Constraints on cosmology from quantum gravity

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based on Kontsevich & Segal [2105.10161]

Witten [2111.06514]

Hertog, OJ, Karlsson [2305.15440], [2408.02652]

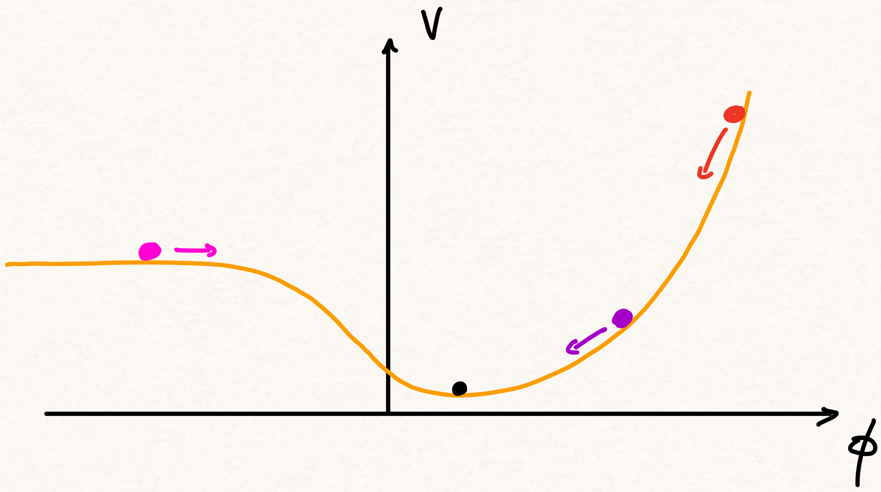
Maldacena [2403.10510]

OJ [2406.08422]

... Hartle, Hawking [1983] ...

1. (Grand) Motivation

• Theory of initial conditions for cosmology •

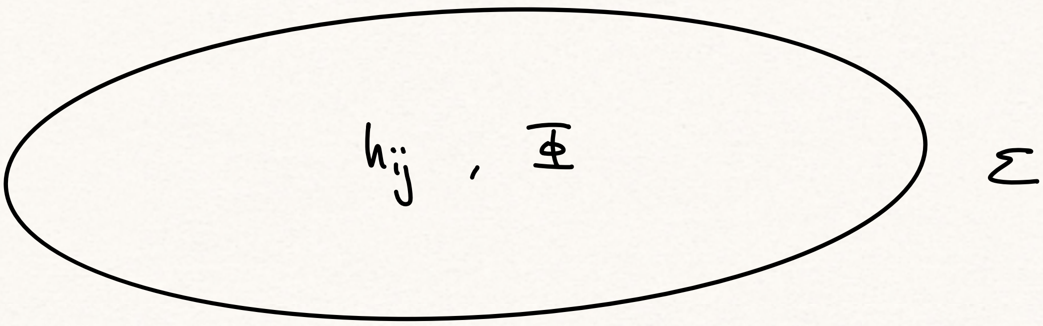


- why was there inflation in the first place?
- how long did it last / where did it start?
- is there a measure on the zoo of pheno models?

  $\rightsquigarrow$ measure on observables $(n_s, r, \ldots)$?

- what is the spatial topology / geometry of the universe?

approach in quantum cosmology : $\Psi_{\text{universe}}[h_{ij}, \Phi]_\Sigma$

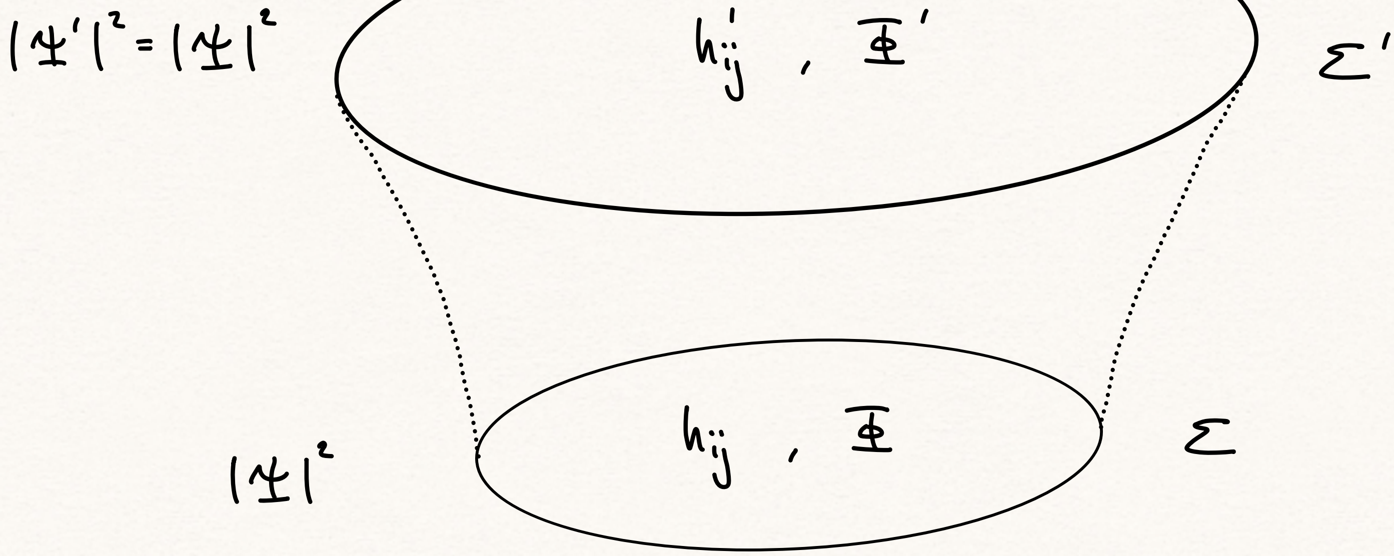


diff. invariance of classical theory

$\rightsquigarrow$ constraints on physical states in quantum theory

one of them : $\hat{H}\,\Psi = 0$ (Wheeler - DeWitt equation)

for large, classical configurations, $|\Psi|^2$ gives probability and is conserved along classical evolution



notice : boundary conditions are imposed at "late times". this is equivalent to initial conditions because probability is conserved

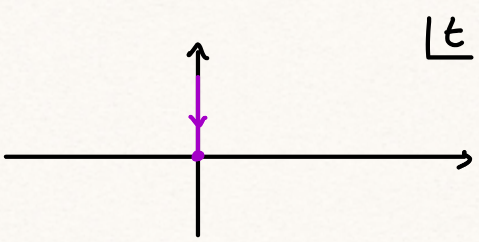
how should we compute Ψ ?

in QFT we have the Euclidean path integral for the ground state :

(spectral decomp. of prop.)

$$\Psi_0[\phi(\vec{x})] = \int D\tilde{\phi}(t,\vec{x})\, e^{iS[\tilde{\phi}]}$$

Euclidean time defined by $t = it_E$, let $t_E : +\infty \longrightarrow 0$



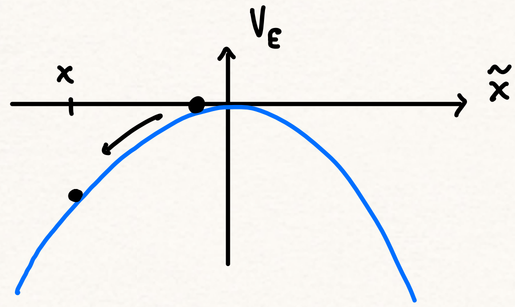
boundary conditions : $\Psi \to 0$ as $t_E \to +\infty$

$\tilde{\phi}(t=0,\vec{x}) = \phi(\vec{x})$

NRQM example : $\mathcal{L} = \frac{1}{2}\dot{\tilde{x}}^2 - \frac{1}{2}\tilde{x}^2$

$$iS[\tilde{x}] = i\int dt\ \frac{1}{2}\dot{\tilde{x}}^2 - \frac{1}{2}\tilde{x}^2$$

$$= \int_{+\infty}^{0} dt_E\ \frac{1}{2}\tilde{x}'^2 + \frac{1}{2}\tilde{x}^2$$



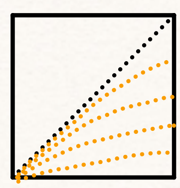
$E = 0$ solution : $\tilde{x}(t_E) = x \cdot e^{-t_E}$

$$\rightsquigarrow \quad \psi_0(x) \sim P \cdot \exp\left(\int dt_E\, x^2 e^{-2t_E}\right)$$

$$= P \cdot e^{-x^2/2}$$

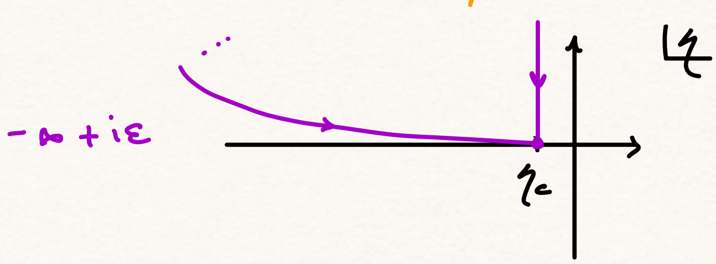
massless scalar field in de Sitter :

metric in flat slicing $ds^2 = \frac{1}{\eta^2}\left(-d\eta^2 + d\vec{x}^2\right)$



scalar field $S = -\frac{1}{2}\int d^4x\, \sqrt{-g}\, (\partial\tilde{\phi})^2$

$$= \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3} \int^{\eta_c} d\eta\ \frac{1}{2\eta^2}\left(\dot{\tilde{\phi}}_{-\vec{k}}\dot{\tilde{\phi}}_{\vec{k}} - \vec{k}^2\tilde{\phi}_{-\vec{k}}\tilde{\phi}_{\vec{k}}\right)$$



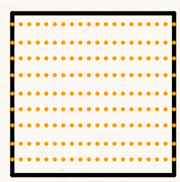
we can calculate the ground state wave functional $\Psi_0[\phi(\vec{x})\,;\,\eta_c]$ by a saddle point solution (here : exact , interacting theory : tree-level approximation) which decays along

$$\tilde{\phi}_{\vec{k}}(\eta) = \phi_{\vec{k}}\, e^{i|\vec{k}|(\eta-\eta_c)}\,\frac{1 - i|\vec{k}|\eta}{1 - i|\vec{k}|\eta_c} \sim \phi_{\vec{k}}\left(1 + \frac{(k\eta)^2}{2} + \frac{i(k\eta)^3}{3} + \dots\right)$$

$$\text{get} \quad \Psi_0 \sim e^{iS} = \exp\left(\int d^3\vec{k}\, \frac{1}{1+(k\eta_c)^2}\left(\frac{ik^2}{2\eta_c} - \frac{k^3}{2}\right)\phi_{-\vec{k}}\,\phi_{\vec{k}}\right)$$

note : solution is complex (in position space!)

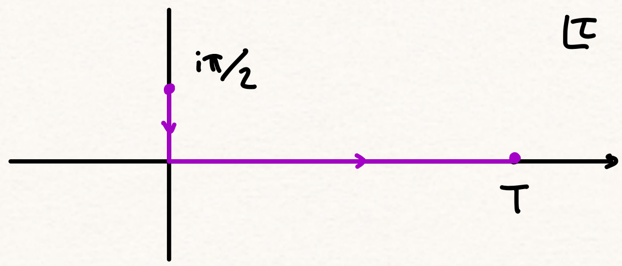
metric in closed slicing $\quad ds^2 = -d\tau^2 + \cosh^2\tau\, d\Omega_3^2$

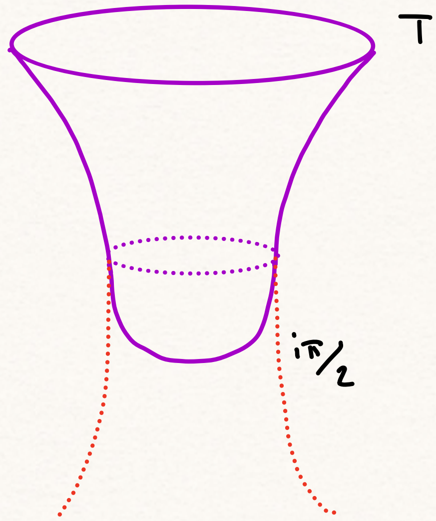


same thing , expand $\tilde{\phi}(\tau, \vec{\Omega}) = \sum_{\ell=0}^{\infty} \tilde{\phi}_\ell(\tau)\, Y^\ell(\vec{\Omega})$

$$\rightsquigarrow \quad S = -\frac{1}{2}\int d^4x\, \sqrt{-g}\, (\partial\tilde{\phi})^2$$

$$= \frac{1}{2}\sum_\ell \int d\tau\, \cosh^3\tau\, \dot{\tilde{\phi}}_\ell^2 - \ell(\ell+2)\cosh\tau\, \tilde{\phi}_\ell^2$$





regular solution: $\tilde{\phi}_\ell(\tau) = \phi_\ell \cdot \dfrac{f_r(\tau)}{f_r(T)}$

$f_r(\tau) =$

$$(1+e^{-2\tau})^\ell \left[{}_2F_1\left(\tfrac{3}{2}+\ell, \ell, -\tfrac{1}{2}; -e^{-2\tau}\right) - \frac{8i}{3}\ell(\ell+1)(\ell+2)e^{-3\tau}\, {}_2F_1\left(\tfrac{3}{2}+\ell, 3+\ell, \tfrac{5}{2}; -e^{-2\tau}\right)\right]$$

at late times $\quad S[\tilde{\phi}] \sim -\frac{1}{2}\sum_\ell \left[\frac{1}{2}\ell(\ell+2)e^{T} - i\,\ell(\ell+1)(\ell+2)\right]\phi_\ell^2$

scalar perturbations in inflation: [Maldacena 2002]

- spatial metric $d\Sigma^2 = a^2 e^{2\zeta(\vec{x})} d\vec{x}^2$, interested in $\Psi_0[\zeta]$
- write $ds^2 = -dt^2 + a(t)^2 e^{2\tilde{\zeta}(t,\vec{x})} d\vec{x}^2$, $\phi(t)$

then $S[\tilde{\zeta}] = \int d^4x\, \varepsilon(t)\left(a^3\dot{\tilde{\zeta}}^2 - a(\nabla\tilde{\zeta})^2\right) + \dots$

here $\varepsilon(t) := -\dfrac{\dot{H}}{H^2}$, $H = \dfrac{\dot{a}}{a}$

- Same procedure : $|\Psi_0[\zeta]|^2 \sim \exp\left(- \int d^3\vec{k} \, \frac{2\epsilon_*}{H_*^2} k^3 \zeta_{-\vec{k}} \zeta_{\vec{k}} \right)$

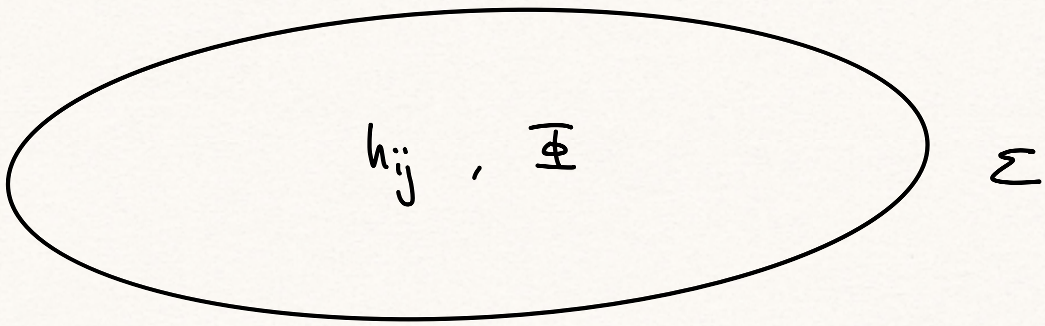
  where $k_{phys} = \frac{k}{a_*} \sim H_*$ (time at which $k_{phys} \gtrsim$ horizon)

  computed by complexifying time as above , agreement with observations

2. The no-boundary state

Recall

$$\Psi_{universe}[h_{ij}, \Phi]_\Sigma$$

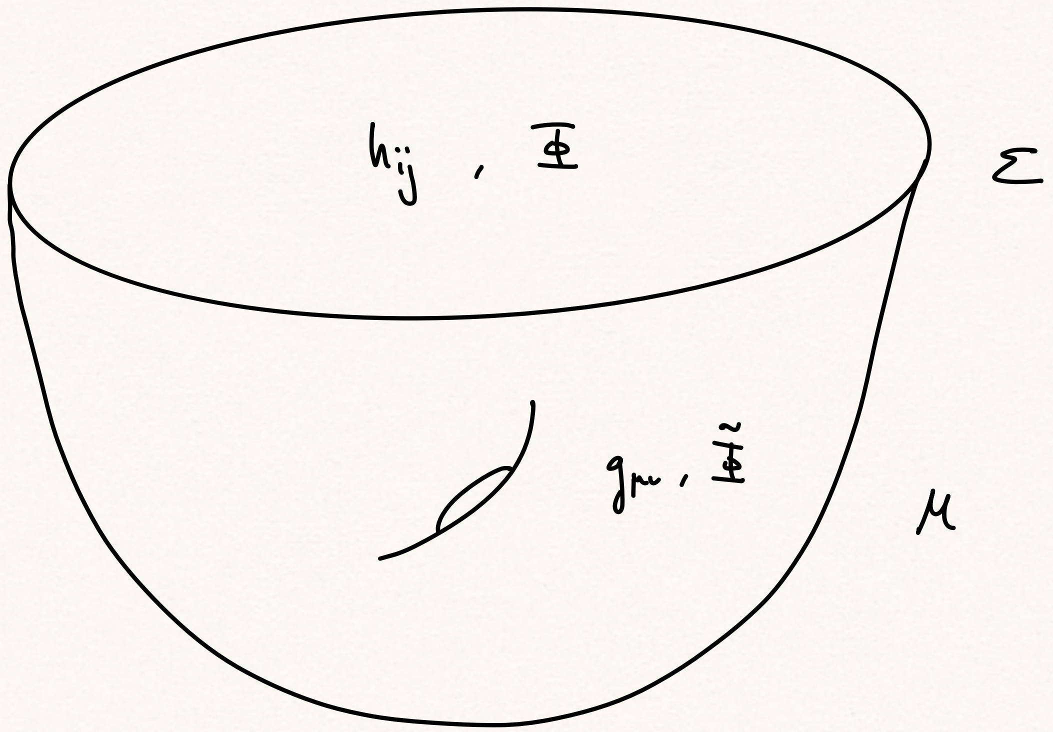


Hartle & Hawking's proposal for closed Σ :

$$\Psi_{HH}[h_{ij}, \Phi]_\Sigma = \int Dg_{\mu\nu} D\tilde{\Phi} \; e^{iS[g, \tilde{\Phi}]}$$

appropriate set of complex configurations on manifolds M with only Σ as boundary .

NO boundary "to the past"



in the semiclassical regime : $\Psi_{HH} \sim \sum_{j \in A} e^{i S_j}$

regular geometries on $\mathcal{M}$ belonging to some class A

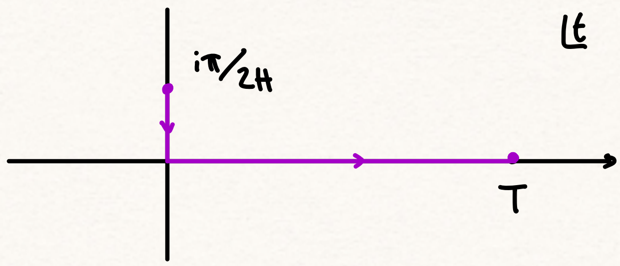
simplest example : pure de Sitter

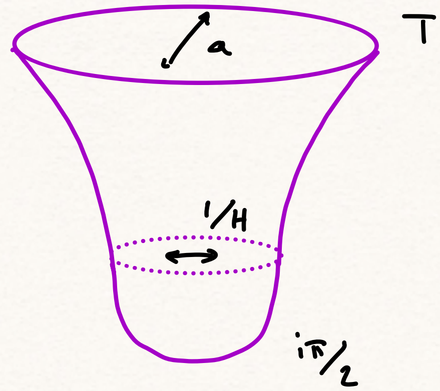
consider $\Psi[h_{ij} = a^2 \Omega_{ij}]$

round S^3

- $S = \int d^4x \sqrt{-g} \left(\frac{R}{2} - \Lambda \right) + \text{(boundary)}$

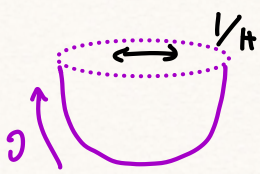
solution $ds^2 = -dt^2 + \tilde{a}(t)^2 d\Omega_3^2$, $\tilde{a}(t) = \frac{1}{H}\cosh(Ht)$, $3H^2 = \Lambda$



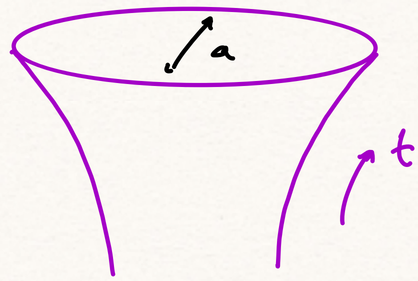


topologically $\mathcal{M} = B^4$, $\partial\mathcal{M} = \Sigma = S^3$

answer : $\Psi_{HH}(a) \sim e^{\frac{4\pi^2}{H^2}} \cdot e^{-\frac{4\pi^2}{H^2} i (aH)^3} + \dots$

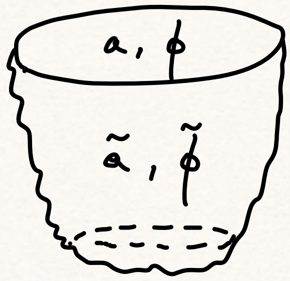


$$ds^2 = \frac{1}{H^2}\left(d\theta^2 + \sin^2\theta \, d\Omega_3^2\right)$$



$$ds^2 = -dt^2 + \frac{1}{H^2}\cosh^2 Ht \, d\Omega_3^2$$

second simplest : inflation

$\Psi_{HH}[a, \phi] \sim \exp\Big(i$  $\Big)$

round S^3 of size a

covered homogeneously with ϕ

assume saddle $(\tilde{a}, \tilde{\phi})$ has maximal symmetry on B^4

Ansatz :

$$ds^2 = -dt^2 + \tilde{a}(t)^2 d\Omega_3^2$$

$$\tilde{\phi} = \tilde{\phi}(t)$$

EoM :

$$\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)^2 = -\frac{1}{\tilde{a}^2} + \frac{1}{3}\left(\frac{\dot{\tilde{\phi}}^2}{2} + V\right)$$

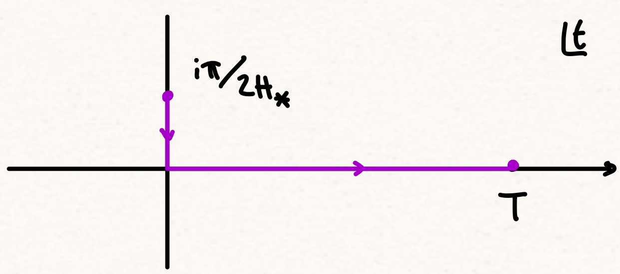
$$\ddot{\tilde{\phi}} + 3\frac{\dot{\tilde{a}}}{\tilde{a}}\dot{\tilde{\phi}} + V' = 0$$

for a while the solution looks like [Maldacena 2403.10510]

$$\tilde{a}(t) = \frac{1}{H_*}\cosh(H_* t)\left(1 + \varepsilon_* \gamma(t) + \dots\right)$$

small

has an imaginary part that decays as $e^{-3H_* t}$

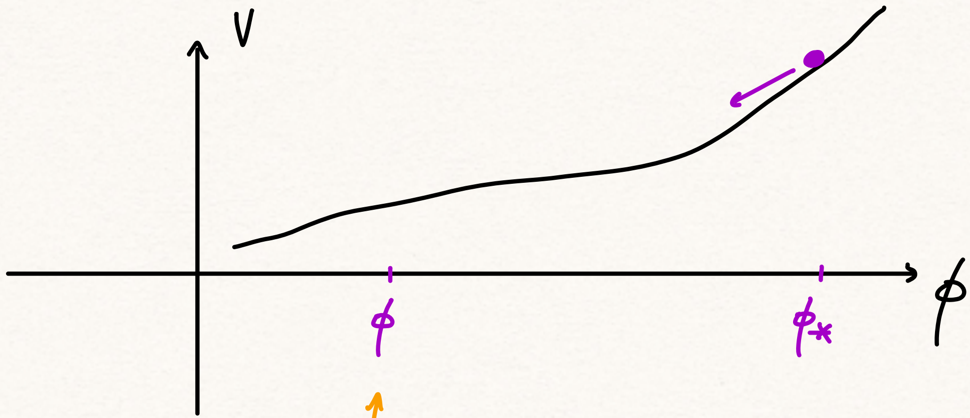


$$\tilde{\phi}(t) = \phi_* + \sqrt{2\varepsilon_*}\,\varphi(t) + \dots \quad , \quad \varphi(t) \sim -H_* t + i\,\mathcal{O}(e^{-3H_* t})$$

when $H_* t = \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon_*}}\right)$ this breaks down [OJ 2406.08422]

change to slow-roll description : $(\mathrm{Re}(\tilde{a}), \mathrm{Re}(\tilde{\phi}))$ follow slow-roll
$(\mathrm{Im}(\tilde{a}), \mathrm{Im}(\tilde{\phi}))$ continue to decay

pictorially :

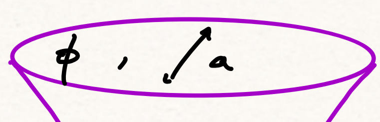


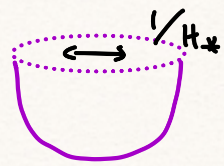
here the universe is very large , $a H(\phi) \sim e^{60}$

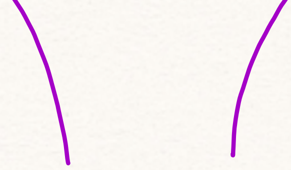
we want to calculate $\Psi_{HH}(a, \phi)$

we backtrack the classical slow-roll solution $(\tilde{a}(t), \tilde{\phi}(t))$ until $\tilde{a} H(\tilde{\phi}) \sim 1$. we call this point $*$, "start of inflation"
$H(\tilde{\phi}(t_*)) = H_*$. the answer is

$$\Psi_{HH}(a, \phi) \sim e^{\frac{4\pi^2}{H_*^2}} \cdot e^{-\frac{4\pi^2}{H^2} i (a H_\phi)^3 + \dots}$$







this gives an alarming result : $|\Psi_{HH}|^2 \sim \exp\left(+\frac{8\pi^2}{H_*^2}\right)$

favouring small H_* $\Longrightarrow$ small amounts of inflation

$\Longrightarrow$ large curvature $\Omega_K \sim \frac{1}{(aH)^2}$!

we won't be able to solve this completely, but we make a step

3. Kontsevich, Segal & Witten

recall : $$\Psi_{HH} = \sum_{j \in A} e^{iS_j}$$

regular geometries on $\mathcal{M}$ belonging to some class A

Witten [2111.06514] proposed what $\mathcal{A}$ should be :

$$\mathcal{A} = \{ \text{allowable complex metrics} \}$$

$$= \{ \text{those } g \text{ on which the Euclidean path integral for free p-form matter converges} \}$$

$$\boxed{\int_{A_p \text{ real}} DA_p \, e^{-S[A_p;g]} < \infty}$$

why only p-forms?
1) free massless bosons exist other than p-forms but do not have gauge-inv. $T_{\mu\nu}$ so cannot be defined on curved spacetime
2) no non-free W compl. theory of massless fields known except p-forms

$\rightsquigarrow$ correlators $\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_m) \rangle = \int D\Phi \, \mathcal{O}(x_1) \cdots \mathcal{O}(x_m) \, e^{-S[\Phi;g]}$

are well-defined on (M,g), $g \in \mathcal{A}$ $\quad (g : T_x \times T_x \longrightarrow \mathbb{C})$

$$S = \frac{1}{2} \int F_{p+1} \wedge * F_{p+1} \quad , \quad F_{p+1} = dA_p$$

$$= \frac{1}{2(p+1)!} \int_M d^D x \sqrt{\det g} \, g^{\alpha_1 \beta_1} \cdots g^{\alpha_{p+1} \beta_{p+1}} F_{\alpha_1 \cdots \alpha_{p+1}} F_{\beta_1 \cdots \beta_{p+1}}$$

$$\rightsquigarrow \operatorname{Re} \left(\sqrt{\det g} \, g^{\alpha_1 \beta_1} \cdots g^{\alpha_{p+1} \beta_{p+1}} F_{\alpha_1 \cdots \alpha_{p+1}} F_{\beta_1 \cdots \beta_{p+1}} \right) > 0$$

$\forall$ real antisymmetric objects $F_{\mu_1 \cdots \mu_{p+1}}(x)$ and all $x \in M$

($\leftrightarrow$ all A, $F = dA$)

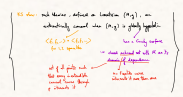
$$\forall p \in \{-1, \ldots, D-1\}$$

Kontsevich & Segal [2105.10161] proved (theorem 2.2)

g allowable $\iff$ $\exists$ real basis such that g is diagonal,

$g_{\mu\nu} = \mathrm{diag}(\lambda_i)$, with

$$\boxed{\sum_{i=0}^{D-1} |\arg(\lambda_i)| < \pi}$$

note: Euclidean metrics lie in the interior of A, 

Lorentzian metrics lie on the boundary

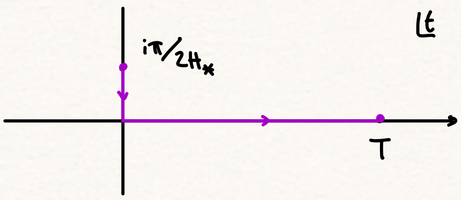
4. Implications for Ψ_{HH}

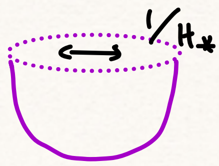
example 1 : inflation

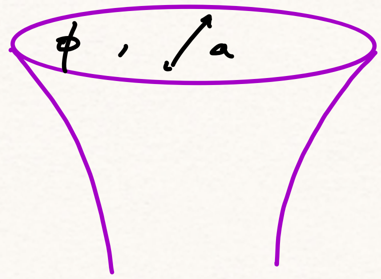
we calculated

$$\Psi_{HH}(a,\phi) \sim e^{iS[\tilde{a},\tilde{\phi}]}$$

$$\sim e^{\frac{4\pi}{H_*^2}} \cdot e^{-\frac{\pi}{H^2} i (a H_\phi)} + \dots$$







here metric is $ds^2 = -dt^2 + \tilde{a}(t)^2 d\Omega_3^2$. is this allowable ?

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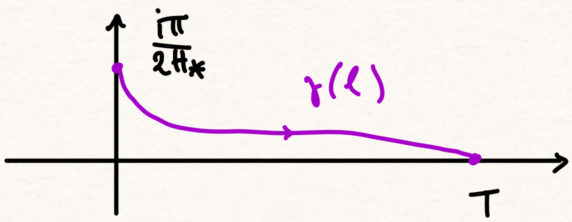
technically : connect "south pole" $\left(t = \frac{i\pi}{2H_*}\right)$ to endpoint $(t = T)$ via a curve $t = \gamma(\ell)$. metric :



$$ds^2 = -\gamma'(\ell)^2 d\ell^2 + a(\gamma(\ell))^2 d\Omega_3^2$$

does there exist a curve $\gamma : \frac{i\pi}{2H_*} \longrightarrow T$ such that

$$\sum_{i=0}^{3} |\arg(\lambda_i)| = |\arg(-\gamma'^2)| + 3|\arg a^2| < \pi \;\; ?$$

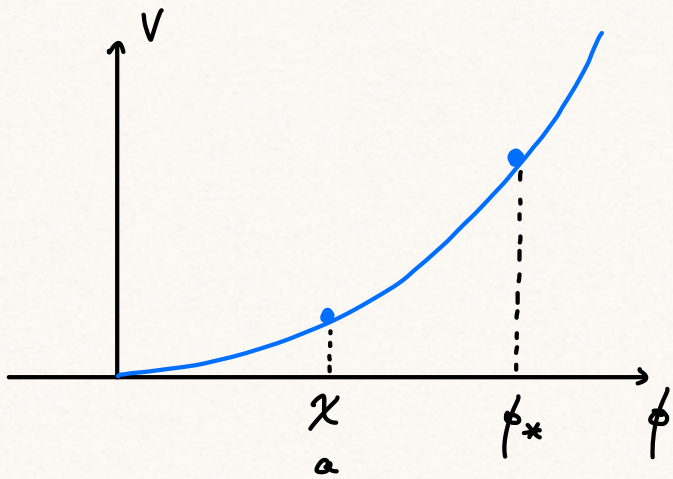
---

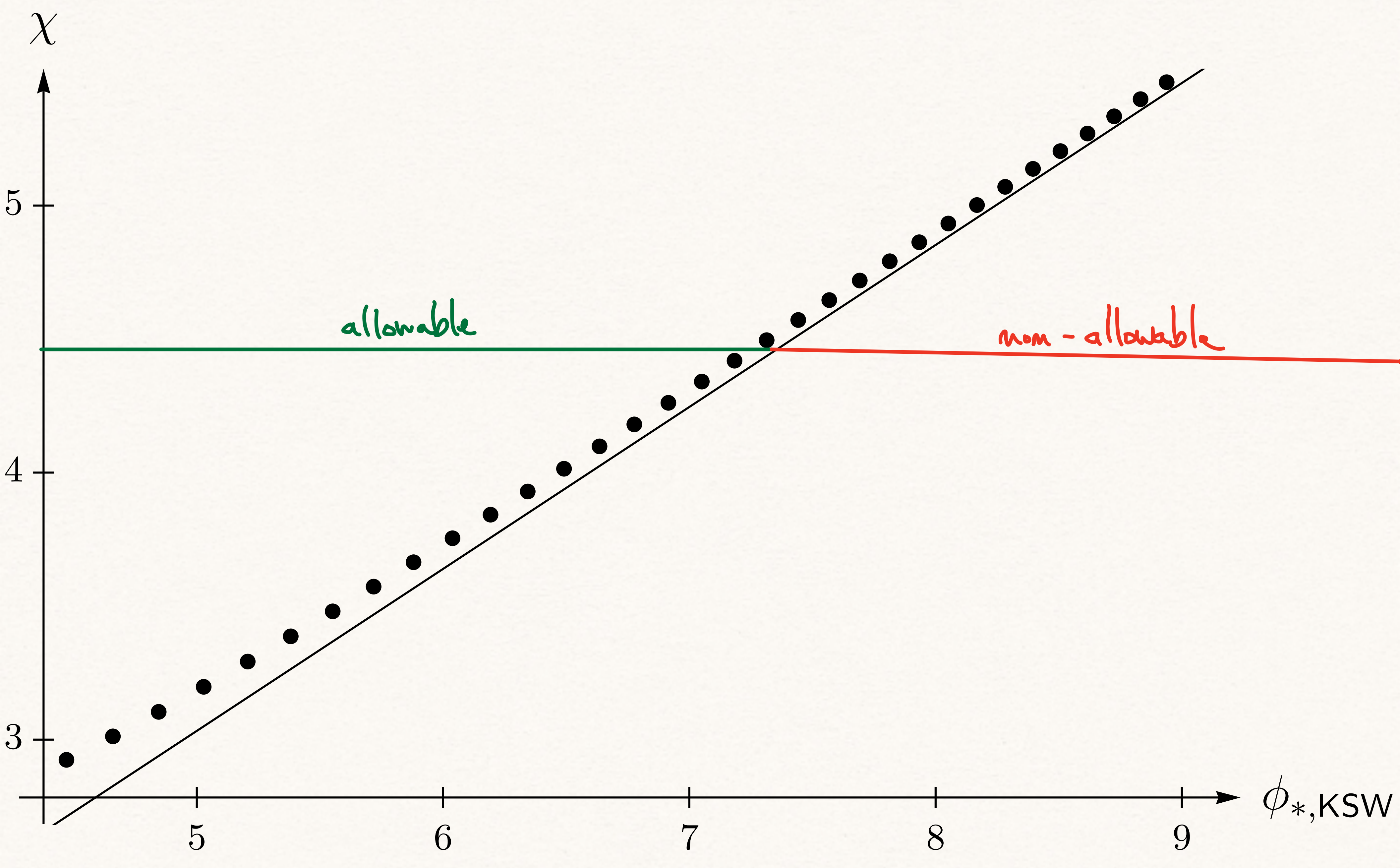
answer [oJ 2406.08422] : allowable $\Longleftrightarrow$

$$A(\phi_*, \chi) = \boxed{\frac{V'_*}{V_*} \int_{\phi_*}^{\chi} |d\phi| \frac{V'_*}{V'(\phi)} < 1}$$

some examples:

① $V(\phi) = \frac{1}{2} m^2 \phi^2$

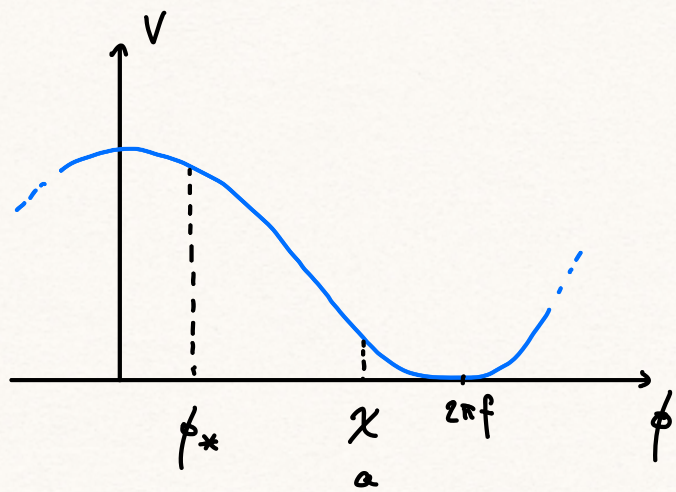


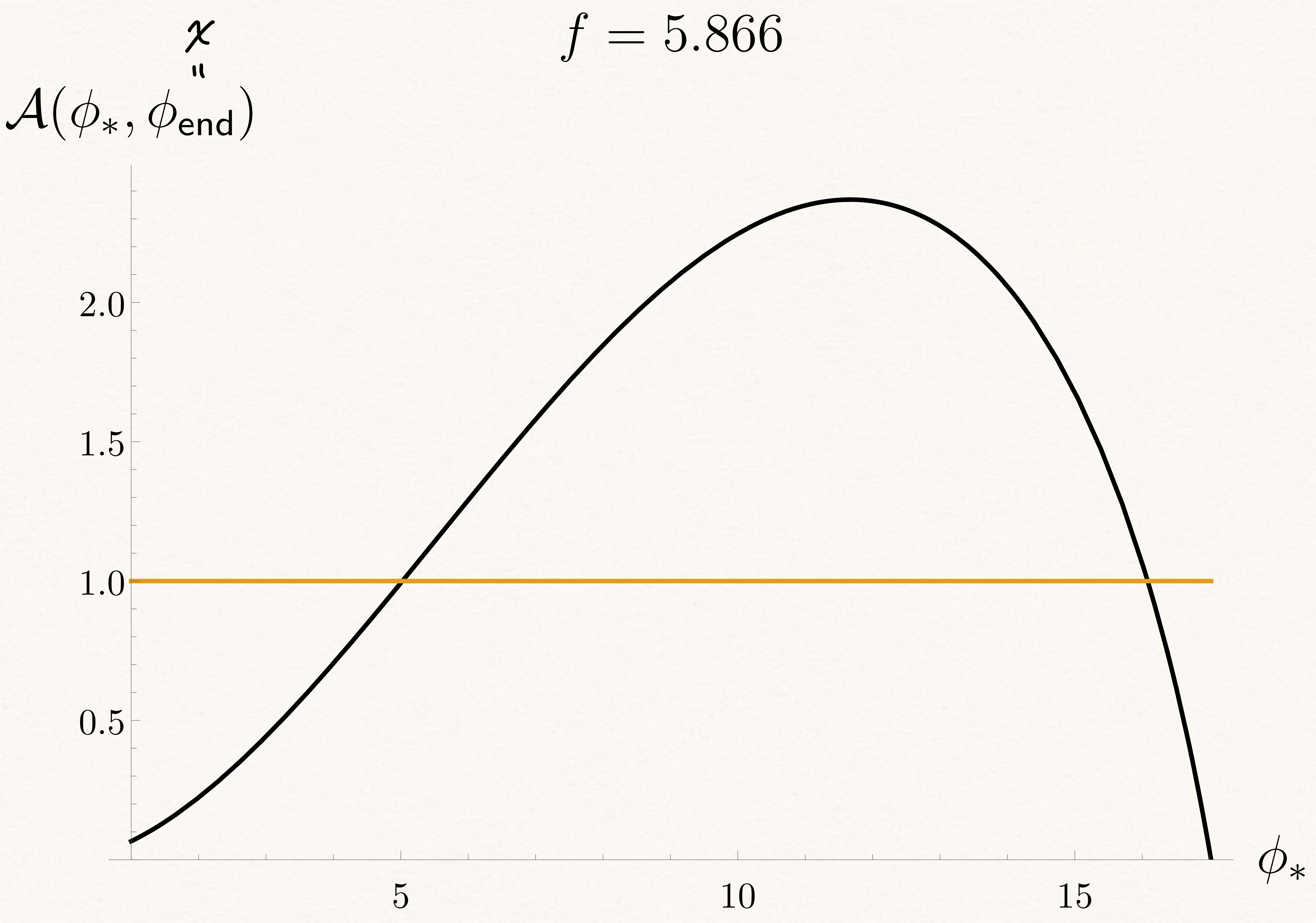


χ
5
4
3
allowable
non-allowable
5
6
7
8
9
$\phi_{*,\mathrm{KSW}}$

conclusion: no slow-roll history that reaches the end of inflation is allowable

② $V(\phi) = 1 + \cos\left(\frac{\phi}{f}\right)$

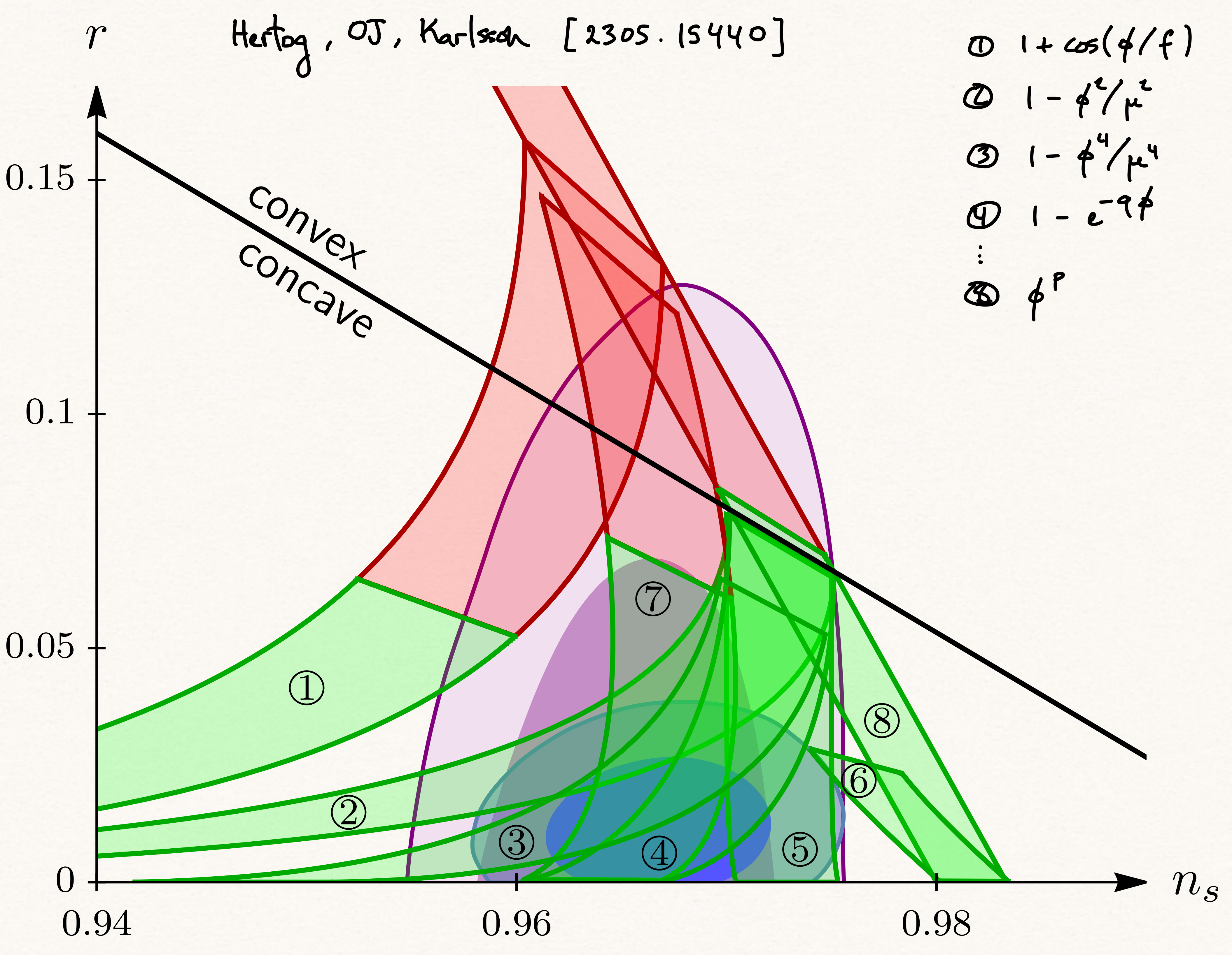




$f = 5.866$
χ
"
$\mathcal{A}(\phi_*, \phi_{\text{end}})$
2.0
1.5
1.0
0.5
5
10
15
ϕ_*

conclusion: if we condition on some inflation having taken place, then a non-trivial minimal amount is predicted by $\Psi_{HH}\big|_{KSW}$

a summary of results for all models discussed in [Planck 1807.06211]

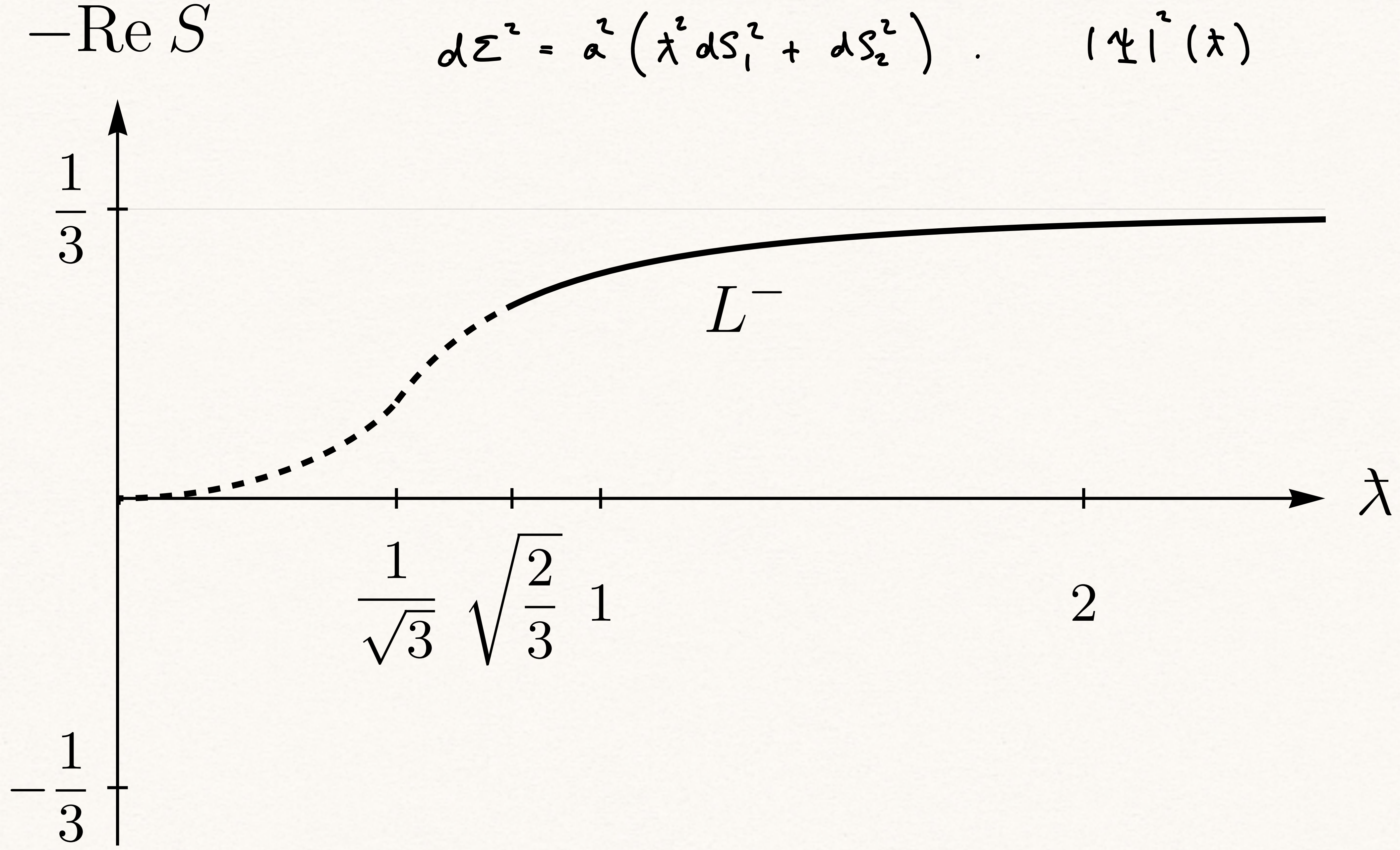


Hertog, OJ, Karlsson [2305.15440]
r
0.15
0.1
0.05
0
0.94
0.96
0.98
n_s
convex
concave
① 1 + cos(φ/f)
② 1 − φ²/μ²
③ 1 − φ⁴/μ⁴
④ 1 − e^{−qφ}
⋮
⑧ φ^p
①
②
③
④
⑤
⑥
⑦
⑧

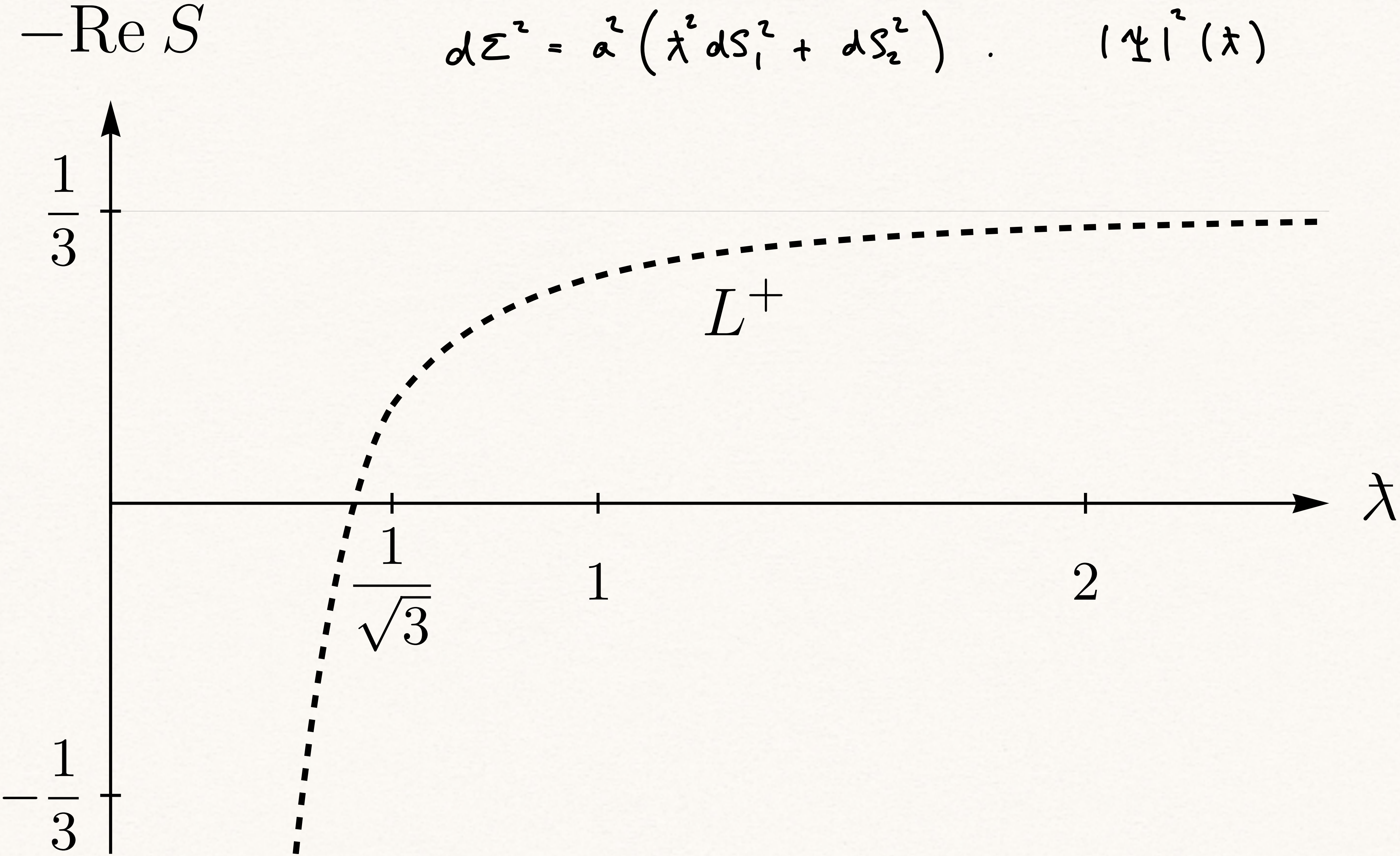
example 2 : anisotropy

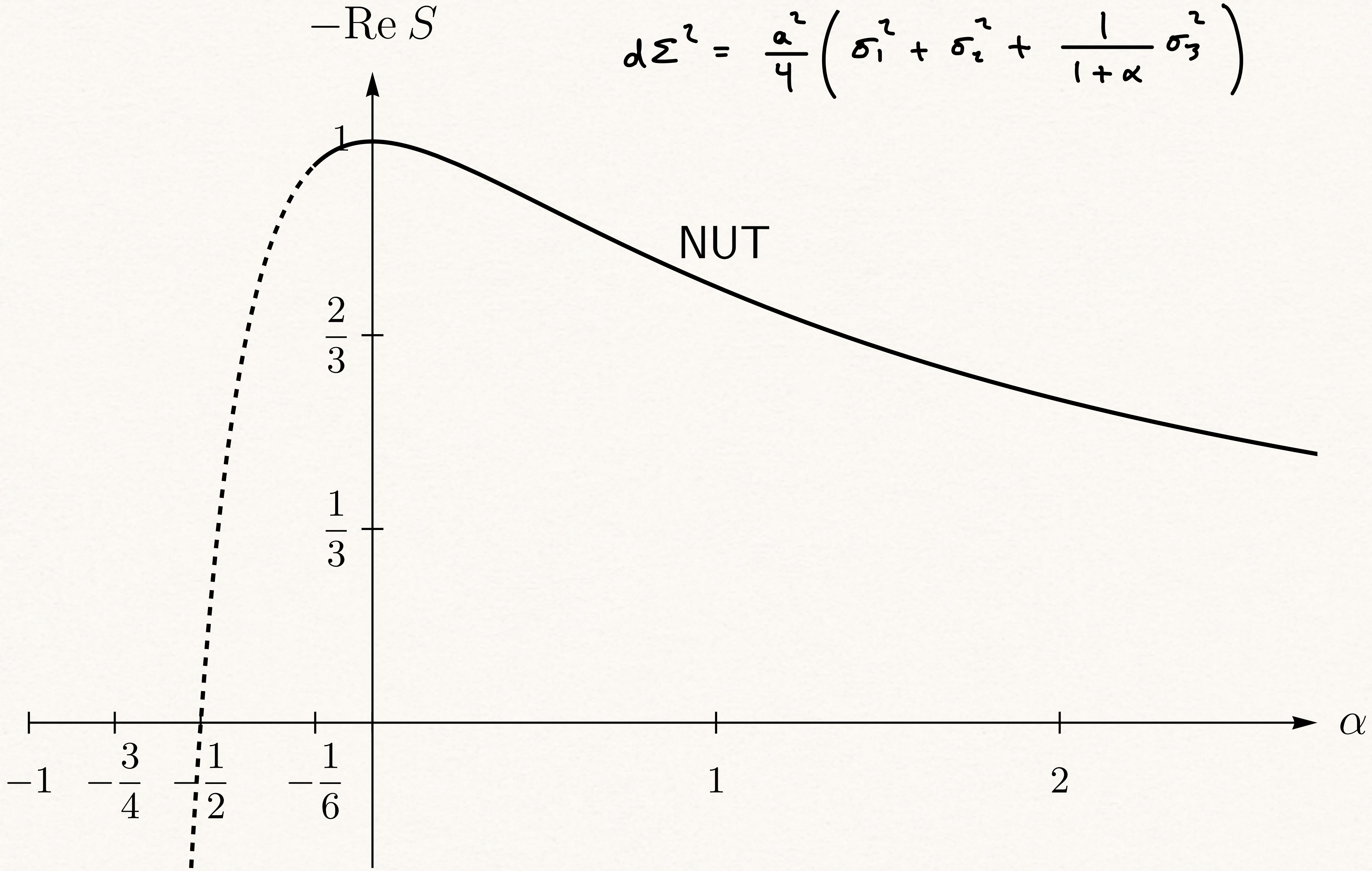
Hertog, OJ, Karlsson [2408.02652]

$d\Sigma^2 = a^2\left(\lambda^2 dS_1^2 + dS_2^2\right)$. $|\Psi|^2(\lambda)$



$d\Sigma^2 = a^2 \left(\lambda^2 dS_1^2 + dS_2^2 \right)$. $|\Psi|^2(\lambda)$





$-\mathrm{Re}\, S$
$d\Sigma^2 = \frac{a^2}{4}\left(\sigma_1^2 + \sigma_2^2 + \frac{1}{1+\alpha}\sigma_3^2\right)$
1
$\frac{2}{3}$
$\frac{1}{3}$
NUT
α
-1
$-\frac{3}{4}$
$-\frac{1}{2}$
$-\frac{1}{6}$
1
2

5. Recap

- We introduced the wave function of the universe, which is supposed to provide a theory of initial conditions for cosmology
- We reviewed the no-boundary proposal, motivated by computation of wave functions in QFT. We highlighted a puzzle: no inflation is predicted
- We reviewed the Kontsevich-Segal-Witten criterion and saw how, when applied to Ψ_{HH}, it leads to an interesting theoretical prior on cosmological observables