Structure functions for semileptonic tau decays including heavy new physics effects.

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KM Formalism

The following calculations extend the results of the paper published on 1992 by Kühn and Mirkes (KM), [1], which is based in the study of the structure functions for the semileptonic decay $\tau \to 3h\nu_\tau$ for any light mesons h in the context of a general V-A Fermi-like theory whose amplitude is given by

$$
\begin{aligned}
\mathcal{M} &= \left\{ \begin{matrix} \cos\theta_c \\ \sin\theta_c \end{matrix} \right\} \frac{G}{\sqrt{2}} H_\mu M^\mu \\
M^\mu &= \overline{u}(p, s_\nu)\gamma^\mu(\epsilon_V - \epsilon_A\gamma_5)u(P, s) \\
H^\mu &= \left\langle \pi^i \pi^j \pi^k | \overline{D}\gamma^\mu(1-\gamma_5)u|0 \right\rangle
\end{aligned}
\tag{1}
$$

Axes

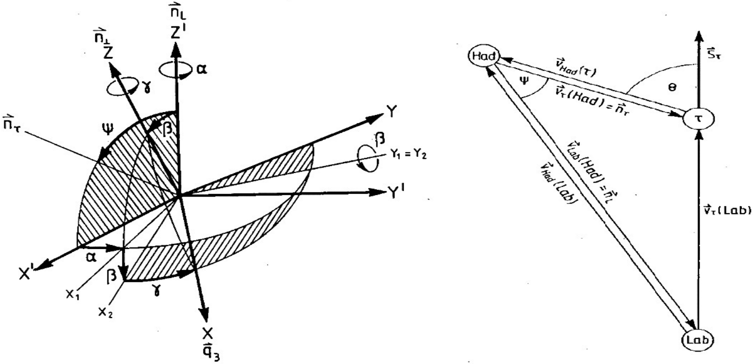

Figure: Definition of Euler Angles (left), Non-relativistic ilustration of ψ, θ, $\vec{\ell}$ (right)

Euler Angles

The frames S and S' will be related by the rotation

$$\vec{x} = R(\alpha, \beta, \gamma)\vec{x}', \tag{2}$$

with $R(\alpha, \beta.\gamma)$ given by the product of the matrices

$$\begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

with the angles defined by the relations:

- $cos\beta = \vec{n}_L \cdot \vec{n}_\perp$,
- $cos\gamma = -\frac{\vec{n}_L \cdot \hat{q}_3}{|\vec{n}_L \times \vec{n}_\perp|}$,
- $cos\alpha = \frac{(\vec{n}_L \times \vec{n}_\tau) \cdot (\vec{n}_L \times \vec{n}_\perp)}{|\vec{n}_L \times \vec{n}_\tau||\vec{n}_L \times \vec{n}_\perp|}$.

Lorentz Invariant Phase Space Measure

Thus, with the latter considerations and after quite an effort, the Lorentz invariant phase space measure reads

$$\begin{aligned} dPS^{(4)} &= (2\pi)^{-8}\delta^4(P-p-q_1-q_2-q_3)\frac{d^3p}{2E_\nu}\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3} \\ &= \frac{1}{64(2\pi)^7}\frac{M_\tau^2-Q^2}{M_\tau^2}\frac{dQ^2}{Q^2}\,ds_1\,ds_2\,d\alpha\,d\gamma\,\frac{d\cos\beta}{2}\,\frac{d\cos\theta}{2} \end{aligned} \tag{3}$$

Lepton Kinematics

Under the choice of reference frame just described and sketched, the leptonic degrees of freedom can be represented as follows

$$
\begin{aligned}
P &= (E, |\vec{P}| \sin\psi, 0, |\vec{P}| \cos\psi) \\
p &= (|\vec{P}|, |\vec{P}| \sin\psi, 0, |\vec{P}| \cos\psi) \\
\ell &= P^{\tau}_{pol}(-\frac{|\vec{P}|}{M_\tau} \cos\theta, -\frac{E}{M_\tau} \cos\theta \sin\psi + \sin\theta \cos\psi, 0, \qquad (4) \\
&\quad -\frac{E}{M_\tau} \cos\theta \cos\psi - \sin\theta \sin\psi) \qquad (5)
\end{aligned}
$$

with

$$
\begin{aligned}
E &= \frac{M_\tau^2 + Q^2}{2\sqrt{Q^2}} \\
|\vec{P}| &= \frac{M_\tau^2 - Q^2}{2\sqrt{Q^2}} \qquad (6)
\end{aligned}
$$

Hadronic kinematics

By chance of the same choice of frames, the hadronic degrees of freedom read

$$\begin{aligned} s_i &= (p_j + p_k)^2 \\ E_i &= \frac{Q^2 - s_i + m_i}{2\sqrt{Q^2}} \\ q_3^x &= \sqrt{E_3^2 - m_3^2} \\ q_2^x &= \frac{(2E_2 E_3 - s_1 + m_2^2 + m_3^2)}{2q_3^x} \\ q_1^x &= \frac{(2E_1 E_3 - s_2 + m_1^2 + m_3^2)}{2q_3^x} \\ q_2^y &= -\sqrt{E_2^2 - (q_2^x)^2 - m_2^2} \end{aligned} \tag{7}$$

Polarization Angle

Even though the polarization angle θ of the tau is defined on its rest frame, and the polar angle ψ in the hadronic rest frame, one very important trait of the present description is that they are constrained by means of their relation with the enery E_h of hte hadronic system with respect to the laboratory frame

$$\cos\theta = \frac{2xm_\tau^2 - m_\tau^2 - Q^2}{(m_\tau^2 - Q^2)\sqrt{x^2 - 4m_\tau^2/s}} \quad (8)$$

$$\cos\psi = \frac{x(m_\tau^2 - Q^2) - 2Q^2}{(m_\tau^2 - Q^2)\sqrt{x^2 - 4Q^2/s}} \quad (9)$$

with $x = \frac{2E_h}{s}$, and $s = 4E_{beam}^2$

Hadronic Matrix element

The hadronic matrix element for the V-A theory H_μ is given by

$$H^\mu = F_1 V_1^\mu + F_2 V_2^\mu + iF_3 V_3^\mu + F_4 V_4^\mu$$

with the basis of vectors associated to every form factor $F_{i=1,2,3,4}$ defined by

- $V_1^\mu = \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) (p_{1\nu} - p_{3\nu})$
- $V_2^\mu = \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) (p_{2\nu} - p_{3\nu})$
- $V_3^\mu = \epsilon^{\mu p_1 p_2 p_3}$
- $V_4^\mu = Q^\mu$

Structure functions

Finally, with all the information and conventions displayed, we can write the relevant contraction among lepton and hadron tensors as

$$
\begin{aligned}
L_{\mu\nu}H^{\mu\nu} &= \sum_X \overline{L}_X H_X = (\epsilon_V^2 + \epsilon_A^2)(m_\tau^2 - Q^2) \sum_X L_X H_X, &(10)\\
& X \in \{A, B, C, .., I, SA, SB, ..., SG\} &(11)
\end{aligned}
$$

Structure functions (lepton part)

After integrating the unobservable (*yet*) angle α

- $L_A = 2/3K_1 + K_2 + 1/3(\overline{K}_1 - K_1)(3\cos^2\beta - 1)/2$
- $L_B = 2/3K_1 + K_2 - 2/3(\overline{K}_1 - K_1)(3\cos^2\beta - 1)/2$
- $L_C = -1/2\overline{K}_1 \sin^2\beta \cos 2\gamma$
- $L_D = 1/2\overline{K}_1 \sin^2\beta \sin 2\gamma$
- $L_E = \overline{K}_3 \cos\beta$
- $L_F = 1/2\overline{K}_1 \sin 2\beta \cos\gamma$
- $L_G = -\overline{K}_3 \sin\beta \sin\gamma$
- $L_H = 1/2\overline{K}_1 \sin 2\beta \sin\gamma$
- $L_I = -\overline{K}_3 \sin\beta \cos\gamma$
- $L_{SA} = K_2$

where the $K_i, \overline{K}_i$ coefficients are functions of the polarization angle and the hadronic invariant mass Q^2 only.

Hadronic Structure Functions

As for the hadronic structure functions we have (some of them)

- $W_A = (x_1^2 + x_3^2)|F_1|^2 + (x_2^2 + x_3^2)|F_2|^2 + 2(x_1x_2 - x_3^2)Re(F_1F_2^*)$
- $W_B = x_4|F_3|^2$
- $W_C = (x_1^2 - x_3^2)|F_1|^2 + (x_2^2 - x_3^2)|F_2|^2 + 2(x_1x_2 + x_3^2)Re(F_1F_2^*)$
- $W_D = 2[(x_1x_3)|F_1|^2 + (x_2x_3)|F_2|^2 + x_3(x_2 - x_1)Re(F_1F_2^*)]$
- $W_E = -2x_3(x_1 + x_2)Im(F_1F_2^*)$
- $W_F = 2x_4(x_1Im(F_1F_3^*) + x_2Im(F_2F_3^*))$
- $W_G = -2x_4(x_1Re(F_1F_3^*) + x_2Re(F_2F_3^*))$
- $W_H = 2x_3x_4(Im(F_1F_3^*) - Im(F_2F_3^*))$
- $W_I = -2x_3x_4(Re(F_1F_3^*) - Re(F_2F_3^*))$
- $W_{SA} = Q^2|F_4|^2$

with $x_1 = V_1^x, x_2 = V_2^x, x_3 = V_1^y,$ and $x_4 = \sqrt{Q^2}x_3V_3^z$

Heavy New Physics extension

We will study the heavy new physics effects that can be obtained from the mass dimension-6 Lagrangian

$$\mathcal{L} = -\frac{G_F V_{uD}}{\sqrt{2}} \Big\{ \bar{\tau}\gamma^\mu(1-\gamma_5)\nu_\tau \cdot [\bar{u}\gamma_\mu(1-\gamma_5)D + \bar{u}\gamma_\mu(\epsilon_V^\tau - \epsilon_A^\tau\gamma_5)D] + \bar{\tau}(1-\gamma_5)\nu_\tau \cdot \bar{u}(\epsilon_S^\tau - \epsilon_P^\tau\gamma_5)D + \epsilon_T^\tau \bar{\tau}\sigma^{\mu\nu}(1-\gamma_5)\nu_\tau \cdot \bar{u}\sigma_{\mu\nu}D + \text{h.c.} \Big\}$$

To make use of the latter piece of Lagrangian, we need to consider the following hadronic matrix elements

$$\begin{aligned}
H^{\mu} &= \langle \pi^i \pi^j \pi^k | \overline{D}\gamma^{\mu}(1-\gamma_5)u|0\rangle = H^{\mu}_V - H^{\mu}_A \qquad (12)\\
&= F_1(Q,s_1,s_2)V_1^{\mu} + F_2(Q,s_1,s_2)V_2^{\mu} + iF_3(Q,s_1,s_2)V_3^{\mu} + F_4(Q,s_1,s_2)Q^{\mu}\\
H^{\mu\nu} &= \langle \pi^i \pi^j \pi^k | \overline{D}\sigma^{\mu\nu}(1+\gamma_5)u|0\rangle\\
&= \sum_{\vec{\alpha}} C_{\alpha}\epsilon^{\mu\nu\gamma\delta} p_{\gamma}^{\alpha_1} p_{\delta}^{\alpha_2}\\
L_{\mu} &= \overline{u}(p)\gamma_{\mu}\cdot(1-\gamma_5)\cdot u(P),\\
L_{\mu\nu} &= \overline{u}(p)\sigma_{\mu\nu}\cdot(1+\gamma_5)\cdot u(P).
\end{aligned}$$

Lepton and Hadron Tensors

The new lepton and hadron tensors are defined as

$$L_{\mu\nu}^{V-A} = \sum_s L_\mu L_\nu^\dagger, L_{\mu\nu\delta\eta}^{TT} = \sum_s L_{\mu\nu} L_{\delta\eta}^\dagger, \quad L_{\mu\nu\eta}^{I} = \sum_s L_{\mu\nu} L_\eta^\dagger$$
$$H_{\mu\nu}^{V-A} = H_\mu H_\nu^\dagger, \quad H_{\mu\nu\gamma\delta}^{TT} = H_{\mu\nu} H_{\gamma\delta}^\dagger, \quad H_{\mu\nu\eta}^{I} = H_{\mu\nu} H_\eta^\dagger$$

where

$$L_{\mu\nu\delta\eta}^{TT} = Tr[\sigma_{\mu\nu}\not{P}\sigma_{\delta\eta}\not{p}] + Tr[\sigma_{\mu\nu}\gamma_5\not{P}\sigma_{\delta\eta}\not{p}] + sM_\tau Tr[\sigma_{\mu\nu}\not{\ell}\sigma_{\delta\eta}\not{p}] + sM_\tau Tr[\sigma_{\mu\nu}\gamma_5\not{\ell}\sigma_{\delta\eta}\not{p}]$$
$$L_{\mu\nu\eta}^{I} = -M_\tau[Tr[\sigma_{\mu\nu}\gamma_\eta\not{p}] + Tr[\sigma_{\mu\nu}\gamma_5\gamma_\eta\not{p}]] - s[Tr[\sigma_{\mu\nu}\not{\ell}\not{P}\gamma_\eta\not{p}] + Tr[\sigma_{\mu\nu}\gamma_5\not{\ell}\not{P}\gamma_\eta\not{p}]].$$

Tensor-Tensor terms

First of all, notice that the contraction between the lepton tensor $L^{TT}_{\mu\nu\delta\eta}$ and the hadron tensor $H^{TT}_{\mu\nu\delta\eta}$, can be written as the contraction between the tensors

$$L^{TT}_{\alpha\beta} = P_\alpha p_\beta + P_\beta p_\alpha - \frac{1}{2}(P \cdot p) g_{\alpha\beta} + (P \to M_\tau \ell) \, ,$$
$$H^{TT}_{\alpha\beta} = -p_k^\alpha \left(p_i^\beta (p_j \cdot p_l) - p_j^\beta (p_i \cdot p_l) \right) + p_l^\alpha \left(p_i^\beta (p_j \cdot p_k) - p_j^\beta (p_i \cdot p_k) \right) \, .$$

i.e.

$$\mathcal{M}_{TT} = 4\epsilon_T^2 L^{TT}_{\mu\nu\delta\eta} H^{\mu\nu} H^{\dagger\delta\eta} = 4\epsilon_T^2 H^{TT}_{\alpha\beta} L^{TT}_{\alpha\beta} \tag{13}$$

Notice also that the lepton tensor can be written as

$$L^{TT}_{\alpha\beta} = L_S^{V-A}{}_{\alpha\beta} - \frac{1}{2}(P \cdot p) g_{\alpha\beta} \, , \tag{14}$$

where $L_S^{V-A}{}_{\alpha\beta}$ stands for the symmetric component of the lepton tensor defined in the V-A theory.

New Structure Functions (TT-Terms)

$$
\begin{aligned}
\delta W_A &= \frac{1}{81}(-9m_1^2(x_1^2 - 4x_2x_1 + 4x_2^2 + 9x_3^2) - 9m_2^2(4x_1^2 - 4x_2x_1 + x_2^2 + 9x_3^2) \\
&\quad -2(2x_1^2 - 5x_2x_1 + 2x_2^2 + 9x_3^2)(-2\alpha_1^2 q^2 - 2\alpha_2^2 q^2 + 5\alpha_1\alpha_2 q^2 + 2x_1^2 + 2x_2^2 + 9x_3^2 - 5x_1x_2)) \\
\delta W_B &= -\frac{1}{9}x_3(3m_1^2(x_1 - 2x_2) + m_2^2(6x_1 - 3x_2) \\
&\quad +(x_1 - x_2)(-2\alpha_1^2 q^2 - 2\alpha_2^2 q^2 + 5\alpha_1\alpha_2 q^2 + 2x_1^2 + 2x_2^2 + 9x_3^2 - 5x_1x_2)) \\
\delta W_D &= \delta W_C \\
\delta W_{SA} &= \frac{1}{81}q^2(-9(\alpha_2 - 2\alpha_1)^2 m_2^2 - 9(\alpha_1 - 2\alpha_2)^2 m_1^2 + 2(2\alpha_1 - \alpha_2)(\alpha_1 - 2\alpha_2) \\
&\qquad (2\alpha_1^2 q^2 + 2\alpha_2^2 q^2 - 5\alpha_1\alpha_2 q^2 - 2x_1^2 - 2x_2^2 - 9x_3^2 + 5x_1x_2)) \\
\delta W_{SB} &= \delta W_{SA} \\
\delta W_{SD} &= \frac{1}{81}q(-9(\alpha_1 - 2\alpha_2)m_1^2(x_1 - 2x_2) - 9(2\alpha_1 - \alpha_2)m_2^2(2x_1 - x_2) \\
&\quad +((4\alpha_2 - 5\alpha_1)x_2 + (4\alpha_1 - 5\alpha_2)x_1)(2\alpha_1^2 q^2 + 2\alpha_2^2 q^2 - 5\alpha_1\alpha_2 q^2 - 2x_1^2 - 2x_2^2 - 9x_3^2 + 5x_1x_2)) \\
\delta W_{SF} &= \frac{1}{9}qx_3(-3(\alpha_1 - 2\alpha_2)m_1^2(3\alpha_2 - 6\alpha_1)m_2^2 \\
&\qquad +(\alpha_1 - \alpha_2)(2\alpha_1^2 q^2 + 2\alpha_2^2 q^2 - 5\alpha_1\alpha_2 q^2 - 2x_1^2 - 2x_2^2 - 9x_3^2 + 5x_1x_2))
\end{aligned}
$$

where $\alpha_2 = \frac{q^2 - s_2 + m_2^2}{2q^2}, \quad \alpha_3 = \frac{s_1 + s_2 - m_3^2 - m_1^2}{2q^2}, \quad q = \sqrt{Q^2}$

Interference Terms

The tensor-vector interference terms are given by

$$\mathcal{M}_I = 2\epsilon_T L_{\mu\nu\eta} H^{\mu\nu} H^{\dagger\eta} + h.c.$$
$$L_{\mu\nu\eta} = Tr\Big[\sigma_{\mu\nu}(1+\gamma_5)u(P)\overline{u}(P)(1+\gamma_5)\gamma_\eta(-p_\nu)\Big]. \quad (15)$$

After some Dirac algebra

$$\sum_s L_{\mu\nu\eta} = \quad -M_\tau\Big[Tr[\sigma_{\mu\nu}\gamma_\eta \not{p}] + Tr[\sigma_{\mu\nu}\gamma_5\gamma_\eta \not{p}]\Big] \quad (16)$$
$$-s\Big[Tr[\sigma_{\mu\nu}\not{\ell}\not{P}\gamma_\eta \not{p}] + Tr[\sigma_{\mu\nu}\gamma_5\not{\ell}\not{P}\gamma_\eta \not{p}]\Big]. \quad (17)$$

Interference terms

Notice that the terms in the latter equation involving the γ_5 matrix can be recovered from the others by considering the identity $\gamma_5 \sigma_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\sigma^{\alpha\beta}$, so that they read

$$\frac{1}{2}\left\{ Tr\left[\epsilon_{\mu\nu\alpha\beta}\sigma^{\alpha\beta}\gamma_\eta \not{p}\right] + s\left[Tr[\epsilon_{\mu\nu\alpha\beta}\sigma^{\alpha\beta}\not{\ell}\not{P}\gamma_\eta \not{p}]\right]\right\} . \tag{18}$$

Meanwhile, $L_{\mu\nu\eta}$ can be written as the self-dual projection of $l^{l}_{\mu\nu\eta}$

$$L_{\mu\nu\eta} = \left(l^{l}_{\mu\nu\eta} + \frac{i}{2}\epsilon_{\mu\nu\alpha\beta} l^{l\,\alpha\beta}{}_{\eta} \right) , \tag{19}$$

where

$$l^{l}_{\mu\nu\eta} = [-8\left(-\ell_\mu g_{\nu\eta}(P\cdot p) + P_\mu g_{\nu\eta}(p\cdot\ell) + P_\eta p_\nu \ell_\mu + P_\nu p_\eta \ell_\mu + P_\nu p_\mu \ell_\eta\right)] - (\mu \to \nu) . \tag{20}$$

Self-Dual decomposition of the interference tensors

We can decompose the tensors $L_{\mu\nu\eta}$, $H_{\mu\nu\eta}$ in the following SO(3) irreps

$$\begin{aligned}
\vec{L}_0 &= [(P\cdot p) - 2P_0 p_0]\,\vec{\ell} + [(P_0+p_0)\ell_0 - (\ell\cdot p)]\,\vec{P} + i(P_0+p_0)\vec{\ell}\times\vec{P} \\
\vec{L}_A &= -i(P\cdot p)\vec{\ell} + i(\ell\cdot p)\vec{P} - 2p_0\vec{\ell}\times\vec{P} + 2i\left(\vec{P}(\vec{\ell}\cdot\vec{P}) - \vec{\ell}(\vec{P}\cdot\vec{P})\right) \\
\hat{L}_S &= \ell_0(P\cdot p)I - P_0(\ell\cdot p)I + 2\ell_0(\vec{P}\otimes\vec{P})_S - 2p_0(\vec{\ell}\otimes\vec{P})_S + 2i\left((\vec{\ell}\times\vec{P})\otimes\vec{P}\right)_S
\end{aligned}$$

$$\begin{aligned}
H_0 &= i(\sqrt{Q^2}F_4^*[E_i\vec{p}_j - E_j\vec{p}_i + i(p_i\times p_j)]) \\
H_A &= i(E_i(\vec{p}_j\times\vec{H}^\dagger) - E_j(\vec{p}_i\times\vec{H}^\dagger) + i[\vec{P}_j(\vec{p}_i\cdot\vec{H}^\dagger) - \vec{P}_i(\vec{p}_j\cdot\vec{H}^\dagger)]) \\
H^I &= i(E_i[\vec{p}_j\otimes\vec{H}^\dagger]_S - E_j[\vec{p}_i\otimes\vec{H}^\dagger]_S + i[(\vec{p}_i\times\vec{p}_j)\otimes\vec{H}^\dagger]_S)
\end{aligned} \tag{21}$$

so that

$$\mathcal{M}_I = \vec{H}_0\cdot\vec{L}_0 + \vec{H}_A\cdot\vec{L}_A + Tr\{H^I L^I\} + c.c. \tag{22}$$

New structure functions (I-Terms)

$$
\begin{aligned}
L_{0z} &= 2\pi 4q^3 M_\tau \left(\sin(\beta)(-\cos(\gamma)) l_{0x} + \sin(\beta)\sin(\gamma) l_{0y} + \cos(\beta) l_{0z}\right) \\
L_{Az} &= 2\pi 4q^2 M_\tau \left(\sin(\beta)(-\cos(\gamma)) l_{Ax} + \sin(\beta)\sin(\gamma) l_{Ay} + \cos(\beta) l_{Az}\right)
\end{aligned}
$$

$$
\begin{aligned}
l_{0x} &= qs\beta_{\mathrm{Qs}} \left(2\sin(\theta) M_\tau^3 \cos(\psi) + q\cos(\theta)\gamma_{\mathrm{Qs}}\sin(\psi)\right) \\
l_{0y} &= -2iqs\sin(\theta) M_\tau^3 \beta_{\mathrm{Qs}} \\
l_{0z} &= -qs\beta_{\mathrm{Qs}} \left(2\sin(\theta) M_\tau^3 \sin(\psi) + q\cos(\theta)\cos(\psi)\left(M_\tau^2 + q^2\right)\right) \\
l_{Ax} &= -2iqsM_\tau^2 \beta_{\mathrm{Qs}} \left(q\cos(\theta)\sin(\psi) - \sin(\theta) M_\tau \cos(\psi)\right) \\
l_{Ay} &= -2qs\sin(\theta) M_\tau \beta_{\mathrm{Qs}}^2 \\
l_{Az} &= -2iqsM_\tau^2 \beta_{\mathrm{Qs}} \left(\sin(\theta) M_\tau \sin(\psi) + q\cos(\theta)\cos(\psi)\right)
\end{aligned}
$$

Structure functions (I-Terms)

Let us define the structure functions for the unobservable τ's case

$$
\begin{aligned}
L_{+}^{I} &= L_{11}^{I} + L_{33}^{I} \\
L_{-}^{I} &= L_{11}^{I} - L_{33}^{I} \\
W_{+}^{I} &= H_{11}^{I} + H_{22} + H_{33}^{I} \\
W_{-}^{I} &= H_{11}^{I} + H_{22}^{I} - H_{33}^{I}
\end{aligned} \tag{23}
$$

Then we can write the contraction between lepton and hadron tensors as

$$
Tr\left\{H^{I} L^{I}\right\} = L_{+}^{I} H_{+}^{I} + L_{-}^{I} H_{-}^{I} \tag{24}
$$

where, after integrating the rather unobservable angle α

$$
\begin{aligned}
H_{\pm} = \quad & \pm\frac{1}{3} F_3^{*} e_3 \left(F_1^{*} - F_2^{*}\right) x_3^2 \left(x_1 + x_2\right) x_4 x_3 \\
& -\frac{1}{3}\left(\left(e_2 - e_3\right) x_1 + \left(e_2 + 2 e_3\right) x_2\right)\left(F_1^{*} x_1 + F_2^{*} x_2\right) \\
L_{+} = \quad & \frac{1}{4}\pi\left(-4 A_{12} \sin^2(\beta)\sin(2\gamma) + A_{11}\left(2\sin^2(\beta)\cos(2\gamma) - \cos(2\beta) + 5\right) - A_{22}\left(2\sin^2(\beta)\cos(2\gamma) + \cos(2\beta) - 5\right)\right. \\
L_{-} = \quad & \frac{1}{4}\pi\left(12 A_{12}\sin^2(\beta)\sin(2\gamma) - 12 A_{23}\sin(2\beta)\sin(\gamma) + 6\left(A_{22} - A_{11}\right)\sin^2(\beta)\cos(2\gamma)\right. && (25) \\
& \left. + 12 A_{13}\sin(2\beta)\cos(\gamma) + \left(A_{11} + A_{22} - 2 A_{33}\right)\left(3\cos(2\beta) + 1\right)\right) && (26)
\end{aligned}
$$

Structure functions I-Term

$$
\begin{aligned}
A_{11} &= qs\beta_{\text{Qs}}\sin(\psi)\left(\gamma_{\text{Qs}}\left(2q\sin(\theta)M_\tau\cos(\psi)-\cos(\theta)\gamma_{\text{Qs}}\sin(\psi)\right)+\cos(\theta)\beta_{\text{Qs}}^2\sin(\psi)\right)\\
A_{12} &= iq^2 s\sin(\theta)M_\tau\beta_{\text{Qs}}^2\sin(\psi)\\
A_{13} &= qs(\beta_{\text{Qs}}\gamma_{\text{Qs}}\left(q\sin(\theta)M_\tau\cos(2\psi)-\cos(\theta)\gamma_{\text{Qs}}\sin(\psi)\cos(\psi)\right)\\
&\qquad +\cos(\theta)\beta_{\text{Qs}}^3\sin(\psi)\cos(\psi))\\
A_{23} &= iq^2 s\sin(\theta)M_\tau\beta_{\text{Qs}}^2\cos(\psi)\\
A_{33} &= -qs\beta_{\text{Qs}}\cos(\psi)\left(\gamma_{\text{Qs}}\left(2q\sin(\theta)M_\tau\sin(\psi)+\cos(\theta)\gamma_{\text{Qs}}\cos(\psi)\right)-\cos(\theta)\beta_{\text{Qs}}^2\cos(\psi)\right)
\end{aligned}
$$

with the associations $M_\tau^2 - q^2 \to \beta_{\text{Qs}}$, $M_\tau^2 + q^2 \to \gamma_{Qs}$

Structure functions for the I-Terms (hadron tensor components)

In the case of observable τ's, we need all the components of the hadronic tensor

- $H_{0x} = -\frac{1}{3} F_4^* q \left(\left(e_2 - e_3 \right) x_1 + \left(e_2 + 2e_3 \right) x_2 \right),$
- $H_{0y} = e_3 F_4^* q x_3,$
- $H_{0z} = -\frac{-i}{3} F_4^* q \left(x_1 + x_2 \right) x_3,$
- $H_{Ax} = \frac{i}{3} x_3 \left(F_1^* \left(x_1 + x_2 \right) x_3 - F_2^* \left(x_1 + x_2 \right) x_3 + 3 e_3 F_3^* x_4 \right),$
- $H_{Ay} = \frac{-i}{3} \left(F_1^* x_1 \left(x_1 + x_2 \right) x_3 + F_2^* x_2 \left(x_1 + x_2 \right) x_3 - F_3^* \left(\left(e_2 - e_3 \right) x_1 + \left(e_2 + 2e_3 \right) x_2 \right) x_4 \right),$
- $H_{Az} = \frac{1}{3} \left(\left(e_2 + 2e_3 \right) F_1^* + \left(e_3 - e_2 \right) F_2^* \right) \left(x_1 + x_2 \right) x_3,$
- $H^I_{11} = -\frac{1}{3} \left(\left(e_2 - e_3 \right) x_1 + \left(e_2 + 2e_3 \right) x_2 \right) \left(F_1^* x_1 + F_2^* x_2 \right),$
- $H^I_{12} = \frac{1}{2} \left(e_3 x_3 \left(F_1^* x_1 + F_2^* x_2 \right) - \frac{1}{3} \left(F_1^* - F_2^* \right) \left(\left(e_2 - e_3 \right) x_1 + \left(e_2 + 2e_3 \right) x_2 \right) x_3 \right),$
- $H^I_{13} = \frac{-i}{2} \left(-\frac{1}{3} \left(x_1 + x_2 \right) x_3 \left(F_1^* x_1 + F_2^* x_2 \right) - \frac{1}{3} F_3^* \left(\left(e_2 - e_3 \right) x_1 + \left(e_2 + 2e_3 \right) x_2 \right) x_4 \right),$
- $H^I_{22} = e_3 \left(F_1^* - F_2^* \right) x_3^2,$
- $H^I_{23} = \frac{-i}{2} \left(e_3 F_3^* x_3 x_4 - \frac{1}{3} \left(F_1^* - F_2^* \right) \left(x_1 + x_2 \right) x_3^2 \right)$
- $H^I_{33} = \frac{1}{3} F_3^* \left(x_1 + x_2 \right) x_3 x_4$

These 12 functions should be paired with their leptonic counterpart but we still have to talk about a more convenient representation for them

A new representation for the lepton tensors

We can obtain the list of independent components of a rotated symmetric lepton tensor $(L_{11} \quad L_{12} \quad L_{13} \quad L_{22} \quad L_{23} \quad L_{33})$ by means of the following linear transformation

$$\begin{pmatrix} (D_{2,2}-D_{2,0}+D_{2,-2}) & -2(D_{1,2}-\sqrt{\frac{2}{3}}D_{1,0}+D_{1,-2}) & -(\sqrt{\frac{2}{3}}D_{0,2}-\frac{2}{3}D_{0,0}+D_{2,-2}) & \text{h.c.} & 1 \\ i(D_{2,2}-D_{2,-2}) & -2i(D_{1,2}-D_{1,-2}) & -i\sqrt{\frac{2}{3}}D_{0,2} & \text{h.c.} & 0 \\ (D_{2,1}-D_{2,-1}) & 2(D_{1,1}-D_{1,-1}) & \sqrt{\frac{2}{3}}D_{0,1} & \text{h.c.} & 0 \\ -(D_{2,2}+\sqrt{\frac{2}{3}}D_{2,0}+D_{2,-2}) & 2(D_{1,2}+\sqrt{\frac{2}{3}}D_{1,0}+D_{1,-2}) & \sqrt{\frac{2}{3}}D_{0,2}+\frac{2}{3}D_{0,0} & \text{h.c.} & 1 \\ -(D_{2,1}+D_{2,-1}) & 2i(D_{1,1}+D_{1,-1}) & i\sqrt{\frac{2}{3}}D_{0,1} & \text{h.c.} & 0 \\ 2\sqrt{\frac{2}{3}}D_{2,0} & -4\sqrt{\frac{2}{3}}D_{1,0} & -\frac{4}{3}D_{0,0} & \text{h.c.} & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \overline{\beta}_1 \\ \overline{\beta}_2 \\ \beta_3 \\ L/3 \end{pmatrix}$$

where, as functions of the non-transformed components L'_{ij}, $\beta_1 = L'_{11} - 2iL'_{12} - L'_{22}$, $\beta_2 = L'_{13} - iL'_{23}$, $\beta_3 = L'_{11} - L'_{22} - 2L'_3$, $\overline{\beta}_1 = L'_{11} + 2iL'_{12} + L'_{22}$, $\overline{\beta}_2 = L'_{13} + i'L_{23}$ and the trace $L = L'_{ii}$ of the symmetric 3-tensor.

Where the $\eta_i, \overline{\eta}_i$ coefficients for the l-tensors and TT-tensors $\lambda_i, \overline{\lambda_i}$ read

- $\eta_1 = \frac{qs\beta_{Qs}\sin(\psi)\left(2q\sin(\theta)M_\tau\left(q^2+\gamma_{Qs}\cos(\psi)\right)-2q\sin(\theta)M_\tau^3-\cos(\theta)M_\tau^2\beta_{Qs}\sin(\psi)+\cos(\theta)\sin(\psi)\left(q^2\beta_{Qs}-\gamma_{Qs}^2\right)\right)}{4M_\tau}$,
- $\eta_2 = \frac{1}{4}s\left(\frac{q\beta_{Qs}\left(\gamma_{Qs}\left(q\sin(\theta)M_\tau\cos(2\psi)-\cos(\theta)\gamma_{Qs}\sin(\psi)\cos(\psi)\right)+\cos(\theta)\beta_{Qs}^2\sin(\psi)\cos(\psi)\right)}{M_\tau}+q^2\sin(\theta)\beta_{Qs}^2\cos(\psi)\right)$,
- $\eta_3 = -\frac{qs\beta_{Qs}\left(\cos(\theta)\beta_{Qs}^2(3\cos(2\psi)+1)-\gamma_{Qs}\left(6q\sin(\theta)M_\tau\sin(2\psi)+\cos(\theta)\gamma_{Qs}(3\cos(2\psi)+1)\right)\right)}{8M_\tau}$,
- $\overline{\eta_1} = -\frac{qs\beta_{Qs}\sin(\psi)\left(2q\sin(\theta)M_\tau\left(q^2-\gamma_{Qs}\cos(\psi)\right)-2q\sin(\theta)M_\tau^3+\cos(\theta)M_\tau^2\beta_{Qs}\sin(\psi)+\cos(\theta)\sin(\psi)\left(\gamma_{Qs}^2-q^2\beta_{Qs}\right)\right)}{4M_\tau}$,
- $\overline{\eta_2} = \frac{-qs\beta_{Qs}\left(q\sin(\theta)M_\tau\left(q^2\cos(\psi)-\gamma_{Qs}\cos(2\psi)\right)-q\sin(\theta)M_\tau^3\cos(\psi)+\cos(\theta)M_\tau^2\beta_{Qs}\sin(\psi)\cos(\psi)+\cos(\theta)\sin(\psi)\cos(\psi)\left(\gamma_{Qs}^2-q^2\beta_{Qs}\right)\right.}{4M_\tau}$
- $\lambda_1 = \frac{1}{4}q^2\sin(\psi)\left(q^2-M_\tau^2\right)\left(-2qs\sin(\theta)M_\tau\cos(\psi)+M_\tau^2(-\sin(\psi))+\sin(\psi)\left(q^2+s\cos(\theta)\gamma_{Qs}\right)\right)$,
- $\lambda_2 = \frac{1}{4}q^2\left(q^2-M_\tau^2\right)\left(-qs\sin(\theta)M_\tau\cos(2\psi)+M_\tau^2\sin(\psi)(-\cos(\psi))+\sin(\psi)\cos(\psi)\left(q^2+s\cos(\theta)\gamma_{Qs}\right)\right)$,
- $\lambda_3 = -\frac{1}{8}q^2\left(q^2-M_\tau^2\right)\left(6qs\sin(\theta)M_\tau\sin(2\psi)-\left(M_\tau^2(3\cos(2\psi)+1)\right)+(3\cos(2\psi)+1)\left(q^2+s\cos(\theta)\gamma_{Qs}\right)\right)$,
- $\overline{\lambda_1} = \lambda_1$,
- $\overline{\lambda_2} = \lambda_2$

with the associations $M_\tau^2 + q^2 \rightarrow \beta_{Qs}$, $(M_\tau^2 - q^2) \rightarrow \gamma_{Qs}$

A new representation for the lepton tensors

We can also obtain the independent components $(L_1 \quad L_2 \quad L_3)$ of any transformed **SO(3)** structure that transforms a triplet irrep, that includes antisymmetric 3D- tensors, spacelike vectors, self-dual four-tensors, and the axial(polar) components of an antisymmetric four-tensor

$$\begin{pmatrix} \frac{1}{2}(D^1_{11} - D^1_{0-1}) & \frac{1}{2}(D^1_{-1-1} - D^1_{1-1}) & \frac{\sqrt{2}}{2}(D^1_{-10} - D^1_{10}) \\ \frac{-i}{2}(D^1_{11} + D^1_{0-1}) & \frac{i}{2}(D^1_{-1-1} + D^1_{1-1}) & \frac{i\sqrt{2}}{2}(D^1_{-10} + D^1_{10}) \\ -\frac{\sqrt{2}}{2}D^1_{10} & \frac{\sqrt{2}}{2}D^1_{-10} & D^1_{00} \end{pmatrix} \begin{pmatrix} \Upsilon_+ \\ \Upsilon_- \\ \Upsilon_3 \end{pmatrix} \tag{27}$$

where as functions of the non-transformed components L'_i;
$\Upsilon_\pm = L'_x \pm iL'y$ and $\Upsilon_3 = L'_z$

Null observables, disentangling the new physics

In the same spirit as in the formalism developed by K&M, we can extract the structure functions coming from the interference terms by measuring the averages $\left\langle D^2_{km} \right\rangle$ and solving the system of equations

$$\Re e\left\{\begin{pmatrix} \langle D^2_{22}\rangle \\ \langle D^2_{20}\rangle \\ \langle D^2_{2-2}\rangle \\ \langle D^2_{12}\rangle \\ \langle D^2_{10}\rangle \\ \langle D^2_{1-2}\rangle \end{pmatrix}\right\} = \begin{pmatrix} \lambda_+ & \lambda_+ & -\lambda_+ & \eta_+ & \eta_+ & -\eta_+ \\ -\lambda_+ & 0 & -\sqrt{\frac{2}{3}}\lambda_+ & \eta_+ & 0 & -\sqrt{\frac{2}{3}}\eta_+ \\ \lambda_+ & -\lambda_+ & -\lambda_+ & \eta_+ & -\eta_+ & \eta_+ \\ -2\lambda_- & -2\lambda_- & 2\lambda_- & -2\eta_- & -2\eta_- & 2\eta_- \\ 2\sqrt{\frac{2}{3}}\lambda_- & 0 & 2\sqrt{\frac{2}{3}}\lambda_- & 2\sqrt{\frac{2}{3}}\eta_- & 0 & 2\sqrt{\frac{2}{3}}\eta_- \\ -2\lambda_- & 2\lambda_- & 2\lambda_- & -2\eta_- & 2\eta_- & -2\eta_- \end{pmatrix} \Re e\left\{\begin{pmatrix} H^{VT}_{11} \\ iH^{VT}_{12} \\ H^{VT}_{22} \\ H^{I}_{11} \\ iH^{I}_{12} \\ H^{I}_{22} \end{pmatrix}\right\}$$

whose solution is assured as long as $\eta_-\lambda_+ - \eta_+\lambda_- \neq 0$, where

$$\begin{aligned} \lambda_+ &= \lambda_1 + \overline{\lambda}_1 \\ \eta_+ &= \eta_1 + \overline{\eta}_1 \\ \lambda_- &= \lambda_2 + \overline{\lambda}_2 \\ \eta_- &= \eta_2 + \overline{\eta}_2 \end{aligned} \tag{28}$$

Angular information visualization

Two of the Euler angles α, β, γ obey the same equivalence relations than the spherical coordinates, then the information displayed by two of these angles must lie somewhere in the *2-sphere*, but we still have a radial direction to play with, which can be associated to the remaining angle, but we need a mapping taking the interval $[0, 1]$ to the interval $[-\pi, \pi]$,

$$\alpha(r) = 2\arcsin(-2\sqrt{1 - r^2} + 1) \tag{29}$$

Angular information visualization

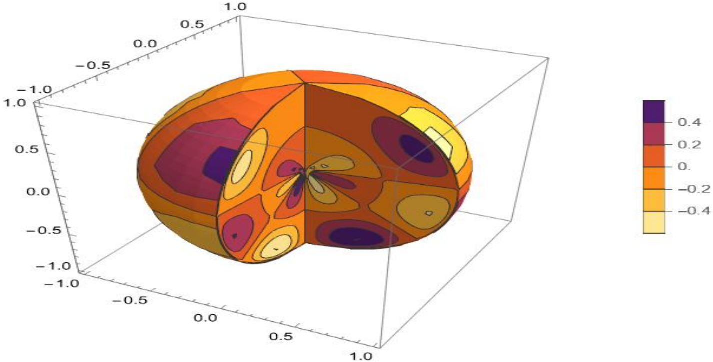

Figure: Wigner-D function $D^4_{2-1}(\alpha, \beta, \gamma)$

Wigner-D functions

How to visualize them ?

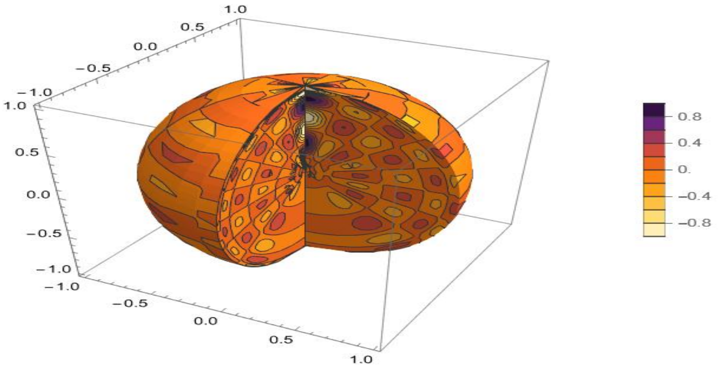

Figure: Wigner-D function $D^{10}_{33}(\alpha, \beta, \gamma)$

Finally D:

Thanks a lot, uwu !

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