Regularization methods in four-point interaction theories: handling magnetic field and chiral imbalance effects

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- André Gonçalves (undergrad student at UFSM)

Regularization of four-point interaction theories

In the set of the two lectures we will study:

- **First Lecture on 27 November: Handling Chiral Imbalance Effects**

- **Second Lecture on 28 November: Handling Magnetic Field Effects**

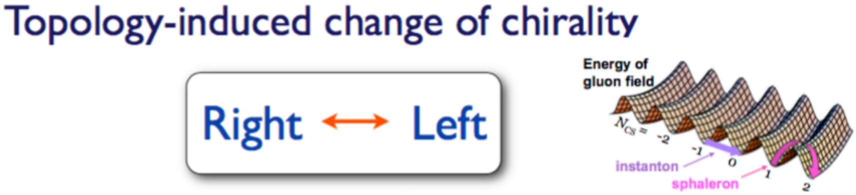
We will apply these effects in the **two-flavor Nambu–Jona-Lasinio model**[1][2], however, several of these procedures can be applied in other effective models or theories.

[1] Y. Nambu and G,.J. Lasinio. Phys. Rev. 122, 345 – Published 1 April, 1961.
[2] Y. Nambu and G,.J. Lasinio. Phys. Rev. 124, 246 – Published 1 October, 1961

Content of the first lecture

Physical Motivations

- The nontrivial nature of the vacuum of non-Abelian gauge theories in general, and of QCD in particular, allows for the existence of topological solutions like **instantons** and **sphalerons**.



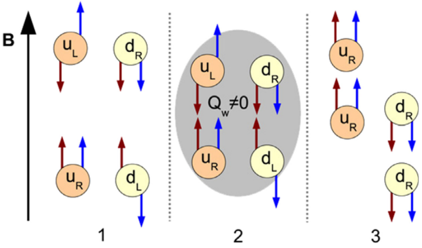
- from the Adler-Bell-Jackiw anomaly in the context of QCD, they can generate an asymmetry between the number of left- and right-handed quarks.

R. W. Jackiw, Int.J.Mod.Phys. A25 (2010) 659-667
G. 't Hooft, PRD 14, 3432 (1976); F. R. Klinkhamer
N. S. Manton, PRD 30, 2212 (1984).

Physical Motivations

- **The chiral imbalance is the difference between right- and left-handed quarks.**
- **No sign problem in LQCD!**
- **Event-by-event C and CP violating processes in heavy-ion collisions;** (Adler-Jackiw anomaly $\rightarrow$ chiral imbalanced medium)[2];
- **chiral anomaly transport model[3]:** $\sim \mu_5 = 2.1T + \sqrt{eB} > 300\text{MeV}$

- **Chiral Magnetic Effect[1]:** $\boxed{\vec{J} = \frac{e\mu_5}{2\pi^2} \int d^3x\vec{B}}$



[1] K. Fukushima, D. Kharzeev, H. Warringa. Phys.Rev.D 78 (2008) 074033
[2] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. 53, 43 (1981).
[2] H. Aoyama, H. Goldberg, and Z. Ryzak, Phys. Rev. Lett. 60, 1902 (1988).
[3] Zilin Yuan, Anping Huang, Wen-Hao Zhou, Guo-Liang Ma and Mei Huang. Phys. Rev. C 109, L031903
Figure: D. Kharzeev, L. Mclerran and H. Warringa. Nucl.Phys.A 803 (2008) 227-253

Physical Motivations

The CME is not applied just to Quantum Chromodynamics, but also to a range of different physical environments, e.g,

- condensed matter systems and hydrodynamics;
- has been observed in recent condensed matter experiments

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Letter | Published: 08 February 2016

Chiral magnetic effect in $ZrTe_5$

Qiang Li, Dmitri E. Kharzeev, Cheng Zhang, Yuan Huang, I. Pletikosić, A. V. Fedorov, R. D. Zhong, J. A. Schneeloch, G. D. Gu & T. Valla

Nature Physics **12**, 550–554 (2016) | Cite this article

- CME has been observed in zirconium pentatelluride.

Physical Motivations

- The **chiral imbalance effects** in the QCD phase diagram can be implemented by means of the **canonical ensemble** with the introduction of a chiral chemical potential μ_5.

$$\mu_5\bar{\psi}\gamma_0\gamma_5\psi \Rightarrow \text{QCD Lagrangian density} \Rightarrow \text{quark models of QCD!} \qquad (1)$$

PHYSICAL REVIEW D **78,** 074033 (2008)

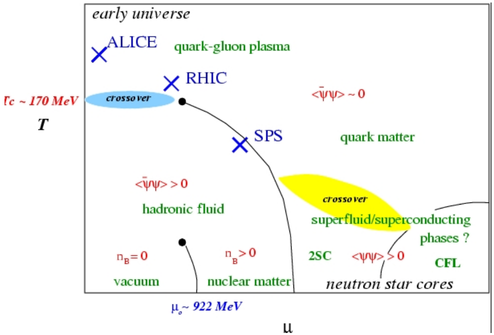
Chiral magnetic effect

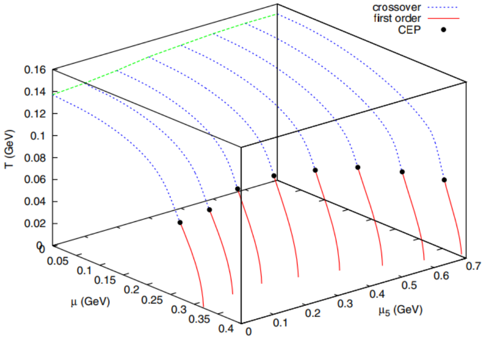
Kenji Fukushima,[1,*] Dmitri E. Kharzeev,[2,+] and Harmen J. Warringa[2,‡]
[1]*Yukawa Institute, Kyoto University, Kyoto, Japan*
[2]*Department of Physics, Brookhaven National Laboratory, Upton New York 11973, USA*
(Received 2 September 2008; published 31 October 2008)

Topological charge changing transitions can induce chirality in the quark-gluon plasma by the axial anomaly. We study the equilibrium response of the quark-gluon plasma in such a situation to an external magnetic field. To mimic the effect of the topological charge changing transitions we will introduce a chiral chemical potential. We will show that an electromagnetic current is generated along the magnetic field. This is the chiral magnetic effect. We compute the magnitude of this current as a function of magnetic field, chirality, temperature, and baryon chemical potential.

DOI: 10.1103/PhysRevD.78.074033 PACS numbers: 12.38.−t, 12.38.Mh

QCD Phase Diagram





Left Figure: B.-J. Schaefer and M. Wagner, Prog.Part.Nucl.Phys. 62 (2009) 381
Right Figure: B. Wang, Y. L. Wang, Z. F. Cui and H. S. Zong. Phys. Rev. D **91** (2015) no.3, 034017

Phase structure of a chiral imbalanced medium

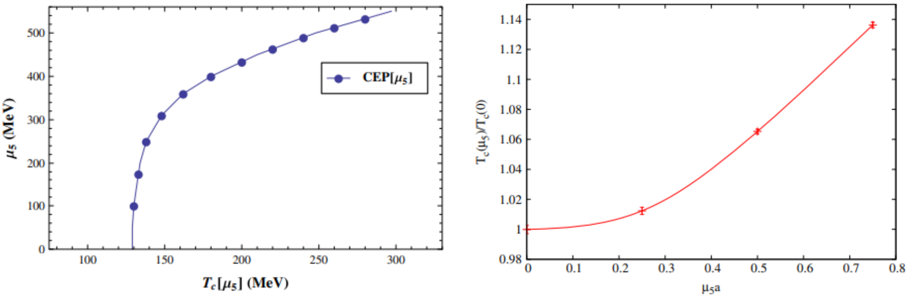


Figure: Pseudocritical temperatures of chiral transition as a function of the chiral chemical potential in LSMq (left) and LQCD (right).

V. V. Braguta, E.-M. Ilgenfritz, A. Yu. Kotov, B. Petersson and S. A. Skinderev. D 93, 034509 (2016)
V. V. Braguta, et al. JHEP06(2015)094

Inverse Chiral catalysis?

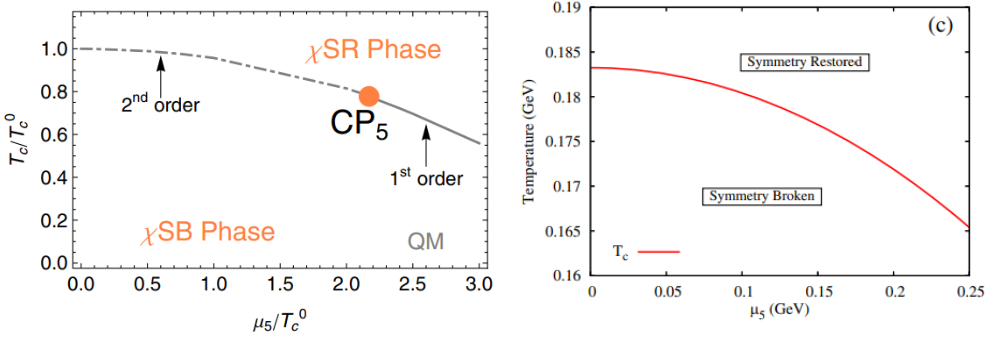
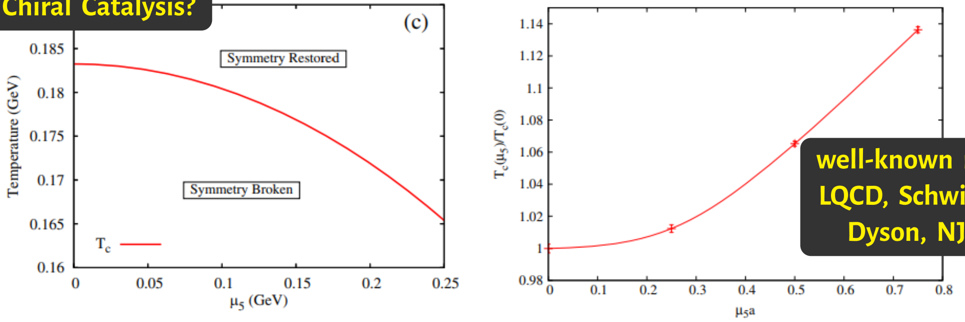


Figure: Left: Pseudocritical temperatures of chiral transition as a function of the chiral chemical potential in the quark-meson model[4] and Nambu–Jona-Lasinio model [5].

Left Plot: M. Ruggieri. Phys. Rev. D 84, 014011 (2011)
Right Plot: Snigdha Ghosh et al. Phys. Rev. D 109, 016021 (2024)

Phase structure of a chiral imbalanced medium

Inverse Chiral Catalysis?



well-known result LQCD, Schwinger-Dyson, NJL...

Figure: Pseudocritical temperatures of chiral transition as a function of the chiral chemical potential in Nambu–Jona-Lasinio model (left) and LQCD (right).

Left Plot: Snigdha Ghosh et al. Phys. Rev. D 109, 016021 (2024)

Right Plot: V. V. Braguta, E.-M. Ilgenfritz, A. Yu. Kotov, B. Petersson and S. A. Skinderev. D 93, 034509 (2016)

Nambu–Jona-Lasinio model SU(2)

The lagrangian of the two-flavor NJL model within chiral imbalanced medium is given by

$$\mathcal{L} = \overline{\psi}\left[i\gamma^{\mu}\partial_{\mu} - \tilde{m} + \gamma_0\gamma_5\mu_5\right]\psi + G\left[(\overline{\psi}\psi)^2 + (\overline{\psi}i\gamma_5\vec{\tau}\psi)^2\right], \tag{2}$$

- **Basic definitions:** the current quark masses matrix $\tilde{m} = \mathrm{diag}(m_u, m_d)$ in which $m_u = m_d = m_0$; τ are the Pauli matrices; ψ is the spinor representing the quark fields $\psi = (\psi_u \quad \psi_d)^T$; G is the coupling constant and μ is the quark chemical potential.
- **This is a non-renormalizable model:** the coupling constant G has dimension of MeV^{-2}.
- **This model has three parameters to set:** the cutoff Λ, the current quark mass m_0 and the coupling G.
  The parameters are chosen in order to obtain the vacuum values of the pion decay constant, $f_\pi \approx 93$ MeV; the pion mass $m_\pi \approx 135$ MeV and the quark condensate $\langle\bar{u}u\rangle \approx (-250 \quad \mathrm{MeV})^3$.

S.P. Klevansky. Rev.Mod.Phys. 64 (1992) 649-708

Nambu–Jona-Lasinio SU(2): some details

To find the quark dispersion relation we start by looking to the functional generator:

$$Z = \mathcal{N} \int D\bar{\psi} D\psi \exp\left[i \int d^4x \mathcal{L}(\bar{\psi}, \psi) \right] \Rightarrow \text{H.S.} \Rightarrow \begin{cases} \sigma = -2G\bar{\psi}\psi \to -2G\langle\bar{\psi}\psi\rangle \\ \vec{\pi} = -2G\bar{\psi} i\gamma_5 \vec{\tau}\psi \end{cases}$$

where we performed the Hubbard-Stratanovich transformation (H.S.) with $\vec{\pi} = 0$ due to isospin symmetry. The partition function becomes

$$Z = \exp\left[-i\frac{\sigma^2}{4G} \int d^4x \right] \exp\left[N_c \int d^4x \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}_{f,D} \ln(\not{k} - m_c - \sigma + \mu_5\gamma_0\gamma_5) \right]. \tag{3}$$

with $N_c = 3$. Evaluating the trace over the flavor and Dirac indices using $\mathrm{Tr}\ln\mathcal{O} \equiv \ln\mathrm{Det}\,\mathcal{O}$

$$\omega_s(k, \mu_5) = \sqrt{(k + s\mu_5)^2 + M^2} \quad \text{where} \quad \begin{cases} M = m_0 + \sigma \\ s = \pm 1 \end{cases}$$

D. Bailin, A. Love. INTRODUCTION TO GAUGE FIELD THEORY. 1986.

Thermodynamic Potential

We will include temperature by means of the Matsubara formalism

$$\int_{\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \Rightarrow iT \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}, \quad \text{where} \quad \begin{cases} (k_0, \vec{k}) \Rightarrow (i\omega_n, \vec{k}) \\ \omega_n = (2n+1)\pi T \end{cases} \tag{4}$$

The thermodynamics of this model will be evaluated through the Landau Free energy, $\mathcal{F}$,

$$\mathcal{Z} = \exp\left[-\beta \int d^3x \mathcal{F}\right], \quad \beta = \frac{1}{T} \tag{5}$$

Then the thermodynamic potential will be given by

$$\mathcal{F}(T, \mu_5) = \frac{(M - m_0)}{4G} - \underbrace{N_c N_f \sum_{s=\pm 1} \int_{\Lambda} \frac{d^3k}{(2\pi)^3} \omega_s(k)}_{\text{needs regularization!}} - 2N_c N_f T \sum_{s=\pm 1} \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \log\left(1 + e^{-\frac{\omega_s(k)}{T}}\right) \tag{6}$$

$\boxed{\Lambda \text{ is the 3D Sharp cutoff} \Rightarrow \textbf{Traditional Regularization Scheme - TRS}}$

D. Bailin, A. Love. INTRODUCTION TO GAUGE FIELD THEORY. 1986.

Traditional Regularization Scheme (TRS)

We will consider a set of regularizations as Traditional Regularization Scheme (TRS), **if the regularization is applied to the entire vacuum contribution which entangles itself with the chiral imbalanced medium**. For instance, in the free energy we have

$$\mathcal{F}(\mu_5) = N_c N_f \sum_{s=\pm 1} \int_0^\infty \frac{d^3k}{(2\pi)^3} \omega_s(k) \Rightarrow N_c N_f \sum_{s=\pm 1} \int_0^\infty \frac{d^3k}{(2\pi)^3} \omega_s(k) U(k,\Lambda) \tag{7}$$

where $U(\Lambda)$ is a **form-factor** function that regularizes the above expression. There are several choices, for example

$$\begin{aligned} U(k,\Lambda) &= \theta(-|\vec{k}| + \Lambda), \quad \text{sharp-3D curtoff}, \\ U(k,\Lambda) &= \left[1 + \left(\frac{k^2}{\Lambda^2}\right)^N\right], \quad \text{Lorentzian function, usually with } N = 5. \end{aligned} \tag{8}$$

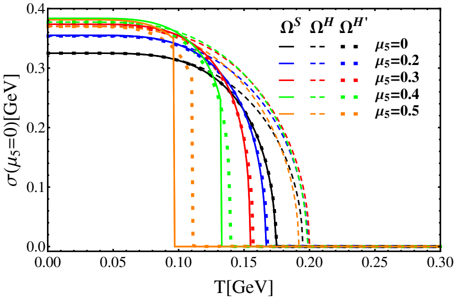
S. Avancini, R. Farias, N. Scoccola and **W. Tavares**. Phys.Rev.D 99 (2019) 11, 116002 (2019).

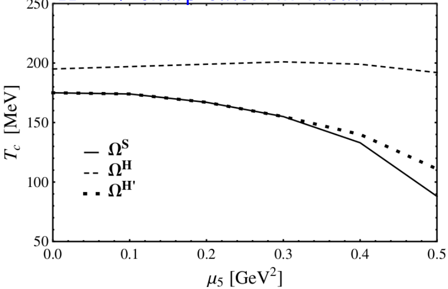
Traditional Regularization Scheme (TRS)

$\Omega^S \Rightarrow$ soft-cutoff in vacuum

$\Omega^H \Rightarrow$ sharp- cutoff in vacuum and medium

$\Omega^{H'} \Rightarrow$ sharp-cutoff in vacuum





Lang Yu, Hao Liu, Mei Huang .Phys. Rev. D 94, 014026 (2016)

Cutoff Independent Regularization

In perturbative QCD the two-flavor spin-0 superconducting gap Δ can be written as[1]

$$\Delta \sim \frac{\mu}{g^5} \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right) \tag{9}$$

This result is inapplicable for densities typically found in the interiors of neutron stars. However, four-fermion models at the one-loop level

$$1 = \lambda G i \int_\Lambda \frac{d^4k}{(2\pi)^4}\left[\frac{1}{k_0^2 - (k+\mu)^2 - \Delta^2} + (\mu \to -\mu)\right],$$

predict vanishing superconducting gaps at high densities, a feature that is caused by the use of a regularizing momentum cutoff Λ

PHYSICAL REVIEW C **73**, 018201 (2006)

Cutoff-independent regularization of four-fermion interactions for color superconductivity

R. L. S. Farias,[1] G. Dallabona,[1] G. Krein,[1] and O. A. Battistel[2]
[1]*Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405-900 São Paulo, SP, Brazil*
[2]*Departamento de Física, Universidade Federal de Santa Maria, 97119-900 Santa Maria, RS, Brazil*
(Received 31 October 2005; published 27 January 2006)

We implement a cutoff-independent regularization of four-fermion interactions to calculate the color-superconducting gap parameter in quark matter. The traditional cutoff regularization has difficulties for chemical potentials μ of the order of the cutoff Λ, predicting in particular a vanishing gap at $\mu \sim \Lambda$. The proposed cutoff-independent regularization predicts a finite gap at high densities and indicates a smooth matching with the weak coupling QCD prediction for the gap at asymptotically high densities.

DOI: 10.1103/PhysRevC.73.018201 PACS number(s): 24.85.+p, 12.38.Mh, 21.65.+f, 26.60.+c

[1] D. T. Son, Phys. Rev. D 59, 094019 (1999).
Farias R., Dallabona G, Krein G., Battistel O.Phys. Rev. C 73, 018201 (2006)

Cutoff Independent Regularization

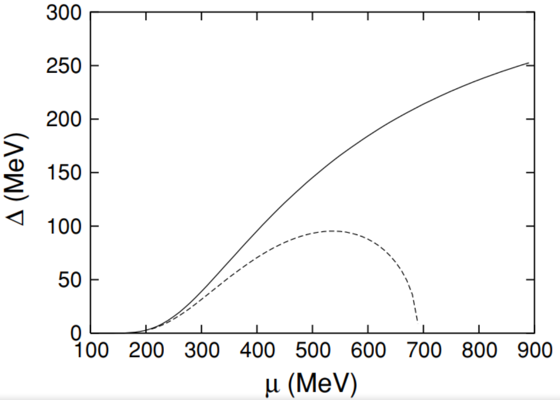
$$1 = \lambda G i \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{k_0^2-(k+\mu)^2-\Delta^2} + (\mu \to -\mu)\right],$$

$$1 = 8\lambda G\{2[i\, I_{\text{quad}}(\Delta^2)] - 4\mu^2[i\, I_{\text{log}}(\Delta^2)] + I_{\text{fin}}(\Delta^2,\mu) + I_{\text{fin}}(\Delta^2,-\mu)\},$$

where

$$I_{\text{quad}}(\Delta^2) = \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{k_0^2-k^2-\Delta^2},$$

$$I_{\text{log}}(\Delta^2) = \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k_0^2-k^2-\Delta^2)^2},$$



Farias R., Dallabona G, Krein G., Battistel O.Phys. Rev. C 73, 018201 (2006)

Medium separation Scheme (MSS)

First, we can work with a more treatable integral by evaluating (euclidean space)

$$\frac{\partial}{\partial M^2}\left[\int^{\Lambda}\frac{d^3k}{(2\pi)^3}\omega_s(k)\right]=\int_{-\infty}^{+\infty}\frac{dk_4}{2\pi}\int^{\Lambda}\frac{d^3k}{(2\pi)^3}\frac{1}{k_4^2+\omega_s^2(k)}\equiv\int_k^{\Lambda}\frac{1}{k_4^2+\omega_s^2(k)} \tag{10}$$

(11)

Then we can split the integration function as

$$\frac{1}{k_4^2+\omega_s^2(k)}=\frac{1}{k_4^2+\omega_0^2(k)}+\frac{k^2+M_0^2-\omega_s(k)^2}{\left(k_4^2+\omega_0^2(k)\right)\left[k_4^2+\omega_s^2(k)\right]}, \tag{12}$$

where

$$A_s(k)=\mu_5+2sk\mu_5+M^2-M_0^2, \tag{13}$$

where M_0 is the effective quark mass in the vacuum and $\omega_0(k)=\sqrt{k^2+M_0^2}$.

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R. Farias, D. Duarte, G. Krein & R. Ramos. Phys.Rev.D 94, 074011 (2016)

Medium separation scheme (MSS)

Now, by using three times the same identity Eq. (12) identity three times

$$\frac{1}{k_4^2+\omega_s^2(k)} = \frac{1}{k_4^2+\omega_0^2(k)} - \frac{A_s(k)}{\left(k_4^2+\omega_0^2(k)\right)^2} + \frac{A_s^2(k)}{\left(k_4^2+\omega_0^2(k)\right)^3} - \frac{A_s^3(k)}{\left(k_4^2+\omega_0^2(k)\right)^3\left[k_4^2+\omega_s^2(k)\right]},$$

therefore,
$$\int_k^\Lambda \frac{1}{k_4^2+\omega_s^2(k)} = \overbrace{\int_k^\Lambda \frac{1}{k_4^2+\omega_0^2(k)}}^{\text{Quad. Div.}} - \overbrace{\int_k^\Lambda \frac{A_s(k)}{\left(k_4^2+\omega_0^2(k)\right)^2}}^{\text{Log.Div.}} + \overbrace{\int_k^\Lambda \frac{A_s^2(k)}{\left(k_4^2+\omega_0^2(k)\right)^3}}^{\text{Log.Div.}}$$
$$\underbrace{-\int_k^\infty \frac{A_s^3(k)}{\left(k_4^2+\omega_0^2(k)\right)^3\left[k_4^2+\omega_s^2(k)\right]}}_{\text{Convergent!}}, \tag{14}$$

however, **we regulate just the divergent contributions**.

R. Farias, D. Duarte, G. Krein and R. Ramos. Phys.Rev.D 94, 074011 (2016)

Medium Separation Scheme

The effective quark masses can be obtained by the minimization of the free energy, i.e., $\frac{\partial F}{\partial M} = 0$, which is

$$\frac{M - m_c}{4 N_f N_c G M} = \underbrace{I_{\text{quad}}\left(\Lambda, M_0\right)}_{\text{Quad. Div.}} + \left(2\mu_5^2 - M^2 + M_0^2\right) \underbrace{I_{\log}\left(\Lambda, M_0\right)}_{\text{Log. Div.}} - \frac{2\mu_5^2 + M^2 - M_0^2}{8\pi^2}$$
$$+ \frac{M^2 - 2\mu_5^2}{8\pi^2} \ln\left(\frac{M^2}{M_0^2}\right) - \sum_{s=\pm 1} \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_s(k)} \frac{1}{e^{\omega_s(k)/T} + 1}, \tag{15}$$

where $I_{\text{quad}}\left(\Lambda, M_0\right)$ and $I_{\log}\left(\Lambda, M_0\right)$ denote the quadratically and logarithmically UV divergent integrals, respectively,

$$I_{\text{quad}}\left(\Lambda, M_0\right) = 2 \int_0^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k_4^2 + k^2 + M_0^2}, \quad I_{\log}\left(\Lambda, M_0\right) = -\frac{\partial}{\partial M_0^2} I_{\text{quad}}\left(\Lambda, M_0\right), \tag{16}$$

R. Farias, D. Duarte, G. Krein and R. Ramos. Phys.Rev.D 94, 074011 (2016)

MSS versus TRS results

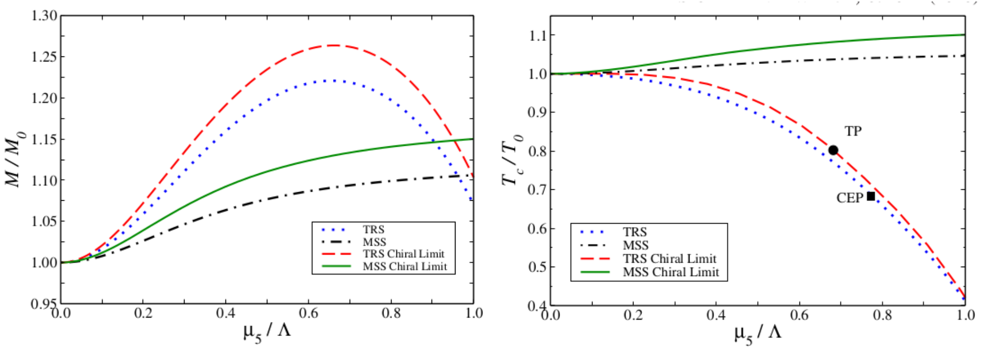

Figure: Left: Effective quark masses as a function of the normalized chiral chemical potential for both TRS and MSS regularizations. Right: Transition temperature as a function of the normalized chiral chemical potential for both MSS and TRS regularizations.

R. Farias, D. Duarte, G. Krein and R. Ramos. Phys.Rev.D 94, 074011 (2016)

Thermodynamics with chiral imbalance

We can obtain the MSS regularized free energy by integration of the gap equation. For instance, take the general form of the gap equation, at $T = 0$,

$$\frac{M - m_0}{2G} - N_c N_f \left[I_{\mathrm{MSS}} \right] = 0, \tag{17}$$

where I_{MSS} is the regulated vacuum contribution with MSS prescription for the gap equation.

$$I_{\mathrm{MSS}} = M \left[I_{\mathrm{quad}}(M_0) - (M^2 - M_0^2 - 2\mu_5^2) I_{\log}(M_0) - \frac{2\mu_5^2 + M^2 - M_0^2}{8\pi^2} + \frac{M^2 - 2\mu_5^2}{8\pi^2} \ln\left(\frac{M^2}{M_0^2}\right) \right]. \tag{18}$$

Therefore, the free energy can be evaluated by

$$\mathcal{F} = \int_{\mathcal{M}_0}^{M} dM \left[\frac{M - m_0}{2G} - N_c N_f I_{\mathrm{MSS}} \right], \quad \text{where } \mathcal{M}_0 \text{ is some constant mass}$$

$$\int_{\mathcal{M}_0}^{M} dM I_{\mathrm{MSS}} = \left[M^2 I_{\log} \frac{M_0^2}{2} + \mu_5^2 - \frac{M^2}{4} + \frac{M^2}{2} I_{\mathrm{quad}} - \frac{3M^2}{64\pi^2} + \frac{M_0 M^2}{16\pi^2} + \frac{M^2 (M^2 4\mu_5^2)}{32\pi^2} \log\left(\frac{M^2}{M_0^2}\right) \right]$$

Thermodynamics with chiral imbalance

With the free energy computed within the MSS prescription, we can study the thermodynamics of the model by using standard expressions, for instance

$$\begin{aligned} P &= -\mathcal{F}(\mu_5, T)\big|_M, & P_N &= P(\mu_5, T) - P(0, \mu_5), \\ s &= \left.\frac{\partial p_N}{\partial T}\right|_{\mu_5}, & \langle n_{5_N}\rangle &= \frac{\partial p_N}{\partial \mu_5}, \\ \varepsilon &= Ts - p_N + \mu_5 \langle n_{5_N}\rangle, \end{aligned} \tag{19}$$

It is common in the literature to define the normalized pressure P_N. Also, we applied the normalized chiral density $\langle n_{5_N}\rangle$. Another interesting quantity is the squared speed of sound, also at finite μ_5,

$$c_s^2 = \left.\frac{\partial p_N}{\partial \varepsilon}\right|_{\mu_5} = \frac{s}{C_v}, \quad \text{where} \quad C_v = T\frac{\partial s}{\partial T}\,. \tag{20}$$

F. Azeredo, D. Duarte, R. Farias, Gastão Krein, R. Ramos. Phys. Rev. D 110 (2024) 076007.

Chiral Density

The chiral density corresponds to the difference in densities of quarks with right- and left-handed chirality, $n_5 = n_R - n_L$, encoded in the difference between their chemical potentials, $\mu_5 = \mu_R - \mu_L$[3].

We will start by defining the chiral density at $T = 0$ without a mass term, i.e., $M = 0$.

$$\begin{aligned} \langle n_5 \rangle &= -\frac{\partial \mathcal{F}}{\partial \mu_5} \\ &= 2N_c N_f \int \frac{d^3p}{(2\pi)^3} \theta(\mu_5 - |\vec{p}|) = \frac{N_c N f \mu_5^3}{3\pi^2} \end{aligned} \tag{21}$$

which is very similar to the standard relation between the quark chemical potential μ and n. Therefore, everything works as the Fermi sphere is filled up to the Fermi momentum $|\vec{p}| = \mu_5$.

[3] Marco Ruggieri, Maxim Chernodub and Zhen-Yan Lu. Phys. Rev. D 102, 014031 (2020)

Chiral Density

Now, consider the inclusion of a momentum independent mass term, M, in the expression for the chiral density

$$\langle n_5 \rangle = 2N_cN_f \int \frac{d^3p}{(2\pi)^3}\chi(p) \quad \text{where} \chi(p) = \frac{|\vec{p}| + \mu_5}{2\sqrt{(|\vec{p}| + \mu_5) + M^2}} + \frac{|\vec{p}| - \mu_5}{2\sqrt{(|\vec{p}| - \mu_5) + M^2}}, \tag{22}$$

The effect of including a mass term M is to enlarge the chiral Fermi surface by putting particles above the Fermi momentum. If we assume, for instance, $M \ll \mu_5 \ll |\vec{p}|$, we obtain

$$\chi(p) \approx \frac{M^2\mu_5}{|\vec{p}|^3} \tag{23}$$

Therefore, the inclusion of a mass term introduces a logarithmic divergence in n_5. For these reasons, it is usual to define the normalized chiral density

$$\langle n_{5_N} \rangle = \frac{\partial P_N}{\partial \mu_5}. \tag{24}$$

Marco Ruggieri, Maxim Chernodub and Zhen-Yan Lu. Phys. Rev. D 102, 014031 (2020)

Chiral density in Chiral Perturbation Theory

From chiral perturbation theory (ChPT), the QCD partition function in the presence of μ_5 is modified to

$$Z(\mu_5) = Z_{QCD} \exp\left[\beta V N_f f_\pi^2 \mu_5^2\right] , \tag{25}$$

where Z_{QCD} is the finite-temperature QCD partition function. From this modified partition function one obtains for average of $\langle n_5 \rangle$:

$$\langle n_5 \rangle = \frac{1}{\beta V} \frac{\partial \log[Z(\mu_5)]}{\partial \mu_5} = 2 N_f f_\pi^2 \mu_5 . \tag{26}$$

from which one concludes that $\langle n_5 \rangle$ is a linearly increasing function of μ_5 with with slope $4f_\pi^2$ in the two-flavor case.

N. Y. Astrakhantsev, V. V. Braguta, A. Y. Kotov, D. D. Kuznedelev, and A. A. Nikolaev. Eur. Phys. J. A 57, 15 (2021).

Chiral density in the NJL model

Within NJL model, the chiral density can be evaluated as

$$\langle n_5 \rangle = -\frac{\partial \mathcal{F}}{\partial \mu_5} \,, \tag{27}$$

which leads to:

$$\langle n_5 \rangle^{\mathrm{TRS}} = 2N_c \sum_{s=\pm 1} \int_0^{\Lambda} \frac{dp\, p^2}{2\pi^2} \frac{s(|\mathbf{p}| + s\mu_5)}{\sqrt{(|\mathbf{p}| + s\mu_5)^2 + M^2}} \,. \tag{28}$$

From this result, it is impossible to establish any connection with the ChPT prediction since the μ_5 effect is being regularized together with the logarithmic divergence in the momentum integral. For the MSS, on the other hand, the derivative of the MSS potential results in

$$\langle n_5 \rangle^{\mathrm{MSS}} = 4N_c \left[M^2 I_{\log}(M_0) - \frac{M^2}{4\pi^2} \ln\left(\frac{M^2}{M_0^2}\right)\right] \mu_5 \,,$$

D. C. Duarte, R. L. S. Farias, and R. O. Ramos. Phys. Rev. D 99, 016005 (2019).

Chiral density in the NJL model

By using the *Implicit Regularization Scheme*[4]

$$i\tilde{I}_{\log}(M^2) = -\frac{f_\pi^2}{12M^2} \, . \quad \tilde{I}_{\log}(M^2) = \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k_0^2 - k^2 - M^2)^2} \, , \tag{29}$$

at an arbitrary mass scale M. We note that $4i\tilde{I}_{\log}(x^2) = I_{\log}(x^2)$, is valid at any mass scale M, and can be expressed in terms of the vacuum quark mass, M_0, using the identity,

$$\tilde{I}_{\log}(M^2) = \tilde{I}_{\log}(M_0^2) - \frac{i}{(4\pi)^2} \ln\left(\frac{M^2}{M_0^2}\right) \, . \tag{30}$$

This allows us to rewrite the last equation as

$$\langle n_5 \rangle^{\mathrm{MSS}} = -16 N_c M^2 i\tilde{I}_{\log}(M^2)\mu_5 . = 4 f_\pi^2 \mu_5 \, , \text{Q.E.D} \tag{31}$$

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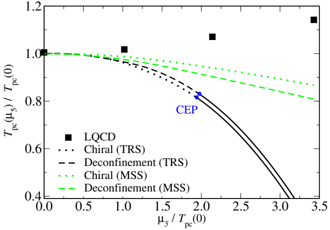
[4] R. L. S. Farias, G. Dallabona, G. Krein, and O. A. Battistel. Phys. Rev. C 73, 018201 (2006). D. C. Duarte, R. L. S. Farias, and R. O. Ramos. Phys. Rev. D 99, 016005 (2019).
F. Azeredo, D. Duarte, R. Farias, Gastão Krein, R. Ramos. Phys. Rev. D 110 (2024) 076007.

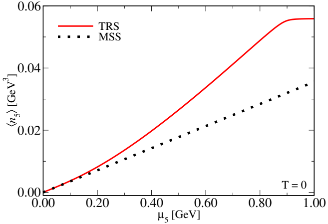
Chiral Chemical Potential + Deconfinement

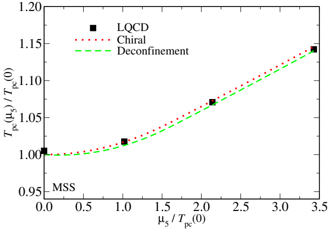
$$\mathcal{U}(\Phi, T, \mu_5) = T^4\left[-\frac{\bar{b}_2(T,\mu_5)}{2}\Phi\Phi^\dagger - \frac{b_3}{6}(\Phi^3 + \Phi^{\dagger 3}) + \frac{b_4}{4}(\Phi\Phi^\dagger)^2\right],$$

in which

$$\bar{b}_2(T,\mu_5) = a_0 + a_1\left(\frac{T_0}{T}\right) + a_2\left(\frac{T_0}{T}\right)^2 + a_3\left(\frac{T_0}{T}\right)^3 + k_1\left(\frac{\mu_5}{T}\right) + k_2\left(\frac{\mu_5}{T}\right)^2 + k_3\left(\frac{\mu_5}{T}\right)^3.$$

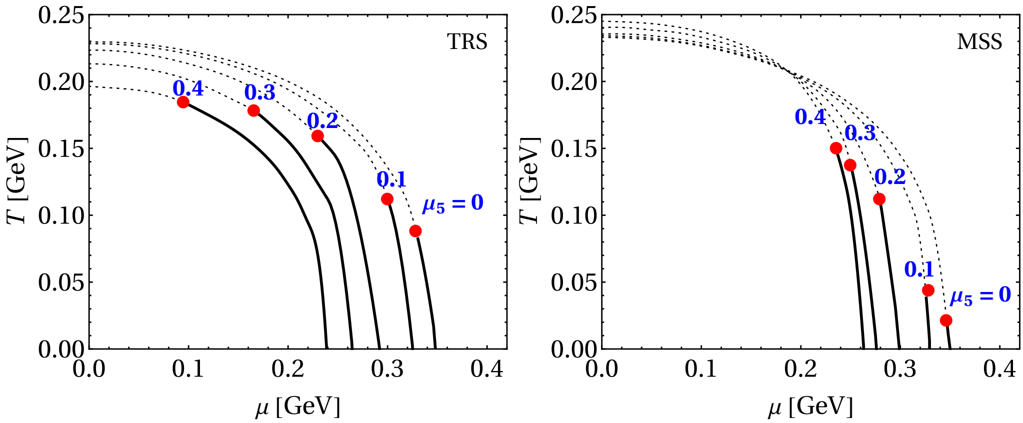






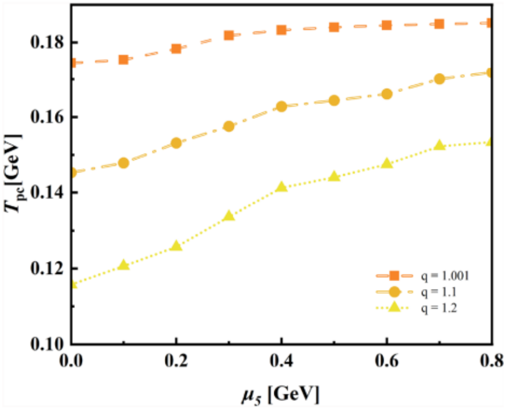
F. Azeredo, D. Duarte, R. Farias, Gastão Krein, R. Ramos. Phys. Rev. D 110 (2024) 076007.

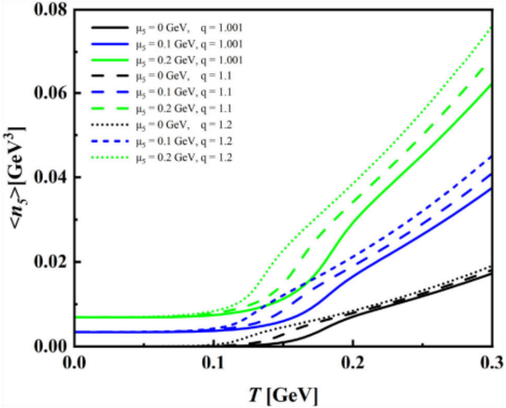
Chiral Chemical Potential + Deconfinement



F. Azeredo, D. Duarte, R. Farias, Gastão Krein, R. Ramos. Phys. Rev. D 110 (2024) 076007.

NJL SU(2)+Chiral Chemical Potential+Tsallis Statistics





Xiang-Qiong Liu. Eur. Phys. J. C (2025) 85:707

Optimized perturbation theory

- Introduction of a fictitious expansion parameter δ:

$$\mathcal{L}(\delta) = (1-\delta)\mathcal{L}_0 + \delta\mathcal{L} = \mathcal{L}_0 + \delta(\mathcal{L} - \mathcal{L}_0) \Rightarrow \begin{cases} \delta = 1 \Rightarrow \text{original lagrangian} \Rightarrow \mathcal{L} \\ \delta = 0 \Rightarrow \text{free lagrangian} \Rightarrow \mathcal{L}_0 \end{cases}$$

- $\mathcal{L}_0$ must have at least two mass parameters η and ζ.

$$\left.\frac{d\mathcal{F}_{\text{OPT}}}{d\eta}\right|_{\bar{\zeta},\sigma,\delta=1} = \left.\frac{d\mathcal{F}_{\text{OPT}}}{d\zeta}\right|_{\bar{\eta},\sigma,\delta=1} = 0 \Rightarrow \left\{ \text{Principle of Minimal Sensitivity (PMS)} \right.$$

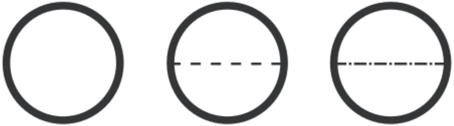
- The Free energy at order δ is given by, (where we defined $\tilde{\mu} = \mu + \zeta$)

$$\mathcal{F}_{\text{OPT}} = \frac{(M-m_c)^2}{4G} - N_f N_c I_1(T,\tilde{\mu},\mu_5) + \delta N_f N_c(\eta + m_c)(\eta - M + m_c) I_2(T,\tilde{\mu},\mu_5)$$
$$+ \delta N_f N_c \eta_0 I_3(T,\tilde{\mu},\mu_5) + \delta G N_f N_c I_3^2(T,\tilde{\mu},\mu_5) - \frac{1}{2}\delta G N_f N_c(\eta + m_c)^2 I_2^2(T,\tilde{\mu},\mu_5) \tag{32}$$

Jean-Loïc Kneur, Marcus B. Pinto and R. Ramos. Phys.Rev.C 81, 065205 (2010)
A. Gonçalves, D. Duarte, R. Farias, M. Pinto, R. Ramos, **W.R. Tavares**. **Phys.Rev.D 112 (2025) 7, 076021.**

Optimized perturbation theory applied to NJL model

The Feynman diagrams at order δ in OPT to the NJL model



The free energy is given by

$$\mathcal{F}_{\rm OPT} = \frac{(M-m_c)^2}{4G} - N_f N_c I_1(T,\tilde{\mu}) + \delta N_f N_c(\eta+m_c)(\eta - M + m_c) I_2(T,\tilde{\mu})$$
$$+\delta N_f N_c \eta_0 I_3(T,\tilde{\mu}) + \delta G N_f N_c I_3^2(T,\tilde{\mu}) - \frac{1}{2}\delta G N_f N_c(\eta+m_c)^2 I_2^2(T,\tilde{\mu})$$

Jean-Loïc Kneur, Marcus B. Pinto and and R. Ramos. Phys.Rev.C 81, 065205 (2010)

Numerical Results

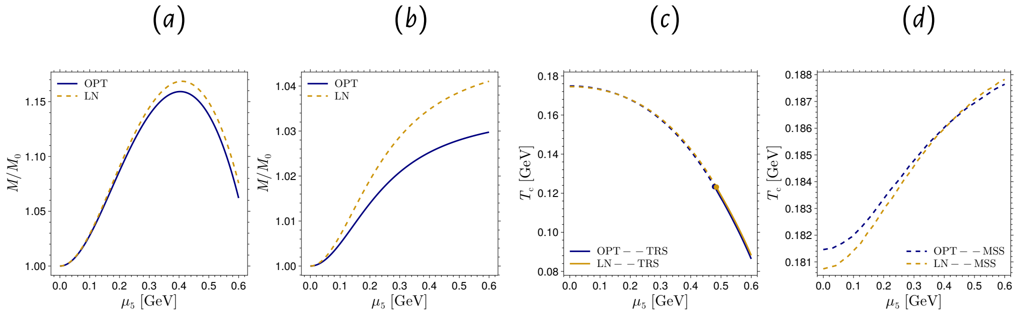


Figure: Effective quark masses as a function of the chiral chemical potential with TRS (a) and MSS (b). Right: Pseudocritical temperature as a function of the chiral chemical potential with TRS (c) and MSS (d).

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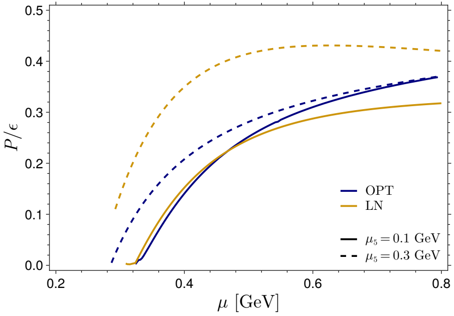
Numerical Results



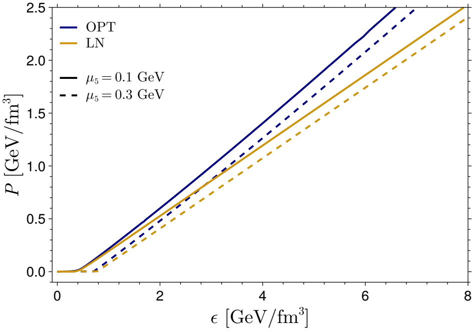


Figure: Left: Ratio P/ϵ as a function of the quark chemical potential for different values of μ_5 in LN and BLN approximations. Right: EOS ($P \times \epsilon$) for different values of μ_5 in LN and BLN approximations.

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Numerical Results

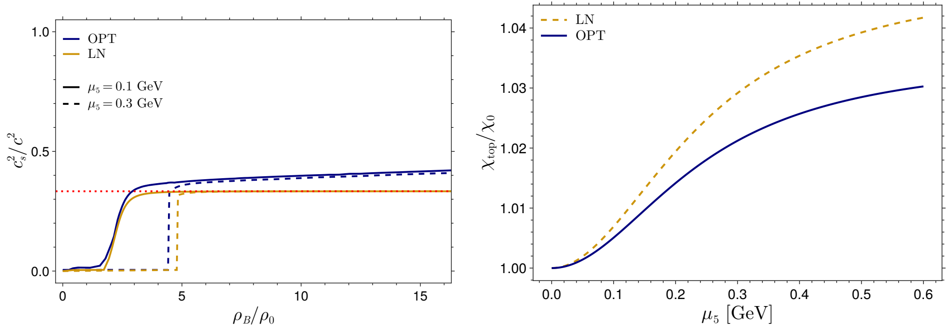

Figure: Left: Sound velocity as a function of the quark chemical potential for different values of μ_5 in LN and BLN approximations. Right: Topological susceptibility as a function of quark chemical potential for different values of μ_5 in LN and BLN approximations.

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Summary

Chiral Imbalanced medium

- **Chiral Imbalance:** The vacuum of QCD has a nontrivial structure with topological charges that, connected with axial anomaly, can induce the difference between the number of right- and left-handed quarks.

- **Chiral Magnetic effect:** Is one of the main effects that can bring experimental evidences of magnetic fields in heavy-Ion collisions.

Regularization

- **Traditional Regularization Scheme :** Is a set of regularization prescriptions that does not separate the vacuum from the medium contribution.

- **Medium Separation Scheme:** Is a regularization prescription, originally applied in the context of high density to study superconductivity. In this procedure, the separation of vacuum and medium is performed.

Conclusions

- **MSS versus TRS:** The MSS procedure avoids that regularization artifacts changes several physical quantities.
- **Examples in literature:** This procedure has been applied in the context of chiral imbalanced media, isospin chemical potential, and color-superconductivity.
- **Future applications:** This procedure can be interesting to separate divergences in renormalizable theories.

Gracias!

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