Time ordered regulator in Schwinger propagator under electomagnetic external fields

Cristián Villavicencio

CCE & dep. Cs. Basicas - Universidad del Bio-Bio

2nd Latin American Workshop on Electromagnetic Effects in QCD

November 2025

Motivation

Schwinger's method:

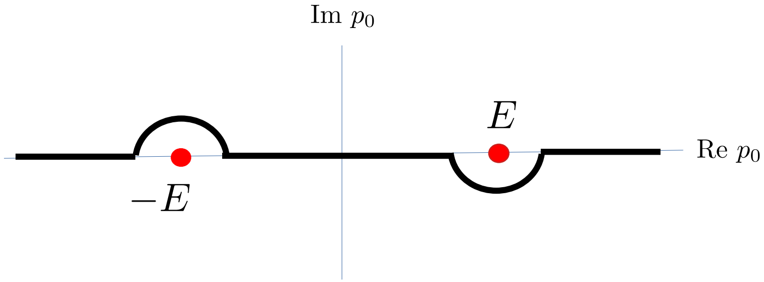
Hamiltionian

$$G(x,x') = \langle x'|\frac{i}{\not{\Pi} - m}|x\rangle = \langle x'|\frac{i(\not{\Pi}+m)}{\not{\Pi}^2 - m^2}|x\rangle = \int_0^\infty ds \langle x'|(\not{\Pi}+m)\, e^{-is(-\not{\Pi}^2+m^2-i\epsilon)}|x\rangle$$

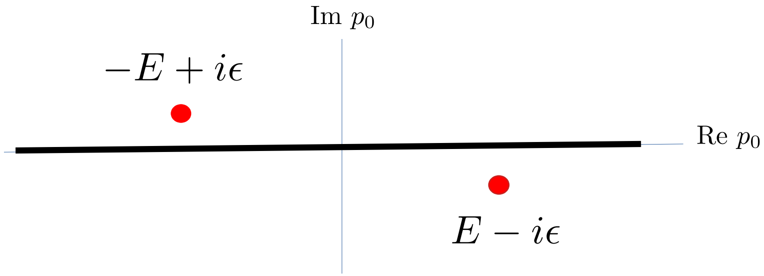
the $i\epsilon$ term is for convergence of proper time integration

N. C. Tsamis and R. P. Woodard,
"Schwinger's propagator is only a Green's function",
Class. Quant. Grav. 18 (2001)

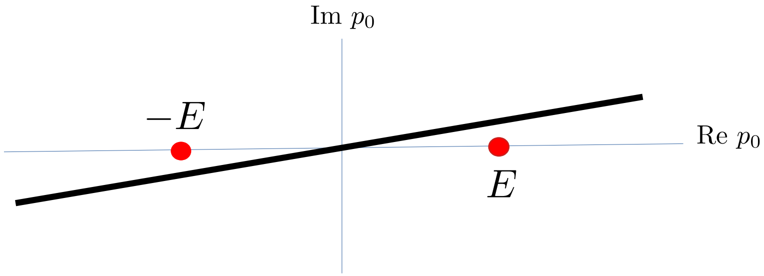
Time ordered regulator



$$\sim \int dp_0 \, \frac{e^{-ip_0(x_0 - x_0')}}{p_0^2 - E^2}$$



$$\sim \int dp_0 \, \frac{e^{-ip_0(x_0 - x_0')}}{p_0^2 - E^2 + i\epsilon}$$

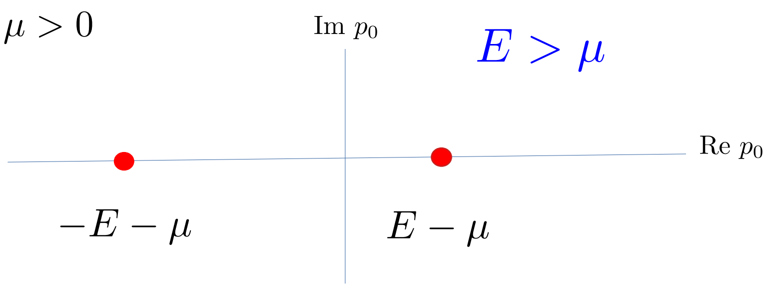


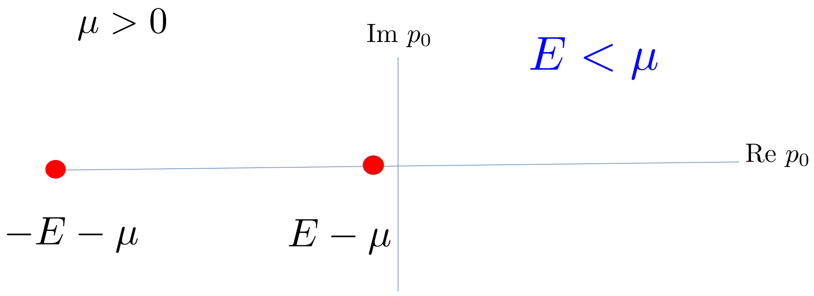
$$\sim \int dp_0 \, \frac{e^{-ip_0(x_0 - x_0')}}{(p_0 + i\epsilon p_0)^2 - E^2}$$

$$G_F(p_0, p) = G(p_0 + i\epsilon p_0, \boldsymbol{p})$$

Finite chemical potential

$$P = (p_0 + \mu, \boldsymbol{p})$$





$$G_F \sim \frac{i}{(P_0 + i\epsilon p_0)^2 - E^2} \approx \frac{i}{P_0^2 - E^2 + 2i\epsilon P_0(P_0 - \mu)}$$

$$= \frac{i}{P_0^2 - E^2 + i\epsilon} - 2\pi\theta(-P_0(P_0 - \mu))\delta(P_0^2 - E^2)$$

is the Dolan-Jackiw propagator in the limit $T \to 0$

$$G_{\rm DJ} \sim \frac{i}{P_0^2 - E^2 + i\epsilon} - 2\pi\left[\theta(P_0)n_F(P_0 - \mu) + \theta(P_0)n_F(\mu - P_0)\right]\delta(P_0^2 - E^2)$$

fermion number density → has to be calculated with TO propagator

$$n = \langle \psi^\dagger \psi \rangle = \mathrm{tr}\left[\gamma_0 G_F(x,x)\right]$$

$$= \int \frac{d^4p}{(2\pi)^4} 4P_0 \theta(-P_0(P_0-\mu))\, 2\pi\delta(P_0^2 - E^2)$$

$$= \frac{\mathrm{sign}(\mu)}{\pi^2} \int_0^\infty dp p^2 \theta(\mu - E)$$

$$= \frac{\mathrm{sign}(\mu)}{3\pi^2} (\mu^2 - m^2)^{1/3} = \frac{p_F^3}{3\pi^2}$$

Schwinger's proper time method

$$\frac{i}{P_0^2 - E^2 + i\epsilon P_0(P_0 - \mu)} = \theta(P_0(P_0 - \mu)) \int_0^\infty ds\, e^{is(P_0^2 - E^2 + i\epsilon)} + \theta(-P_0(P_0 - \mu)) \int_{-\infty}^0 ds\, e^{-is(P_0^2 - E^2 + i\epsilon)}$$

Chodos, Everding, Owen; Phys. Rev. D **42** (1990)

A simplified notation

$$\frac{i}{D + i\eta} = \int_{-\infty}^\infty ds\, r_s(\eta)\, e^{isD} \qquad r_s(\eta) = e^{-s\eta}\, \mathrm{sign}(s)\, \theta(s\eta)$$

Mizher, Raya, Villavicencio; Int. J. Mod. Phys. B **30** (2015) 1550257

Finite temperature with Matsubara frequencies

$$P_0 = i\omega_n + \mu \qquad \frac{i}{-\omega_n^2 + 2i\mu\omega_n + \mu^2 - E^2} = \int_{-\infty}^{\infty} ds\, r_s(2\mu\omega_n)\, e^{is(-\omega_n^2+\mu^2-E^2)}$$

Mizher, Hernández-Ortiz, Raya, Villavicencio; Eur. Phys. J. C **78**, 912, (2018)

Vorticity → rigidly rotating cylinder model

$$P_0 = p_0 + \left(n + \frac{1}{2}\right)\Omega$$

$$G_F(x,x') = \sum_n \int_{-\infty}^{\infty} \frac{d^2 p_\parallel}{(2\pi)^2} \int_0^{\infty} \frac{dp_\perp p_\perp}{2\pi} \int_{-\infty}^{\infty} ds\, r_s(\epsilon\, P_0\, p_0)\, e^{is(P_0^2 - E^2)} V_n(x) V_n^\dagger(x')$$

Electric and magnetic field

$$G(x,x') = \langle x'|\frac{i}{\not{\Pi} - m}|x\rangle \qquad \Pi_\nu = \hat{p}_\nu + qA_\nu(\hat{x}) + \mu g_{\nu 0}$$

Finite magnetic field → no problem

$$\frac{i}{\not{\Pi} - m} \to \frac{i}{(\hat{p}_0 + \mu)\gamma_0 + \hat{p}^3\gamma_3 + \not{\Pi}_\perp - m}$$

In general the procedure is the same if

$$[\Pi_\mu, \hat{p}_0] = 0 \quad \Rightarrow \quad \partial_0 A_\mu = 0$$

The green function becomes local in time:

$$G(x,x') \quad \to \quad G(x_0 - x'_0; \boldsymbol{x}, \boldsymbol{x}') = \int \frac{dp_0}{2\pi} e^{-ip_0(x_0 - x'_0)} \tilde{G}(p_0; \boldsymbol{x}, \boldsymbol{x}')$$

Time-ordered propagator:

$$G_F(p_0; \boldsymbol{x}, \boldsymbol{x}') = \tilde{G}(p_0 + i\epsilon p_0; \boldsymbol{x}, \boldsymbol{x}')$$

Constant electric and magnetic field

$$G(x, x') = e^{i\Phi(x,x')} S(x - x')$$

Schwinger phase

$$\Phi(x, x') = q \int_{x'}^{x} d\xi^\nu (A_\nu(\xi) + g_{\nu 0}\, \mu/q)$$

$$\xi = xt + x'(1 - t)$$

Considering the condition $\partial_0 A(x) = 0$

$$A_0(x) = -E^i x_i, \qquad A_i(x) = \frac{1}{2}\epsilon_{ijk} B^j x^k$$

(symmetric gauge)

$$\Phi(x, x') = (\mu - q\boldsymbol{E} \cdot \bar{\boldsymbol{x}})\, (x_0 - x'_0) - [q\boldsymbol{B} \times \bar{\boldsymbol{x}}] \cdot (\boldsymbol{x} - \boldsymbol{x}')$$

$$\bar{\boldsymbol{x}} \equiv \frac{1}{2}(\boldsymbol{x} + \boldsymbol{x}')$$

careful with non-local terms ~ $\bar{\boldsymbol{x}}$

Wigner transformation

$$G(x, x') = e^{i\Phi(x-x';\,\bar{\boldsymbol{x}})} S(x - x')$$

$$\tilde{G}(p, \bar{\boldsymbol{x}}) = \int d^4r\, e^{ir\cdot p} e^{i\Phi(r;\bar{\boldsymbol{x}})}\, S(r) = \tilde{S}(P)$$

$$P_0 = p_0 + \mu - q\boldsymbol{E}\cdot\bar{\boldsymbol{x}}, \qquad \boldsymbol{P} = \boldsymbol{p} + q\boldsymbol{B}\times\bar{\boldsymbol{x}}$$

$$G_F(p;\bar{\boldsymbol{x}}) = \tilde{S}(P_0 + i\epsilon p_0, \boldsymbol{P})$$

Constant electric field $\boldsymbol{E} = E\hat{\boldsymbol{e}}_3$

$$G_F(p;z) = \int_{-\infty}^{\infty} ds\, r_s\big(\epsilon p_0(p_0 + \mu - qEz)\big)\, G_s(p;z)$$

$$G_s(p;z) = \exp\left\{ is\left[\frac{\tanh(qEs)}{qEs} P_{\parallel}^2 + p_{\perp}^2 - m^2\right]\right\}$$
$$\times \left\{ \not{P}_{\parallel}\left[1 + \tanh^2(qEs) - 2\tanh(qEs)\gamma_0\gamma_3\right] + (\not{p}_{\perp} + m)\left[1 - \tanh(qEs)\gamma_0\gamma_3\right]\right\}$$

$$P_{\parallel} = (p^0 + \mu - qEz, 0, 0, p^3)$$
$$p_{\perp} = (0, p^1, p^2, 0)$$
$$z = \frac{1}{2}(x^3 + x'^3),$$

how to interpret $\tilde{\mu} = \mu - qEz$

Under a gauge transformation, the chemical potential must be redefined to obtain the same physical system.

W. Dittrich and H. Gies, "*Probing the quantum vacuum. Perturbative effective action approach in quantum electrodynamics and its application*", vol. 166 (Springer, 2000)

Also, effective chemical potential can be described through number density

$$n(\tilde{\mu}, E) \qquad \longleftrightarrow \qquad \tilde{\mu}(n, E)$$

Electric current and charge number density expectation values

$$j(\tilde{\mu}, E) = q \langle \bar{\psi}\gamma^3\psi \rangle = -q\,\mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\,\gamma^3 G_F(p;z),$$
$$n(\tilde{\mu}, E) = \langle \psi^\dagger\psi \rangle = -\mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\,\gamma^0 G_F(p;z),$$

- Expectation value from time-ordered propagator → unchanged stable value
- Different from Kubo formula (retarded correlator) → linear response

This current must be considered as quasistatic charge motion

After trace in spin and integrating spatial momentum components

$$n(\tilde{\mu}, E) = \frac{1}{2\pi^{5/2}} \int_{-\infty}^{\infty} ds\, f_s\, \frac{\left[1 + \tau_s{}^2 (qEs)^2\right]}{2s\sqrt{\tau_s}\sqrt{is}}$$

$$\tau_s \equiv \frac{\tanh(qEs)}{qEs}$$

$$j(\tilde{\mu}, E) = \frac{1}{2\pi^{5/2}} \int_{-\infty}^{\infty} ds\, f_s\, \frac{q^2 E \sqrt{\tau_s}}{\sqrt{is}}$$

$$f_s \equiv \int_{-\infty}^{\infty} dp_0\, r_s\big(\epsilon p_0 (p_0 - \tilde{\mu})\big)\, ip_0\, e^{is\left[\tau_s\, p_0^2 - m^2\right]},$$

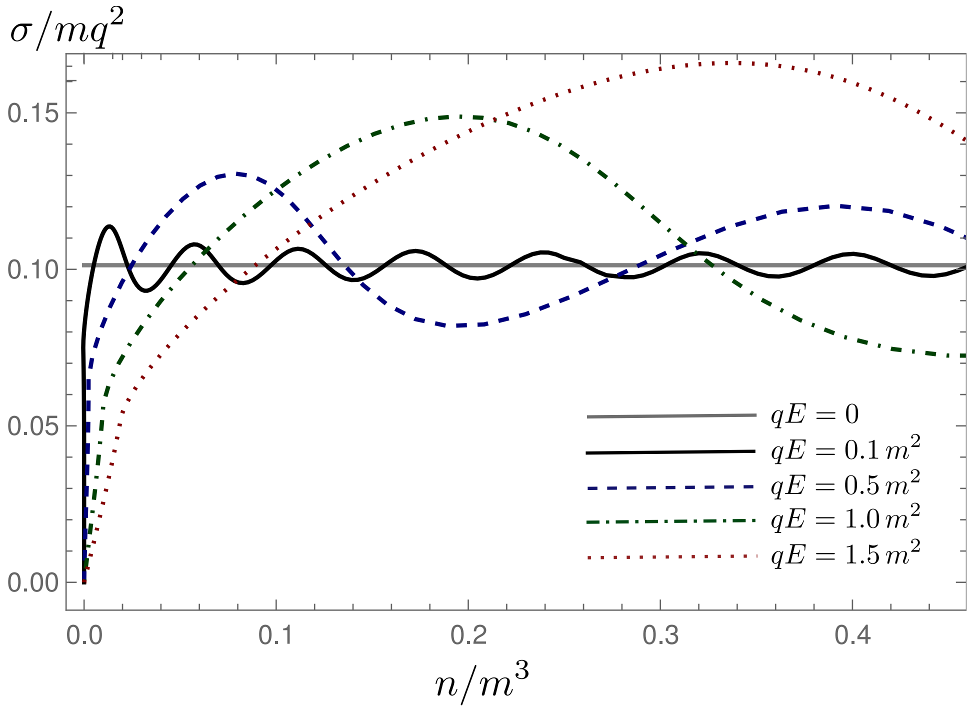
$$j(\tilde{\mu}, E) = \frac{\text{sign}(\tilde{\mu})}{2\pi^{5/2}} \int_0^\infty ds\, g_s\, \frac{q^2 E}{s^{3/2}\sqrt{\tau_s}}$$

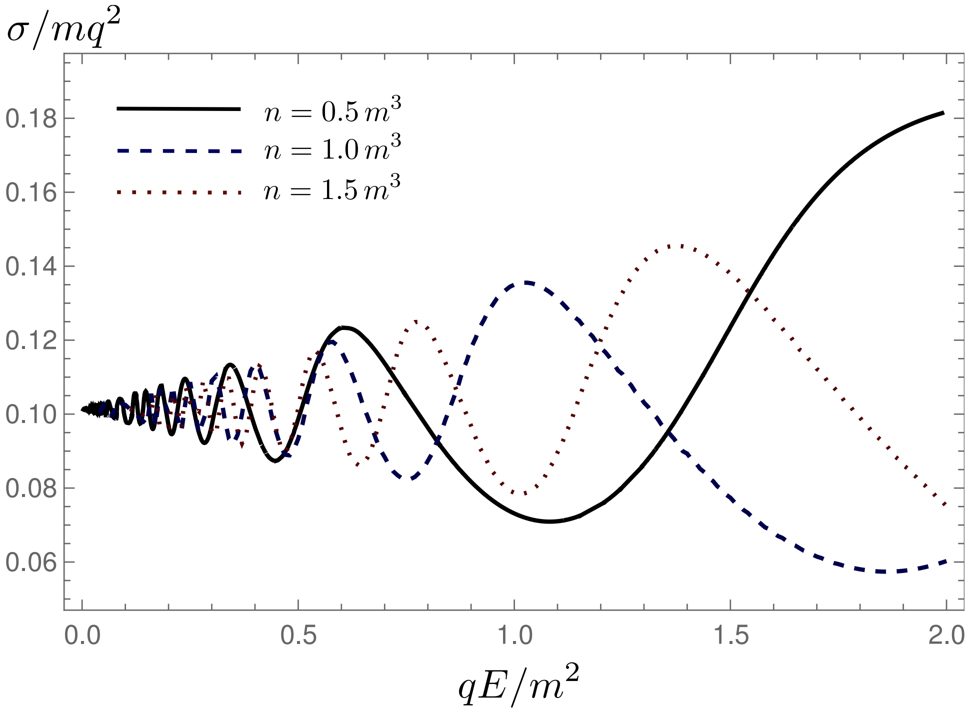
$$n(\tilde{\mu}, E) = n(\tilde{\mu}, 0) + \frac{\text{sign}(\tilde{\mu})}{2\pi^{5/2}} \int_0^\infty ds\, g_s\, \frac{1}{2s^{5/2}} \left[{\tau_s}^{-3/2} - 1 + {\tau_s}^{1/2} (qEs)^2 \right]$$

$$n(\tilde{\mu}, 0) = \frac{\text{sign}(\tilde{\mu})}{3\pi^2} \theta(|\tilde{\mu}| - m)(\tilde{\mu}^2 - m^2)^{3/2}$$

$$g_s \equiv \sin\left(sm^2 - \frac{\pi}{4}\right) - \sin\left(s(m^2 - \tau_s \tilde{\mu}^2) - \frac{\pi}{4}\right),$$

Electric conductivity $\sigma(n, E) = j/E$



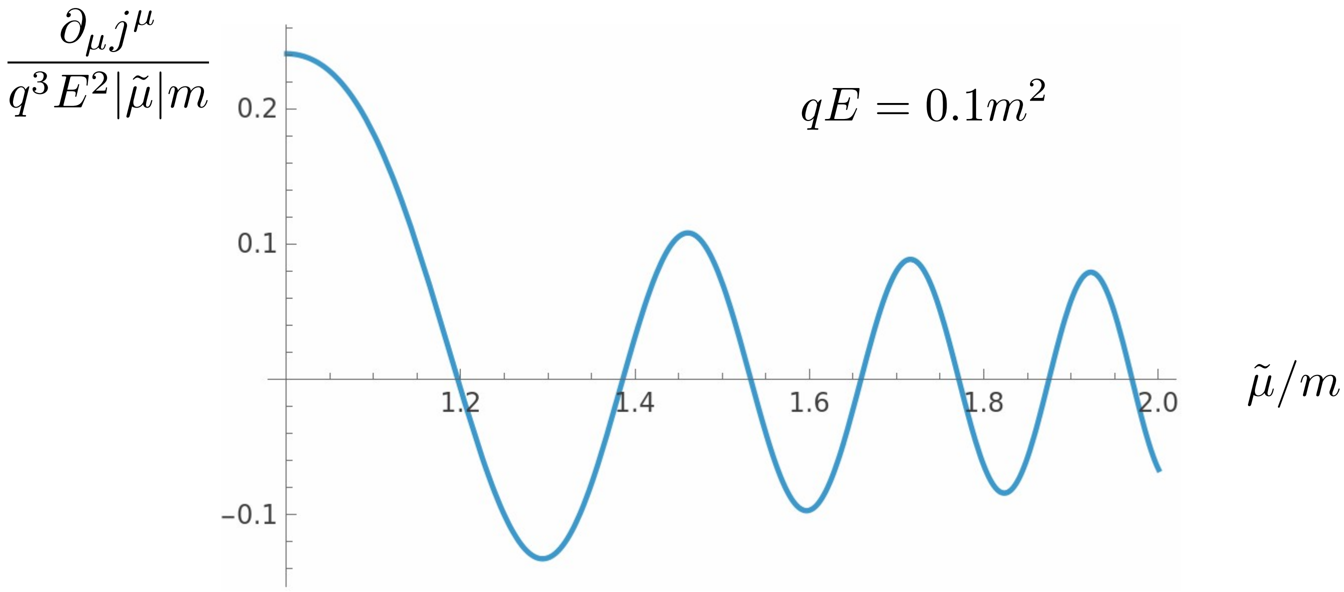


arXiv: 2305.12271 [hep-ph]

$$\sigma(n, 0) = \frac{q^2 m}{\pi^2}$$

Continuity equation → oscillates around zero

$$\partial_\mu j^\mu = -\frac{\partial j}{\partial z} = \frac{q^3 E^2 |\tilde{\mu}|}{\pi^{5/2}} \int_0^\infty \frac{ds}{\sqrt{s}} \sqrt{\tau_s} \cos\left(s(m^2 - \tau_s \tilde{\mu}^2) - \frac{\pi}{4} \right) \neq 0$$



The role of coordinate is not clear

Consider Green function non-local in time $\bar{x}_0 = t$

with the following gauge

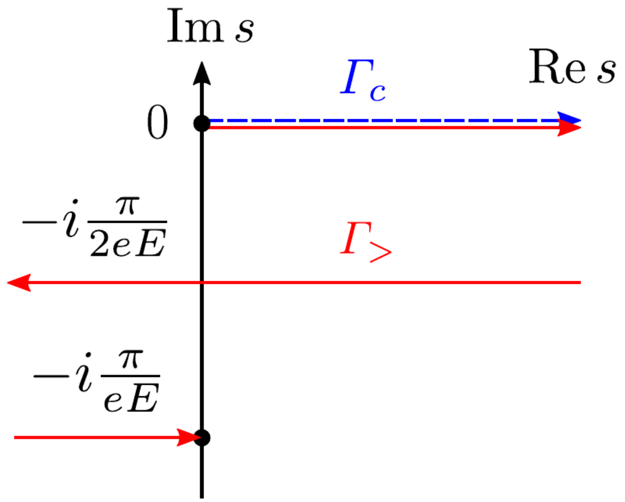
$$A_0 = -aqEz, \qquad A_3 = (1-a)qEt$$

$$\tilde{\mu} = \mu - aqEz$$

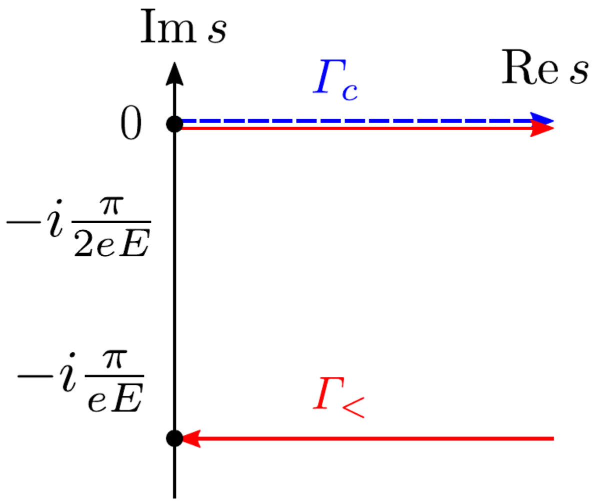
- Time-ordered propagator depends on a.
- Expectation values depends on a only through $\tilde{\mu}$
- $\partial_\mu j^\mu = 0$ if $a = 0$

???

other proposals → Ward identity on axial currents



in-out prescription



in-in prescription

$$\Gamma_{\lessgtr} \quad \Rightarrow \quad z \lessgtr 0$$

Copinger, Fukushima, Pu; Phys. Rev. Lett. **121**, 261602 (2018)

Note is independent of the TO propagator

Summary

- Time-ordered propagator can be defined in Schwinger's proper time formalism at finite electromagnetic field if $\partial_0 A_\mu = 0 \Rightarrow$ local in time coordinates.
- Time-ordered propagator *may* be defined in general considering Wigner transformation
- Electric potential ambiguities must be absorbed in chemical potential.
- Functions depending on numbert density instead of chemical potential is free of ambiguities
- 4-current not conserved in the example (caused by quasistatic approximation?)

Summary

- Time-ordered propagator can be defined in Schwinger's proper time formalism at finite electromagnetic field if $\partial_0 A_\mu = 0 \Rightarrow$ local in time coordinates.
- Time-ordered propagator *may* be defined in general considering Wigner transformation
- Electric potential ambiguities must be absorbed in chemical potential.
- Functions depending on numbert density instead of chemical potential is free of ambiguities
- 4-current not conserved in the example (caused by quasistatic approximation?)

Gracias!