Second order spin-1/2 bosons

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Outline

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Introduction. Homogeneous Lorentz Group

The **Homogeneous Lorentz Group (HLG)** is the group $O(1,3)$, whose elements, $L^{\mu}{}_{\rho}$, are defined by

$$L^{\mu}{}_{\rho} g_{\mu\nu} L^{\nu}{}_{\sigma} = g_{\rho\sigma}. \tag{1}$$

This group can be separated into 4 disconnected components:

- Proper ortochronous: $\det L = 1$ & $L^{0}{}_{0} > 0$. It's the subgroup $SO^{+}(1,3)$, aka **Restricted Lorentz Group (RLG)**. Its elements are $\Lambda^{\mu}{}_{\nu}$.
- Improper ortochronous: $\det L = -1$ & $L^{0}{}_{0} > 0$. Its elements are

$$[\mathcal{P}\Lambda]^{\mu}{}_{\nu}, \qquad \mathcal{P} = \mathrm{Diag}(1,-1,-1,-1). \tag{2}$$

- Improper heterochronous: $\det L = -1$ & $L^{0}{}_{0} < 0$. Its elements are

$$[\mathcal{T}\Lambda]^{\mu}{}_{\nu}, \qquad \mathcal{T} = \mathrm{Diag}(-1,1,1,1). \tag{3}$$

- Proper heterochronous: $\det L = 1$ & $L^{0}{}_{0} < 0$. Its elements are

$$[\mathcal{P}\mathcal{T}\Lambda]^{\mu}{}_{\nu}, \qquad \mathcal{P}\mathcal{T} = \mathrm{Diag}(-1,-1,-1,-1). \tag{4}$$

In general, the RLG is a six-parameter Lie group whose elements can be written as

$$\Lambda(\boldsymbol{\theta}, \boldsymbol{\phi}) = \exp\left[-\frac{i}{2}\Omega_{\mu\nu}J^{\mu\nu}\right] \tag{5}$$

where the generators $J^{\mu\nu}$ satisfy the algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} + \eta^{\nu\sigma}J^{\mu\rho} - \eta^{\mu\sigma}J^{\nu\rho}). \tag{6}$$

It can be rewritten as

$$[J^i, J^j] = i\varepsilon^{ijk}J^k, \qquad [J^i, K^j] = i\varepsilon^{ijk}K^k, \qquad [K^i, K^j] = -i\varepsilon^{ijk}J^k, \tag{7}$$

where $J^i = \frac{1}{2}\varepsilon^{ijk}J^{jk}$, $K^i = J^{i0}$. Defining the operators $\boldsymbol{A}, \boldsymbol{B}$ as

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{J} - i\boldsymbol{K}), \qquad \boldsymbol{B} = \frac{1}{2}(\boldsymbol{J} + i\boldsymbol{K}), \tag{8}$$

it simplifies to two copies of the $SU(2)$ algebra

$$[A^i, A^j] = i\varepsilon^{ijk}A^k, \qquad [A^i, B^j] = 0, \qquad [B^i, B^j] = i\varepsilon^{ijk}B^k. \tag{9}$$

In this sense

$$SO^+(1,3) \simeq SU(2)_A \otimes SU(2)_B \tag{10}$$

The irreps of the RLG are therefore

- Defined by (a, b), where $a, b = 0, 1/2, 1, 3/2, \cdots$.
- Have dimension $(2a+1)(2b+1)$.
- Have eigenstates $\{|a, b, m_a, m_b\rangle = |a, m_a\rangle |b, m_b\rangle\}$, with

$$\boldsymbol{A}^2 |a, m_a\rangle = a(a+1) |a, m_a\rangle \qquad \boldsymbol{B}^2 |b, m_b\rangle = b(b+1) |b, m_b\rangle \tag{11}$$

$$A_3 |a, m_a\rangle = m_a |a, m_a\rangle \qquad B_3 |b, m_b\rangle = m_b |b, m_b\rangle \tag{12}$$

$$(0, 0)$$

$$\left(\tfrac{1}{2}, 0\right) \qquad \left(0, \tfrac{1}{2}\right)$$

$$(1, 0) \qquad \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \qquad (0, 1)$$

$$\left(\tfrac{3}{2}, 0\right) \qquad \left(1, \tfrac{1}{2}\right) \qquad \left(\tfrac{1}{2}, 1\right) \qquad \left(0, \tfrac{3}{2}\right)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

There is an infinite number of irreps for the RLG, however, the Standard Model only uses a few of them:

- $(0,0)$: Higgs.
- $(\frac{1}{2},0)$, $(0,\frac{1}{2})$: Leptons and quarks.
- $(\frac{1}{2},\frac{1}{2})$: Gauge bosons.

$$(0,0)$$

$$\left(\frac{1}{2},0\right) \qquad \left(0,\frac{1}{2}\right)$$

$$(1,0) \qquad \left(\frac{1}{2},\frac{1}{2}\right) \qquad (0,1)$$

$$\left(\frac{3}{2},0\right) \qquad \left(1,\frac{1}{2}\right) \qquad \left(\frac{1}{2},1\right) \qquad \left(0,\frac{3}{2}\right)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

In this presentation we will work with spin-1/2 fields belonging to the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ rep.

A field $\psi \in \left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ has spin-1/2. Its free dynamics are conventionally described by the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \tag{13}$$

- **Dirac solution $\Rightarrow$ KG solution**.
  Every solution of the Dirac equation is a solution to the Klein-Gordon (KG) equation

$$(-i\gamma^\mu \partial_\mu - m)(i\gamma^\mu \partial_\mu - m)\psi(x) = (\partial^2 + m^2)\psi(x) = 0. \tag{14}$$

- **KG solution $\not\Rightarrow$ Dirac solution**.
  The converse is not true, i.e. not every solution of the KG equation solves the Dirac equation. However, the most general solution to the KG can be split into two Dirac solutions[1]

$$\psi = \frac{1}{2}\left(1 + \frac{\not\partial}{m}\right)\psi + \frac{1}{2}\left(1 - \frac{\not\partial}{m}\right)\psi = \frac{1}{2}\left(1 + \frac{\not\partial}{m}\right)\psi + \gamma^5 \frac{1}{2}\left(1 + \frac{\not\partial}{m}\right)\gamma^5\psi = \psi_1 + \gamma^5\psi_2 \tag{15}$$

Is it possible to describe the free dynamics of a spin-1/2 field $\psi \in \left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ by using just the KG equation?

$\Rightarrow$ **Second order spin-1/2 field**

[1] N. Cufaro Petroni et al. "Second order wave equation for spin 1/2 fields". In: *Phys. Rev. D* 31 (1985), pp. 3157–3161. DOI: 10.1103/PhysRevD.31.3157.

Outline

The Lagrangian for the second order spin-1/2 field $\psi \in \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ is

$$\mathcal{L} = \partial^\mu \bar{\psi} \partial_\mu \psi - m^2 \bar{\psi}\psi, \tag{16}$$

where the Dirac dual is $\bar{\psi} = \psi^\dagger \gamma^0$. The conjugated momenta $\pi_\psi, \pi_{\bar{\psi}}$ are

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \dot{\bar{\psi}}, \qquad \pi_{\bar{\psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} = \dot{\psi}. \tag{17}$$

Observations

- The field has 8 degrees of freedom, twice those of the Dirac field.
- The mass dimension of this field is 1, not the 3/2 dimension of the Dirac field.

The Dirac decomposition of the field is

$$\psi = \frac{1}{\sqrt{2m}}\left(\psi_1 + \gamma^5\psi_2\right), \qquad \bar{\psi} = \frac{1}{\sqrt{2m}}\left(\bar{\psi}_1 - \bar{\psi}_2\gamma^5\right), \tag{18}$$

$$\psi(x) = \frac{1}{\sqrt{2m}}\int\frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\left[a^1_{\boldsymbol{p},s} + a^2_{\boldsymbol{p},s}\gamma^5\right]u_{\boldsymbol{p},s}e^{-ip\cdot x} + \left[b^{1\dagger}_{\boldsymbol{p},s} + b^{2\dagger}_{\boldsymbol{p},s}\gamma^5\right]v_{\boldsymbol{p},s}e^{ip\cdot x}\right\}\Big|_{p^0=E_{\boldsymbol{p}}},$$

$$\bar{\psi}(x) = \frac{1}{\sqrt{2m}}\int\frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\bar{u}_{\boldsymbol{p},s}\left[a^{1\dagger}_{\boldsymbol{p},s} - a^{2\dagger}_{\boldsymbol{p},s}\gamma^5\right]e^{ip\cdot x} + \bar{v}_{\boldsymbol{p},s}\left[b^1_{\boldsymbol{p},s} - b^2_{\boldsymbol{p},s}\gamma^5\right]e^{-ip\cdot x}\right\}\Big|_{p^0=E_{\boldsymbol{p}}}.$$

The canonical quantization starts by imposing anticommutation (fermionic) relations at equal times

$$\left\{\psi_a(\boldsymbol{x},t), \pi_{\psi,b}(\boldsymbol{y},t)\right\} = -\left\{\bar{\psi}_a(\boldsymbol{x},t), \pi_{\bar{\psi},b}(\boldsymbol{y},t)\right\} = i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \tag{19}$$

which imply anticommutation relations with the wrong sign ⇒ negative-norm states

$$\{a^1_{\boldsymbol{p},s}, a^{1\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \qquad \{b^1_{\boldsymbol{p},s}, b^{1\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \tag{20}$$

$$\{a^2_{\boldsymbol{p},s}, a^{2\dagger}_{\boldsymbol{q},r}\} = -(2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \qquad \{b^2_{\boldsymbol{p},s}, b^{2\dagger}_{\boldsymbol{q},r}\} = -(2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}). \tag{21}$$

Naive hermitian theory ill-defined

It has been shown recently[2] that this problem can be fixed, and the second order spin-1/2 field can be consistently quantized as a pseudohermitian QFT.

[2] Rodolfo Ferro-Hernández et al. "Quantization of second-order fermions". In: *Phys. Rev. D* 109 (8 Apr. 2024), p. 085003. DOI: 10.1103/PhysRevD.109.085003. URL: https://link.aps.org/doi/10.1103/PhysRevD.109.085003.

Pseudohermiticity in quantum theories was proposed and studied in[3]. An operator H is pseudo-hermitian if it satisfies

$$H^{\#} = \eta^{-1} H^{\dagger} \eta = H, \tag{22}$$

where η is a linear and invertible operator. By redefining the inner product between two states as

$$\langle a(t)|b(t)\rangle_{\eta} = \langle a(t)|\eta|b(t)\rangle , \tag{23}$$

two features emerge:

- The probability amplitudes are preserved in time

$$\langle a(t)|b(t)\rangle_{\eta} = \langle a|e^{-iH^{\dagger}t}\eta e^{iHt}|b\rangle = \langle a|\eta e^{-iHt}e^{iHt}|b\rangle = \langle a|b\rangle_{\eta} . \tag{24}$$

- The energy spectrum is real

$$(E - E^{*})\langle a_E|a_E\rangle_{\eta} = \langle a_E|(\eta H - H^{\dagger}\eta)|a_E\rangle = 0. \tag{25}$$

where $|a_E\rangle$ are energy eigenestates: $H|a_E\rangle = E|a_E\rangle$.

[3] Ali Mostafazadeh. "Pseudo-Hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian". In: *Journal of Mathematical Physics* 43.1 (Jan. 2002), pp. 205–214. ISSN: 0022-2488. DOI: 10.1063/1.1418246. eprint: https://pubs.aip.org/aip/jmp/article-pdf/43/1/205/19019524/205_1_online.pdf. URL: https://doi.org/10.1063/1.1418246.

Second order spin-1/2 fermions. Pseudohermitian theory

The second order spin-1/2 field theory can be turned pseudohermitian by redefining the dual of the field ψ as $\hat{\psi}$

$$\hat{\psi} = \eta^{-1}\bar{\psi}\eta, \qquad \mathcal{L} = \partial^{\mu}\hat{\psi}\partial_{\mu}\psi - m^2\hat{\psi}\psi, \qquad \mathcal{L}^{\#} = \eta^{-1}\mathcal{L}^{\dagger}\eta = \mathcal{L}. \tag{26}$$

The fields are now

$$\psi(x) = \frac{1}{\sqrt{2m}}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\left[a^1_{\boldsymbol{p},s} + a^2_{\boldsymbol{p},s}\gamma^5\right]u_{\boldsymbol{p},s}e^{-ip\cdot x} + \left[b^{1\dagger}_{\boldsymbol{p},s} + b^{2\dagger}_{\boldsymbol{p},s}\gamma^5\right]v_{\boldsymbol{p},s}e^{ip\cdot x}\right\},$$

$$\hat{\psi}(x) = \frac{1}{\sqrt{2m}}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\bar{u}_{\boldsymbol{p},s}\left[\eta^{-1}a^{1\dagger}_{\boldsymbol{p},s}\eta - \eta^{-1}a^{2\dagger}_{\boldsymbol{p},s}\eta\gamma^5\right]e^{ip\cdot x}\right.$$
$$\left. + \bar{v}_{\boldsymbol{p},s}\left[\eta^{-1}b^1_{\boldsymbol{p},s}\eta - \eta^{-1}b^2_{\boldsymbol{p},s}\eta\gamma^5\right]e^{-ip\cdot x}\right\}.$$

Defining the action of η on the creation/anihilation operators as

$$\eta^{-1}a^1_{\boldsymbol{p},s}\eta = a^1_{\boldsymbol{p},s}, \qquad \eta^{-1}b^{1\dagger}_{\boldsymbol{p},s}\eta = b^{1\dagger}_{\boldsymbol{p},s}, \qquad \eta^{-1}a^2_{\boldsymbol{p},s}\eta = -a^2_{\boldsymbol{p},s}, \qquad \eta^{-1}b^{2\dagger}_{\boldsymbol{p},s}\eta = -b^{2\dagger}_{\boldsymbol{p},s}, \tag{27}$$

the dual becomes

$$\hat{\psi}(x) = \frac{1}{\sqrt{2m}}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\bar{u}_{\boldsymbol{p},s}\left[a^{1\dagger}_{\boldsymbol{p},s} + a^{2\dagger}_{\boldsymbol{p},s}\gamma^5\right]e^{ip\cdot x} + \bar{v}_{\boldsymbol{p},s}\left[b^1_{\boldsymbol{p},s} + b^2_{\boldsymbol{p},s}\gamma^5\right]e^{-ip\cdot x}\right\}. \tag{28}$$

An explicit expression for this operator η is

$$\eta = \exp\left[i\pi\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\sum_{s}\left(a^{2\dagger}_{\boldsymbol{p},s}a^2_{\boldsymbol{p},s} + b^{2\dagger}_{\boldsymbol{p},s}b^2_{\boldsymbol{p},s}\right)\right], \qquad \eta^{\dagger} = \eta \qquad \eta^{\dagger}\eta = 1. \tag{29}$$

The Lagrangian is now

$$\mathcal{L} = \partial^\mu \hat{\psi} \partial_\mu \psi - m^2 \bar{\psi} \psi, \tag{30}$$

where $\hat{\psi} = \eta^{-1} \bar{\psi} \eta$. The conjugated momenta $\pi_\psi, \pi_{\hat{\psi}}$ are

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \dot{\hat{\psi}}, \qquad \pi_{\hat{\psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \hat{\psi})} = \dot{\psi}. \tag{31}$$

The canonical anticommutation relations at equal times

$$\{\psi_a(\boldsymbol{x}, t), \pi_{\psi,b}(\boldsymbol{y}, t)\} = -\left\{\hat{\psi}_a(\boldsymbol{x}, t), \pi_{\hat{\psi},b}(\boldsymbol{y}, t)\right\} = i\delta_{ab}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}), \tag{32}$$

imply now the anticommutation relations with the right sign ⇒ no negative-norm states

$$\{a^1_{\boldsymbol{p},s}, a^{1\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \qquad \{b^1_{\boldsymbol{p},s}, b^{1\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \tag{33}$$

$$\{a^2_{\boldsymbol{p},s}, a^{2\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \qquad \{b^2_{\boldsymbol{p},s}, b^{2\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}). \tag{34}$$

Second order spin-1/2 fermions. Pseudohermitian theory

In addition, this pseudohermitian QFT has the following features

- Microcausality

$$\{\psi_a(x), \hat{\psi}_b(y)\} = \Delta(x-y)\delta_{ab}, \qquad \{\psi_a(x), \psi_b(y)\} = 0, \qquad \{\hat{\psi}_a(x), \hat{\psi}_b(y)\} = 0, \tag{35}$$

  where $\Delta(x-y)$ is the Lorentz invariant and causal Schwinger's Green function

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right] \tag{36}$$

- Hamiltonian

$$\begin{aligned} H &= : \int d^3x \left(\partial_0 \hat{\psi} \partial_0 \psi + \partial_i \hat{\psi} \partial_i \psi + m^2 \hat{\psi} \psi \right) : \\ &= \int \frac{d^3q}{(2\pi)^3} E_{\boldsymbol{q}} \sum_{r=1}^{2} \left\{ a^{1\dagger}_{\boldsymbol{q},r} a^1_{\boldsymbol{q},r} + a^{2\dagger}_{\boldsymbol{q},r} a^2_{\boldsymbol{q},r} + b^{1\dagger}_{\boldsymbol{q},r} b^1_{\boldsymbol{q},r} + b^{2\dagger}_{\boldsymbol{q},r} b^2_{\boldsymbol{q},r} \right\} \end{aligned} \tag{37}$$

- Momentum

$$\begin{aligned} P_i &= : - \int d^3x \left(\partial_0 \hat{\psi} \partial_i \psi + \partial_i \hat{\psi} \partial_0 \psi \right) : \\ &= \int \frac{d^3q}{(2\pi)^3} \boldsymbol{q} \sum_{r=1}^{2} \left\{ a^{1\dagger}_{\boldsymbol{q},r} a^1_{\boldsymbol{q},r} + a^{2\dagger}_{\boldsymbol{q},r} a^2_{\boldsymbol{q},r} + b^{1\dagger}_{\boldsymbol{q},r} b^1_{\boldsymbol{q},r} + b^{2\dagger}_{\boldsymbol{q},r} b^2_{\boldsymbol{q},r} \right\} \end{aligned} \tag{38}$$

Second order spin-1/2 fermions. Pseudohermitian theory

- $U(1)$-charge

$$Q_{U(1)} = :i\int d^3x \left(\hat{\psi}\partial_0\psi - \partial_0\hat{\psi}\psi\right): = \int \frac{d^3q}{(2\pi)^3}\sum_{r=1}^{2}\left\{a^{1\dagger}_{\mathbf{q},r}a^{1}_{\mathbf{q},r} + a^{2\dagger}_{\mathbf{q},r}a^{2}_{\mathbf{q},r} - b^{1\dagger}_{\mathbf{q},r}b^{1}_{\mathbf{q},r} - b^{2\dagger}_{\mathbf{q},r}b^{2}_{\mathbf{q},r}\right\} \quad (39)$$

- Discrete symmetries

$$P\psi(x)P^{-1} = i\gamma^0\psi(\mathcal{P}x), \qquad C\psi(x)C^{-1} = \mathcal{C}\hat{\psi}^T(x), \qquad T\psi(x)T^{-1} = \mathcal{C}\gamma^5\psi(\mathcal{T}x),$$

where $\mathcal{C} = -i\gamma^2\gamma^0$. The theory is invariant under C, P, and T.

- Can have dimension-4 fermion self-interactions

$$\mathcal{L}_{\text{int}} = \frac{\lambda_1}{2}\left(\hat{\psi}\psi\right)^2 + \frac{\lambda_2}{2}\left(\hat{\psi}\gamma^5\psi\right)\left(\hat{\psi}\gamma^5\psi\right) + \frac{\lambda_3}{2}\left(\hat{\psi}M^{\mu\nu}\psi\right)\left(\hat{\psi}M_{\mu\nu}\psi\right) \quad (40)$$

- Renormalizable. It has been shown in[4] that its electrodynamics and self-interactions are renormalizable at one-loop.

[4] Carlos A. Vaquera-Araujo, Mauro Napsuciale, and René Ángeles-Martınez. "Renormalization of the QED of self-interacting second order spin 1/2 fermions.". In: *Journal of High Energy Physics* 2013.1 (Jan. 2013), p. 11. ISSN: 1029-8479. DOI: 10.1007/JHEP01(2013)011. URL: https://doi.org/10.1007/JHEP01(2013)011.

Outline

What happens if we try to quantize the second-order spin-1/2 field with bosonic statistics instead of the fermionic one?

The Lagrangian for the second order spin-1/2 field $\psi \in \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ is

$$\mathcal{L} = \partial^\mu \bar{\psi} \partial_\mu \psi - m^2 \bar{\psi}\psi, \tag{41}$$

where the Dirac dual is $\bar{\psi} = \psi^\dagger \gamma^0$. The conjugated momenta $\pi_\psi, \pi_{\bar{\psi}}$ are

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \dot{\bar{\psi}}, \qquad \pi_{\bar{\psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} = \dot{\psi}. \tag{42}$$

Observations

- The field has 8 degrees of freedom, twice those of the Dirac field.
- The mass dimension of this field is 1, not the 3/2 dimension of the Dirac field.

What happens if we try to quantize the second-order spin-$1/2$ field with bosonic statistics instead of the fermionic one?

Dirac decomposition of the field

$$\psi = \frac{1}{\sqrt{2m}}\left(\psi_1 + \gamma^5\psi_2\right), \qquad \bar{\psi} = \frac{1}{\sqrt{2m}}\left(\bar{\psi}_1 - \bar{\psi}_2\gamma^5\right), \tag{43}$$

$$\psi(x) = \frac{1}{\sqrt{2m}}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\left[a^1_{\boldsymbol{p},s} + a^2_{\boldsymbol{p},s}\gamma^5\right]u_{\boldsymbol{p},s}e^{-ip\cdot x} + \left[b^{1\dagger}_{\boldsymbol{p},s} + b^{2\dagger}_{\boldsymbol{p},s}\gamma^5\right]v_{\boldsymbol{p},s}e^{ip\cdot x}\right\}\Big|_{p^0=E_{\boldsymbol{p}}},$$

$$\bar{\psi}(x) = \frac{1}{\sqrt{2m}}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\bar{u}_{\boldsymbol{p},s}\left[a^{1\dagger}_{\boldsymbol{p},s} - a^{2\dagger}_{\boldsymbol{p},s}\gamma^5\right]e^{ip\cdot x} + \bar{v}_{\boldsymbol{p},s}\left[b^1_{\boldsymbol{p},s} - b^2_{\boldsymbol{p},s}\gamma^5\right]e^{-ip\cdot x}\right\}\Big|_{p^0=E_{\boldsymbol{p}}}.$$

This time we impose commutation (bosonic) relations at equal times

$$\left[\psi_a(\boldsymbol{x},t), \pi_{\psi,b}(\boldsymbol{y},t)\right] = \left[\bar{\psi}_a(\boldsymbol{x},t), \pi_{\bar{\psi},b}(\boldsymbol{y},t)\right] = i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \tag{44}$$

which imply commutation relations with the wrong sign $\Rightarrow$ negative-norm states

$$[a^1_{\boldsymbol{p},s}, a^{1\dagger}_{\boldsymbol{q},r}] = (2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \qquad [b^1_{\boldsymbol{p},s}, b^{1\dagger}_{\boldsymbol{q},r}] = -(2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \tag{45}$$

$$[a^2_{\boldsymbol{p},s}, a^{2\dagger}_{\boldsymbol{q},r}] = -(2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \qquad [b^2_{\boldsymbol{p},s}, b^{2\dagger}_{\boldsymbol{q},r}] = (2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}). \tag{46}$$

Naive hermitian theory ill-defined

Let's try the same recipe: pseudohermitian QFT.

Let's turn the theory pseudohermitian by redefining the dual $\hat{\psi}$ as

$$\hat{\psi} = \eta^{-1}\bar{\psi}\eta, \qquad \mathcal{L} = \partial^{\mu}\hat{\psi}\partial_{\mu}\psi - m^{2}\hat{\psi}\psi, \qquad \mathcal{L}^{\#} = \eta^{-1}\mathcal{L}^{\dagger}\eta = \mathcal{L}. \tag{47}$$

The fields are

$$\psi(x) = \frac{1}{\sqrt{2m}}\int\frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\left[a^{1}_{\boldsymbol{p},s} + a^{2}_{\boldsymbol{p},s}\gamma^5\right]u_{\boldsymbol{p},s}e^{-ip\cdot x} + \left[b^{1\dagger}_{\boldsymbol{p},s} + b^{2\dagger}_{\boldsymbol{p},s}\gamma^5\right]v_{\boldsymbol{p},s}e^{ip\cdot x}\right\},$$

$$\hat{\psi}(x) = \frac{1}{\sqrt{2m}}\int\frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\bar{u}_{\boldsymbol{p},s}\left[\eta^{-1}a^{1\dagger}_{\boldsymbol{p},s}\eta - \eta^{-1}a^{2\dagger}_{\boldsymbol{p},s}\eta\gamma^5\right]e^{ip\cdot x}\right.$$
$$\left. + \bar{v}_{\boldsymbol{p},s}\left[\eta^{-1}b^{1}_{\boldsymbol{p},s}\eta - \eta^{-1}b^{2}_{\boldsymbol{p},s}\eta\gamma^5\right]e^{-ip\cdot x}\right\}.$$

Defining the action of η on the creation/anihilation operators as

$$\eta^{-1}a^{1}_{\boldsymbol{p},s}\eta = a^{1}_{\boldsymbol{p},s}, \qquad \eta^{-1}b^{1\dagger}_{\boldsymbol{p},s}\eta = -b^{1\dagger}_{\boldsymbol{p},s}, \qquad \eta^{-1}a^{2}_{\boldsymbol{p},s}\eta = -a^{2}_{\boldsymbol{p},s}, \qquad \eta^{-1}b^{2\dagger}_{\boldsymbol{p},s}\eta = b^{2\dagger}_{\boldsymbol{p},s}, \tag{48}$$

the dual becomes

$$\hat{\psi}(x) = \frac{1}{\sqrt{2m}}\int\frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\bar{u}_{\boldsymbol{p},s}\left[a^{1\dagger}_{\boldsymbol{p},s} + a^{2\dagger}_{\boldsymbol{p},s}\gamma^5\right]e^{ip\cdot x} - \bar{v}_{\boldsymbol{p},s}\left[b^{1}_{\boldsymbol{p},s} + b^{2}_{\boldsymbol{p},s}\gamma^5\right]e^{-ip\cdot x}\right\}. \tag{49}$$

An explicit expression for this operator η is

$$\eta = \exp\left[i\pi\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\sum_{s}\left(a^{2\dagger}_{\boldsymbol{p},s}a^{2}_{\boldsymbol{p},s} + b^{1\dagger}_{\boldsymbol{p},s}b^{1}_{\boldsymbol{p},s}\right)\right], \qquad \eta^{\dagger} = \eta \qquad \eta^{\dagger}\eta = 1. \tag{50}$$

The Klein-Gordon Lagrangian is now

$$\mathcal{L} = \partial^{\mu}\hat{\psi}\partial_{\mu}\psi - m^2\bar{\psi}\psi, \tag{51}$$

where $\hat{\psi} = \eta^{-1}\bar{\psi}\eta$. The conjugated momenta $\pi_{\psi}, \pi_{\hat{\psi}}$ are

$$\pi_{\psi} = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = \dot{\hat{\psi}}, \qquad \pi_{\hat{\psi}} = \frac{\partial\mathcal{L}}{\partial(\partial_0\hat{\psi})} = \dot{\psi}. \tag{52}$$

The canonical commutation relations at equal times

$$\left[\psi_a(\boldsymbol{x}, t), \pi_{\psi,b}(\boldsymbol{y}, t)\right] = \left[\hat{\psi}_a(\boldsymbol{x}, t), \pi_{\hat{\psi},b}(\boldsymbol{y}, t)\right] = i\delta_{ab}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}), \tag{53}$$

imply now the commutation relations with the right sign ⇒ no negative-norm states

$$[a^1_{\boldsymbol{p},s}, a^{1\dagger}_{\boldsymbol{q},r}] = (2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \qquad [b^1_{\boldsymbol{p},s}, b^{1\dagger}_{\boldsymbol{q},r}] = (2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \tag{54}$$

$$[a^2_{\boldsymbol{p},s}, a^{2\dagger}_{\boldsymbol{q},r}] = (2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \qquad [b^2_{\boldsymbol{p},s}, b^{2\dagger}_{\boldsymbol{q},r}] = (2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}). \tag{55}$$

In addition, this pseudohermitian QFT has the following features

- Microcausality

$$[\psi_a(x), \hat{\psi}_b(y)] = \Delta(x-y)\delta_{ab}, \qquad [\psi_a(x), \psi_b(y)] = 0, \qquad [\hat{\psi}_a(x), \hat{\psi}_b(y)] = 0, \tag{56}$$

where $\Delta(x-y)$ is the Lorentz invariant and causal Schwinger's Green function

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right] \tag{57}$$

- Hamiltonian

$$\begin{aligned} H &= : \int d^3x \left(\partial_0 \hat{\psi} \partial_0 \psi + \partial_i \hat{\psi} \partial_i \psi + m^2 \hat{\psi} \psi \right) : \\ &= \int \frac{d^3q}{(2\pi)^3} E_{\boldsymbol{q}} \sum_{r=1}^{2} \left\{ a^{1\dagger}_{\boldsymbol{q},r} a^{1}_{\boldsymbol{q},r} + a^{2\dagger}_{\boldsymbol{q},r} a^{2}_{\boldsymbol{q},r} + b^{1\dagger}_{\boldsymbol{q},r} b^{1}_{\boldsymbol{q},r} + b^{2\dagger}_{\boldsymbol{q},r} b^{2}_{\boldsymbol{q},r} \right\} \end{aligned} \tag{58}$$

- Momentum

$$\begin{aligned} P_i &= : - \int d^3x \left(\partial_0 \hat{\psi} \partial_i \psi + \partial_i \hat{\psi} \partial_0 \psi \right) : \\ &= \int \frac{d^3q}{(2\pi)^3} \boldsymbol{q} \sum_{r=1}^{2} \left\{ a^{1\dagger}_{\boldsymbol{q},r} a^{1}_{\boldsymbol{q},r} + a^{2\dagger}_{\boldsymbol{q},r} a^{2}_{\boldsymbol{q},r} + b^{1\dagger}_{\boldsymbol{q},r} b^{1}_{\boldsymbol{q},r} + b^{2\dagger}_{\boldsymbol{q},r} b^{2}_{\boldsymbol{q},r} \right\} \end{aligned} \tag{59}$$

- $U(1)$-charge

$$
\begin{aligned}
Q_{U(1)} &= : i \int d^3x \left(\hat{\psi} \partial_0 \psi - \partial_0 \hat{\psi} \psi \right) : \\
&= \int \frac{d^3q}{(2\pi)^3} \sum_{r=1}^{2} \left\{ a^{1\dagger}_{\boldsymbol{q},r} a^{1}_{\boldsymbol{q},r} + a^{2\dagger}_{\boldsymbol{q},r} a^{2}_{\boldsymbol{q},r} - b^{1\dagger}_{\boldsymbol{q},r} b^{1}_{\boldsymbol{q},r} - b^{2\dagger}_{\boldsymbol{q},r} b^{2}_{\boldsymbol{q},r} \right\}
\end{aligned}
\tag{60}
$$

- Discrete symmetries

$$
P\psi(x)P^{-1} = i\gamma^0 \psi(\mathcal{P}x), \qquad C\psi(x)C^{-1} = \mathcal{C}\hat{\psi}^T(x), \qquad T\psi(x)T^{-1} = \mathcal{C}\gamma^5 \psi(\mathcal{T}x),
$$

where $\mathcal{C} = -i\gamma^2\gamma^0$. The theory is invariant under C, P, and T.

- Can have dimension-4 boson self-interactions

$$
\mathcal{L}_{\text{int}} = \frac{\lambda_1}{2} \left(\hat{\psi}\psi \right)^2 + \frac{\lambda_2}{2} \left(\hat{\psi}\gamma^5\psi \right) \left(\hat{\psi}\gamma^5\psi \right) + \frac{\lambda_3}{2} \left(\hat{\psi} M^{\mu\nu} \psi \right) \left(\hat{\psi} M_{\mu\nu} \psi \right) \tag{61}
$$

Outline

Conclusions

Second-order spin-1/2 bosonic naive hermitian theory:

- Suffers from a similar problem as the fermionic one: indefinite metric. Inconsistent QFT.

Alternative approach: pseudohermiticity

- Introduce pseudohermiticity by defining a new dual $\hat{\psi} = \eta^{-1}\bar{\psi}\eta$ where $\eta = \exp\left[i\pi \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \sum_s \left(a^{2\dagger}_{\boldsymbol{p},s} a^2_{\boldsymbol{p},s} + b^{1\dagger}_{\boldsymbol{p},s} b^1_{\boldsymbol{p},s}\right)\right]$, $\mathcal{L}^\# = \eta^{-1}\mathcal{L}^\dagger\eta = \mathcal{L}$.

Second-order spin-1/2 bosonic pseudohermitian theory:

- Well defined pseudohermitian QFT: no negative-norm states, causal theory, real energy spectrum, unitary time evolution, Hamiltonian bounded from below, C,P,T invariant.
- The field has 8 degrees of freedom.
- Since the field has mass dimension one, the theory can have renormalizable self-interactions.
- Evades spin-statistics connection. This point has been studied before in[5][6][7], within the context of pseudohermitian QFTs.

[5] André LeClair and Matthias Neubert. "Semi-Lorentz invariance, unitarity, and critical exponents of symplectic fermion models". In: *Journal of High Energy Physics* 2007.10 (Oct. 2007), p. 027. DOI: 10.1088/1126-6708/2007/10/027. URL: https://dx.doi.org/10.1088/1126-6708/2007/10/027.

[6] Dharam Vir Ahluwalia and Cheng-Yang Lee. "Spin-half bosons with mass dimension three-half: Evading the spin-statistics theorem". In: *Europhysics Letters* 140.2 (Oct. 2022), p. 24001. DOI: 10.1209/0295-5075/ac97bd. URL: https://dx.doi.org/10.1209/0295-5075/ac97bd.

[7] Dharam Vir Ahluwalia et al. "Irreducible representations of the Poincaré group with reflections and two-fold Wigner degeneracy". In: *Journal of High Energy Physics* 2024.4 (Apr. 2024), p. 75. ISSN: 1029-8479. DOI: 10.1007/JHEP04(2024)075. URL: https://doi.org/10.1007/JHEP04(2024)075.

Thanks