{\mu\nu}$ are the Christoffel's symbols.
The stationary Dirac equation is therefore

$$\left(-i\hbar v_F \left[\frac{1}{R}\sigma^x \left(\partial_\theta + \frac{\cot(\theta)}{2}\right) + \frac{1}{R\sin(\theta)}\sigma^y \partial_\varphi\right] + m v_F^2 \sigma^z\right)\Psi = E\Psi.$$

Dirac equation on a sphere

For the spinor,

$$\Psi(\theta,\varphi) = \sum_k e^{ik\varphi} \begin{pmatrix} f_k^+(\theta) \\ f_k^-(\theta) \end{pmatrix} = \sum_k e^{ik\varphi}\psi_k(\theta), \qquad k = \frac{1}{2}+n, \quad n \in \mathbb{Z}.$$

Decoupling the components

$$-\left[\frac{1}{\sin(\theta)}\partial_\theta \sin(\theta)\,\partial_\theta + \frac{k\cos(\theta)}{\sin^2(\theta)}\sigma^z - \left(\frac{k}{\sin(\theta)}\right)^2 - \frac{1}{4\sin^2(\theta)} - \frac{1}{4}\right]\psi_k$$

$$= \chi^2\psi_k.$$

with the notation

$$\chi^2 \equiv \tilde{E}^2 - \tilde{m}^2, \qquad \tilde{m} \equiv \frac{m v_F R}{\hbar}, \qquad \tilde{E} \equiv \frac{ER}{\hbar v_F}.$$

Dirac equation on a sphere

Upon performing the change of variables $x = \cos(\theta)$, each of the components of the spinor, it can be written as

$$\left[\frac{d}{dx}\left(1-x^2\right)\frac{d}{dx} - \frac{k^2 \mp kx + \frac{1}{4}}{1-x^2} + \chi^2 - \frac{1}{4}\right] f_k^{\pm} = 0.$$

We propose a solution of the form

$$f_k^{\pm}(x) = (1+x)^a (1-x)^b \, q_k^{\pm}(x),$$

where

$$\left[\left(1-x^2\right)\frac{d^2}{dx^2} + \left(\pm\frac{k}{|k|} - 2\left(|k|+1\right)x\right)\frac{d}{dx} - |k|\left(|k|+1\right) + \chi^2 - \frac{1}{4}\right] q_k^{\pm} = 0.$$

This differential equation has a square integrable solution if only if

$$\chi^2 - \left(|k| + \frac{1}{2}\right)^2 = w\left(w + 2a_k + 2b_k + 1\right), \qquad w \in \mathbb{N}^0,$$

Dirac equation on a sphere

which yields the quantization rule

$$\tilde{E}_{wk} = \sqrt{\left(w + |k| + \frac{1}{2}\right)^2 + \tilde{m}^2},$$

whereas the eigenstates are

$$\psi_k(x) \equiv \begin{pmatrix} C_{1w}(1+x)^{\frac{1}{2}\left|k+\frac{1}{2}\right|}(1-x)^{\frac{1}{2}\left|k-\frac{1}{2}\right|} P_w^{\left|k-\frac{1}{2}\right|,\left|k+\frac{1}{2}\right|}(x) \\ C_{2w}(1+x)^{\frac{1}{2}\left|k-\frac{1}{2}\right|}(1-x)^{\frac{1}{2}\left|k+\frac{1}{2}\right|} P_w^{\left|k+\frac{1}{2}\right|,\left|k-\frac{1}{2}\right|}(x) \end{pmatrix},$$

where $P_w^{\alpha,\beta}(x)$ are the Jacobi polynomials and $x \in [-1, 1]$, whereas C_{1w} and C_{2w} are constants.

Dirac equation a la Ritus

The Dirac equation in curved spacetime can be conveniently recast in the form

$$\left(e^i_{\hat{\jmath}}\gamma^{\hat{\jmath}}v_F p_i + m v_F^2\right)\Psi = -\sigma^z v_F p_0 \Psi,$$

or, assuming that the potentials are time independent,

$$-e^i_{\hat{\jmath}}\gamma^{\hat{\jmath}}p_i\Psi = \left(\frac{E}{v_F}\sigma^z + m v_F\right)\Psi.$$

Acting with $\left(\frac{E}{v_F}\sigma^z - m v_F\right)$ on the left hand side,

$$-e^i_{\hat{\jmath}}\gamma^{\hat{\jmath}}p_i e^a_{\hat{k}}\gamma^{\hat{k}}p_a\Psi = \chi^2\Psi,$$

which might suggestively be expressed as

$$-\frac{1}{2}\left(\left\{e^i_{\hat{\jmath}}\gamma^{\hat{\jmath}}p_i, e^a_{\hat{k}}\gamma^{\hat{k}}p_a\right\} + \left[e^i_{\hat{\jmath}}\gamma^{\hat{\jmath}}p_i, e^a_{\hat{k}}\gamma^{\hat{k}}p_a\right]\right)\Psi = \chi^2\Psi.$$

Dirac equation a la Ritus

After lengthy but straightforward algebra, we can prove that

$$\begin{aligned}\left\{e^i_{\hat{\jmath}}\gamma^{\hat{\jmath}}p_i, e^a_{\hat{k}}\gamma^{\hat{k}}p_a\right\} &= \eta^{\hat{k}\hat{l}}e^i_{\hat{k}}p_i e^a_{\hat{l}}p_a + 2\Sigma^{\hat{k}\hat{l}}\mathfrak{F}_{\hat{k}\hat{l}} \\ &= D^2 + 2\Sigma^{\hat{k}\hat{l}}\mathfrak{F}_{\hat{k}\hat{l}}\end{aligned}$$

where we define the *electromagnetic* and *spin* tensors as

$$\mathfrak{F}_{\hat{k}\hat{l}} \equiv \left[e^i_{\hat{k}}p_i, e^a_{\hat{l}}p_a\right], \qquad \Sigma^{\hat{k}\hat{l}} \equiv \frac{1}{4}\left[\gamma^{\hat{k}}, \gamma^{\hat{l}}\right]$$

Notice that the only nonvanishing component of the field tensor is

$$\mathfrak{F}_{12} = \hbar^2\frac{\cot(\theta)}{R}\frac{\partial_\varphi}{R\sin(\theta)},$$

which, of course, gives

$$\left(-i\hbar v_F\left[\frac{1}{R}\sigma^x\left(\partial_\theta + \frac{\cot(\theta)}{2}\right) + \frac{1}{R\sin(\theta)}\sigma^y\partial_\varphi\right] + mv_F^2\sigma^z\right)\Psi = E\Psi.$$

Dirac equation a la Ritus

From the solutions

$$\begin{aligned}\psi_k(x) &= \begin{pmatrix} C_{1w}(1+x)^{\frac{1}{2}\left|k+\frac{1}{2}\right|}(1-x)^{\frac{1}{2}\left|k-\frac{1}{2}\right|}P_w^{\left|k-\frac{1}{2}\right|,\left|k+\frac{1}{2}\right|}(x) \\ C_{2w}(1+x)^{\frac{1}{2}\left|k-\frac{1}{2}\right|}(1-x)^{\frac{1}{2}\left|k+\frac{1}{2}\right|}P_w^{\left|k+\frac{1}{2}\right|,\left|k-\frac{1}{2}\right|}(x) \end{pmatrix} \\ &= \begin{pmatrix} f^+ \\ f^- \end{pmatrix},\end{aligned}$$

which we can write as

$$\psi_k(x) = \mathbb{A}\left(C_{1w}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_{2w}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right),$$

we define

$$\mathbb{A} \equiv \begin{pmatrix} f^+ & 0 \\ 0 & f^- \end{pmatrix}.$$

Dirac equation a la Ritus

Interestingly

$$\int_{-1}^{1} dx (\mathbb{A})_{mk}(x) (\mathbb{A})_{nk}(x) \equiv \mathbb{I}_{mnk} = \mathbb{I}_{2\times 2}\delta_{mn}$$

and

$$\mathbb{I}_{2\times 2}\delta(x-y) = \sum_{w\in\mathbb{N}^0} (\mathbb{A}_{wk}(y) (\mathbb{A})_{wk}(x) \, .$$

Thus, the $\mathbb{A}$ matrices correspond to the Ritus matrices on curved space.

Fermion propagator in curved space

We write the fermion propagator as

$$\begin{aligned} S(x,y) &= \int dp \mathbb{A}_{pk}(y) S_F(p) \mathbb{A}_{pk}(x) \\ &= \sum_{w\in\mathbb{N}^0} \mathbb{A}_{wk}(y) S_F(p(w)) \mathbb{A}_{wk}(x) \,. \end{aligned}$$

Then, the current density

$$j^{\mu}(x,y) = \lim Tr\left[\gamma^{\mu} S(x,y)\right]$$

becomes

$$\begin{aligned} j^{\mu}(x,y) &= Tr\left[\gamma^{\mu} \sum_{w\in\mathbb{N}^0} \frac{1}{E_{wk}} \mathbb{A}_{wk}(y) \mathbb{A}_{wk}(x)\right] \\ &= \sum_{w\in\mathbb{N}^0} \frac{1}{E_{wk}} Tr\left[\gamma^{\mu} \mathbb{A}_{wk}(y) \mathbb{A}_{wk}(x)\right], \end{aligned}$$

with $\tilde{E}_{wk} = \sqrt{\left(w + |k| + \frac{1}{2}\right)^2 + \tilde{m}^2}$.

Fermion propagator in curved space

The charge density

$$j^0(x,y) = \frac{1}{mv_F^2}\sum_{w\in\mathbb{N}^0}\frac{1}{\left(\frac{\Delta}{R}\right)^2\left(w+|k|+\frac{1}{2}\right)^2+1}\psi_{kw}(y)\cdot\psi_{kw}(x)$$

with

$$\psi_{kw}(x) = \begin{pmatrix} f_{1w}^{\left|k-\frac{1}{2}\right|,\left|k+\frac{1}{2}\right|}(x) \\ f_{2w}^{\left|k+\frac{1}{2}\right|,\left|k-\frac{1}{2}\right|}(x) \end{pmatrix}$$

and $\Delta \equiv \frac{\hbar}{mv_F}$.
Notice that when $R\to\infty$,

$$j^0(x,y;r) = \frac{2}{mv_F^2}\delta(x-y),$$

namely, we recover the flat space charge density.

In summary

- We have solved the Dirac equation on a sphere
- We factorize the solution as a free flat-space spinor multiplied by a matrix that contains all the information of the curvature of space
- Such matrix correspond to the Ritus matrix of curved space
- The propagator and current density have been obtained explicitly
- We recover the behavior of the charge density of flat space as $R \to \infty$

Outlook

- Article on the way!
- Explore other systems with curvature
- Explore other external fields (vorticity)

Thank you!