Dinamical Casimir Effect

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Scope

- Intro.
- Example.
- The model.
- Zeta function.
- Determinant.

Intro I

Zeta functions are often associated with sequences of real numbers $\lambda_1, \lambda_2, \lambda_3, \ldots$, which, for many applications, are eigenvalues of Laplace-type operators. As a generalization of the Riemann zeta function,

$$\zeta_R(s) = \sum_{k=1}^{\infty} k^{-s},$$

we define

$$\zeta(s) = \sum_{k=1}^{\infty} \lambda_k^{-s},$$

where s is a complex parameter whose real part is assumed to be sufficiently large to make the series convergent.

Intro II

To indicate how the zeta function relates to other spectral functions, we discuss the example of a functional determinant. Consider a sequence of finitely many numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. If we consider them as eigenvalues of the matrix P, we have

$$\det P = \prod_{k=1}^{n} \lambda_k,$$

which implies $\ln \det P = \sum_{k=1}^{n} \ln \lambda_k = -\frac{d}{ds} \sum_{k=1}^{n} \lambda_k^{-s} \Big|_{s=0}$.
It is

$$\ln\ \det P = -\zeta'(0), \rightarrow \det P = e^{-\zeta'(0)}.$$

Intro III

Given that $F(\lambda) = 0$ defines the eigenvalues λ_n, then the logarithmic derivative

$$\frac{d}{d\lambda} \ln F(\lambda) = \frac{F'(\lambda)}{F(\lambda)}$$

has poles at the same eigenvalues. If we expand the logarithmic derivative about $\lambda = \lambda_n$, we obtain for $F'(\lambda_n) \neq 0$ that

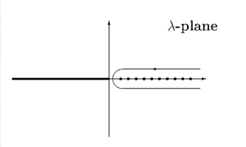
$$\begin{aligned} \frac{F'(\lambda)}{F(\lambda)} &= \frac{F'(\lambda - \lambda_n + \lambda_n)}{F(\lambda - \lambda_n + \lambda_n)} \\ &= \frac{F'(\lambda_n) + (\lambda - \lambda_n) F''(\lambda_n) + \cdots}{(\lambda - \lambda_n) F'(\lambda_n) + (\lambda - \lambda_n)^2 F''(\lambda_n) + \cdots} \\ &= \frac{1}{\lambda - \lambda_n} + \cdots \end{aligned}$$

Intro IV

Cauchy's residue theorem show, given the appropriate behavior of $F(\lambda)$ at infinity, that for $\mathrm{Re}\, s > \frac{1}{2}$,

$$\zeta_P(s) = \frac{1}{2\pi i} \int_\gamma d\lambda \lambda^{-s} \frac{d}{d\lambda} \ln F(\lambda),$$

where the contour γ is shown in the figure



Example I

As an example, we consider the eigenvalue problem

$$-\frac{d^2}{d\tau^2}\phi_n(\tau) = \lambda_n \phi_n(\tau), \quad \phi_n(0) = \phi_n(L) = 0.$$

The eigenfunctions have the form

$$\phi_n(\tau) = a \sin\left(\sqrt{\lambda_n}\tau\right) + b\cos\left(\sqrt{\lambda_n}\tau\right).$$

Example II

The appearance of the cosine is excluded by the boundary value $\phi_n(0) = 0$. The eigenvalues are found from the equation

$$\sin\left(\sqrt{\lambda_n}L\right) = 0.$$

This condition can be solved for analytically, and we find

$$\phi_n(\tau) = A\sin\left(\sqrt{\lambda_n}\tau\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2,$$

Example III

For this example the natural choice for the function F is $F(\lambda) = \sin(\sqrt{\lambda}L)$. This choice has to be modified because $\lambda = 0$ satisfies $F(0) = 0$. To avoid $F(\lambda)$ having more zeros than there are actual eigenvalues we define

$$F(\lambda) = \frac{\sin(\sqrt{\lambda}L)}{\sqrt{\lambda}} = \frac{1}{2i\sqrt{\lambda}}\left(e^{i\sqrt{\lambda}L} - e^{-i\sqrt{\lambda}L}\right).$$

Note that $F(\lambda)$ is an entire function of λ.

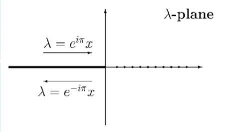
Example IV

The next step in the contour integral formalism is to rewrite the zeta function using Cauchy's integral formula. Given that $F(\lambda) = 0$ defines the eigenvalues λ_n, then the logarithmic derivative

$$\frac{d}{d\lambda} \ln F(\lambda) = \frac{F'(\lambda)}{F(\lambda)}.$$

We next want to shrink the contour to the negative real axis. As λ approaches the negative real axis from above, λ^{-s} picks up the phase $\left(e^{i\pi}\right)^{-s} = e^{-i\pi s}$; the limit from below produces $\left(e^{-i\pi}\right)^{-s} = e^{i\pi s}$.

Example V



Along the contour,the zeta function is given by

$$\zeta_P(s) = \frac{\sin \pi s}{\pi} \int_0^\infty dx x^{-s} \frac{d}{dx} \ln \left(\frac{e^{\sqrt{x}L}}{2\sqrt{x}} \left[1 - e^{-2\sqrt{x}L} \right] \right).$$

Example VI

In order to compute the integral, we need

$$
\begin{aligned}
\zeta_P'(0) &= \left(\frac{d}{ds}\bigg|_{s=0} \frac{\sin \pi s}{\pi} \right) \left(\int_0^\infty dx x^{-s} \right. \\
&\quad \left. \times \frac{d}{dx} \ln \left(\frac{e^{\sqrt{x}L}}{2\sqrt{x}} \left[1 - e^{-2\sqrt{x}L} \right] \right) \right) \Big|_{s=0} + \left(\frac{\sin \pi s}{\pi} \right) \bigg|_{s=0} \\
&\quad \times \left(\frac{d}{ds}\bigg|_{s=0}^{\infty} \int_0^\infty dx x^{-s} \frac{d}{dx} \ln \left(\frac{e^{\sqrt{x}L}}{2\sqrt{x}} \left[1 - e^{-2\sqrt{x}L} \right] \right) \right) \\
&= \int_0^\infty dx \frac{d}{dx} \ln \left(\frac{e^{\sqrt{x}L}}{2\sqrt{x}} \left[1 - e^{-2\sqrt{x}L} \right] \right).
\end{aligned}
$$

Example VII

To analyze the equation, further we split the integral as $\int_0^1 dx + \int_1^\infty dx$. From our previous remarks it follows that $\int_0^1 dx$ can be considered to be in final form, but $\int_1^\infty dx$ needs further manipulation. The pieces needing extra attention are

$$\int_1^\infty dx x^{-s} \frac{d}{dx} \ln e^{\sqrt{x}L} = \frac{L}{2} \int_1^\infty dx x^{-s-1/2} = \frac{L}{2s-1},$$

$$\int_1^\infty dx x^{-s} \frac{d}{dx} \ln \left(\frac{1}{2\sqrt{x}} \right) = -\frac{1}{2} \int_1^\infty dx x^{-s-1} = -\frac{1}{2s}.$$

Example VIII

We end with

$$\zeta_P(s) = \frac{L\sin\pi s}{(2s-1)\pi} - \frac{\sin\pi s}{2s\pi} + \frac{\sin\pi s}{\pi}\int_1^\infty dx x^{-s}\frac{d}{dx}\ln\left(1-e^{-2\sqrt{x}L}\right) + \frac{\sin\pi s}{\pi}\int_0^1 dx x^{-s}\frac{d}{dx}\ln\left(\frac{e^{\sqrt{x}L}}{2\sqrt{x}}\left[1-e^{-2\sqrt{x}L}\right]\right),$$

a form perfectly suited for the evaluation of $\zeta_P'(0)$. We find

$$\zeta_P'(0) = -L - 0 - \ln\left(1-e^{-2L}\right) + \ln\left(\frac{e^L}{2}\left[1-e^{-2L}\right]\right) - \ln L = -\ln(2L).$$

Example IX

It agrees with the answer found from the well known values $\zeta_R(0) = -\frac{1}{2}, \zeta'_R(0) = -\frac{1}{2}\ln(2\pi)$:

$$\zeta_P(s) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^{-2s} = \left(\frac{L}{\pi}\right)^{2s} \zeta_R(2s),$$

which implies that

$$\begin{aligned} \zeta'_P(0) &= 2\ln\left(\frac{L}{\pi}\right)\zeta_R(0) + 2\zeta'_R(0) \\ &= -\ln\left(\frac{L}{\pi}\right) - \ln(2\pi) \\ &= -\ln(2L). \end{aligned}$$

The model I

$$\mathcal{H} := L^2((0,l)) \oplus \mathbb{C},$$
$$\varphi(z) = \begin{pmatrix} \varphi_1(z) \\ \varphi_2 \end{pmatrix}$$

$$\mathcal{D}(A) := \{\varphi(z) \in \mathcal{H} : \varphi_1(z), \varphi_1'(z) \in \mathcal{AC}[0,l], \varphi_1''(z) \in L^2((0,l)),$$
$$\cos\alpha\varphi_1(0) + \theta\sin\alpha\varphi_1'(0) = 0, \varphi_2 = \beta_1'\varphi_1(l) - \beta_2'\varphi_1'(l)\}$$

$$A\varphi(z) := \begin{pmatrix} \left[-\partial_z{}^2 + m^2 + V(z)\right]\varphi_1(z) \\ -\left[\beta_1\varphi_1(l) - \beta_2\varphi_1'(l)\right] \end{pmatrix}.$$

The model II

where $V(z)$ is a bounded function, and the dynamical equation of the coupled system reads as

$$\left(\partial_t{}^2 + A\right)\varphi(z) = 0,$$

which implies that the field satisfy the differential equation

$$\left[\partial_t{}^2 - \partial_z{}^2 + m^2 + V(z)\right]\varphi(z) = 0,$$

with dynamical boundary conditions

$$\cos\alpha\varphi_1(t,0) + \theta\sin\alpha\partial_z\varphi_1(t,0) = 0,$$
$$\partial_t{}^2\left[\beta_1'\varphi_1(t,l) - \beta_2'\partial_z\varphi_1(t,l)\right] = \left[\beta_1\varphi_1(t,l) - \beta_2\partial_z\varphi_1(t,l)\right].$$

The model III

For the stationary states $\varphi(t,z) = e^{-i\omega t}\varphi(z), \varphi \in \mathcal{D}(A)$, we get

$$A\varphi(z) = \omega^2 \varphi(z)$$
$$\cos\alpha\varphi_1(0) + \theta\sin\alpha\partial_z\varphi_1(0) = 0,$$
$$\left(\beta_2 + \omega^2\beta_2'\right)\varphi_1'(l) = \left(\beta_1 + \omega^2\beta_1'\right)\varphi_1(l),$$

A so defined is self-adjoint if $\rho = \beta_1'\beta_2 - \beta_1\beta_2' > 0$, and that A is positive if $m^2 + V(z) > 0$ for $z \in [0,l], \alpha = 0$ or $\alpha \in \left[\frac{\pi}{2}, \pi\right)$, and

$$\beta_1 \geq 0, \quad \beta_1', \beta_2 < 0, \quad \text{for} \quad \beta_2' > 0,$$
$$\beta_1 \leq 0, \quad \beta_1', \beta_2 > 0, \quad \text{for} \quad \beta_2' < 0,$$

The model IV

For simplicity, in the following we take $\alpha = 0$ so as to require that $\varphi_1(0) = 0$. Also we take $V(z) \equiv 0$.
The general solution of

$$\left(-\partial_z{}^2 + m^2\right)\varphi_1(z) = \omega^2\varphi_1(z), \text{ with } \quad \varphi_1(0) = 0,$$

given by

$$\varphi_1(z) \sim \sin\left(z\sqrt{\omega^2 - m^2}\right).$$

The model V

where $\omega^2 > 0$ since we are considering a positive definite operator. Then, the boundary condition at $z = l$

$$(\beta_2 + \omega^2 \beta_2') \sqrt{\omega^2 - m^2} \cos\left(l\sqrt{\omega^2 - m^2}\right) = (\beta_1 + \omega^2 \beta_1') \sin\left(l\sqrt{\omega^2 - m^2}\right).$$

For $\omega^2 < m^2$, the boundary condition at $z = l$ gives

$$(\beta_2 + \omega^2 \beta_2') \, i\sqrt{m^2 - \omega^2} \cosh\left(l\sqrt{m^2 - \omega^2}\right) = (\beta_1 + \omega^2 \beta_1') \, i \sinh\left(l\sqrt{m^2 - \omega^2}\right),$$

or

$$\frac{\tanh\left(l\sqrt{m^2 - \omega^2}\right)}{l\sqrt{m^2 - \omega^2}} = \frac{(\beta_2 + \omega^2 \beta_2')}{l\,(\beta_1 + \omega^2 \beta_1')},$$

The model VI

Which has no solutions since the left hand side is positive for $0 < \omega^2 < m^2$, while the right hand side is a decreasing function of ω^2 that takes the value $\frac{\beta_2}{l\beta_1} < 0$ for $\omega^2 = 0$. Indeed,

$$\frac{\partial}{\partial\omega^2}\left(\frac{(\beta_2 + \omega^2\beta_2')}{l\left(\beta_1 + \omega^2\beta_1'\right)}\right) = \frac{-\rho}{l\left(\beta_1 + \omega^2\beta_1'\right)^2} < 0.$$

For $\omega^2 \geq m^2$, defining $x := l\sqrt{\omega^2 - m^2}$, the spectrum is given by the zeroes of

$$f(x) := x\left(a + bx^2\right)\cos x - \left(c + dx^2\right)\sin x$$

where we have defined the dimensionless parameters

$$a = l\left(\beta_2 + m^2\beta_2'\right), b = \beta_2'/l, c = l^2\left(\beta_1 + m^2\beta_1'\right) \text{ and } d = \beta_1',$$

with $l\rho = (ad - bc) > 0$.

Zeta function I

The analytic extension of the associated ζ-function,

$$\zeta_A(s) := \sum_{n=1}^{\infty} \left(\frac{\omega_n}{\mu} \right)^{-2s},$$

where μ is an arbitrarily chosen mass scale. In terms of $x_n = l\sqrt{\omega_n^2 - m^2}$,

$$\zeta_A(s) = (\mu l)^{2s} \sum_{n=1}^{\infty} \left(l^2 \omega_n^2 \right)^{-s} = (\mu l)^{2s} \sum_{n=1}^{\infty} \left({x_n}^2 + M^2 \right)^{-s} .-$$

Zeta function II

The knowledge of $\zeta_A(s)$ allows for the evaluation of several relevant magnitudes. The determinant of the operator A is defined in this context by

$$\log \mathrm{Det}(A) := -\zeta_A{}'(0) = \log\left(\frac{\mu l}{\pi}\right) + \\ + \frac{l^2 m^2}{6} - \frac{d}{3b} - \frac{\pi^2}{24} - \gamma + \log(2\pi) - \triangle\zeta'(0).$$

The vacuum energy of the quantum system, " $\sum_n \frac{\hbar\omega_n}{2}$ ", is a formally divergent quantity which requires a precise definition through regularization. In the present context, the Casimir energy is defined in terms of the analytic continuation

$$E^{(0)}_{\mathrm{Cas}}(l) := \frac{\hbar\mu}{2}\sum_n \left(\frac{\omega_n}{\mu}\right)^{-2s}\Bigg|_{s\to-\frac{1}{2}} = \frac{\hbar\mu}{2}\zeta_A(s)\Big|_{s\to-\frac{1}{2}}$$

Zeta function III

Notice that $f(z)$ in is an odd function which has no nonreal zeroes, since A is a positive self-adjoint operator. Moreover, the eigenvalues ${\omega_n}^2 > m^2 > 0$ (i.e. $x_n > 0$).
Then, employing the Cauchy's residue theorem we can write, for real s and an integration path encircling clockwise all the $x_n, n \in \mathbb{N}$,

$$(\mu l)^{-2s}\zeta_A(s) = -\frac{1}{2\pi i}\oint_{-i\infty}^{i\infty} dz\left(z^2+M^2\right)^{-s}\frac{d}{dz}\log f(z) =$$

$$= -\frac{M^{-2s}}{2\pi i}\oint_{-i\infty}^{i\infty} dz\left(z^2+1\right)^{-s}\frac{d}{dz}\log f(Mz).$$

Zeta function IV

So,

$$\left(\frac{m}{\mu}\right)^{2s} \zeta_A(s) = \frac{1}{\pi}\Im\left\{ e^{i\pi s}\int_1^\infty dy\,\left(y^2-1\right)^{-s}\frac{d}{dy}\log f(iMy) + \right.$$
$$\left. + \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 dy\,\left(1-y^2\right)^{-s}\frac{d}{dy}\log f(iMy)\right\} +$$
$$-\frac{1}{2\pi i}\lim_{\varepsilon\to 0^+}\int_{-i}^{i}\varepsilon dz\,\left(\varepsilon^2 z^2+1\right)^{-s}\left[\frac{1}{\varepsilon z}+O(\varepsilon)\right],$$

Zeta function V

- Where the last integral is evaluated on the half-circle $|z| = 1, \Re z \geq 0$ and we can write

$$\log(2if(iMy)) = My + \log(P(y)) + \log\left(1 - e^{-2My}\frac{P(-y)}{P(y)}\right),$$

with

$$P(y) := c - aMy - dM^2y^2 + bM^3y^3 =$$
$$= l^2\left\{\left(\beta_1 + m^2\beta_1'\right) - \left(\beta_2 + m^2\beta_2'\right)my - \beta_1'm^2y^2 + \beta_2'm^3y^3\right\}.$$

Zeta function VI

We remark that $P(y)$ is independent of l and the rational function

$$Q(y) := \frac{P(-y)}{P(y)} \asymp \begin{cases} -1 - \frac{2\beta_1'}{\beta_2' m y} + O\left(y^{-2}\right), & \text{if } \beta_2' \neq 0, \\ 1 - \frac{2\beta_2}{\beta_1' m y} + O\left(y^{-2}\right), & \text{if } \quad \beta_2' = 0, \beta_1' \neq 0. \end{cases}$$

So, we have

$$\frac{d}{dy} \log f(iMy) = M + \frac{P'(y)}{P(y)} + \frac{P(y)P'(-y) + \left[P'(y) + 2MP(y)\right]P(-y)}{P(y)\left[e^{2My}P(y) - P(-y)\right]}$$

where the last term is $O\left(e^{-2My}\right)$ and we have the asymptotic behaviors

$$\begin{cases} = M + \frac{3}{y} + \frac{\beta_1'}{m\beta_2' y^2} + O\left(y^{-3}\right) + O\left(e^{-2My}\right), & \text{for} \quad y \gg 1, \beta_2' \neq 0, \\ = M + \frac{2}{y} - \frac{\beta_2}{m\beta_1' y^2} + O\left(y^{-3}\right) + O\left(e^{-2My}\right), & \text{for} \quad y \gg 1, \beta_2' = 0, \\ = \frac{1}{y} + \frac{3a-6b-c+6d}{3(a-c)} M^2 y + O\left(y^2\right), & \text{for} \quad y \ll 1. \end{cases}$$

Zeta function VII

Therefore, for $\frac{1}{2} < s < 1$, we can write

$$\left(\frac{m}{\mu}\right)^{2s} \zeta_A(s) = I_1(s) + I_2(s) + F(s),$$

where

$$I_1(s) := \frac{M}{\pi}\sin(\pi s)\int_1^\infty dy\,\left(y^2-1\right)^{-s} = \frac{M}{2\pi^{3/2}}\sin(\pi s)\Gamma(1-s)\Gamma(s-1/2),$$

$$I_2(s) := \frac{1}{\pi}\Im\left\{e^{i\pi s}\int_1^\infty dy\,\left(y^2-1\right)^{-s}\frac{d}{dy}\log P(y)\right\} =$$

$$= \frac{1}{\pi}\Im\left\{e^{i\pi s}\int_1^\infty dy\,\left(y^2-1\right)^{-s}\sum_{k=1}^{3}\frac{1}{y-z_k}\right\}$$

with $z_k, k = 1, 2, 3$ the zeroes of the cubic polynomial $P(y)$.

Zeta function VIII

and

$$F(s) := -\frac{1}{2} + \frac{\sin(\pi s)}{\pi} \int_1^\infty dy \left(y^2 - 1\right)^{-s} \frac{d}{dy} \log\left(1 - e^{-2My} \frac{P(-y)}{P(y)}\right),$$

The zeroes of $P(y)$ obey

$$z_1 + z_2 + z_3 = \frac{d}{bM} = \frac{\beta_1'}{m\beta_2'},$$

$$z_1 z_2 + z_1 z_3 + z_2 z_3 = \frac{-a}{bM^2} = -\frac{\beta_2 + m^2 \beta_2'}{m^2 \beta_2'},$$

$$z_1 z_2 z_3 = \frac{-c}{bM^3} = -\frac{\beta_1 + m^2 \beta_1'}{m^3 \beta_2'}.$$

Zeta function IX

Now notice that, for $s < 1$ and due to the factor $\Gamma(s-1/2)$ in the expression of $I_1(s)$, the analytic extension of its contribution to $\zeta_A(s)$ presents simple poles at $s=\frac{1}{2}-n$, for $n=0,1,2,\cdots$, with residue

$$\text{Res}\left[\left(\frac{m}{\mu}\right)^{-2s} I_1(s)\right]_{s=\frac{1}{2}-n} = \frac{l\mu}{2\pi^{3/2}n!}\left(\frac{m}{\mu}\right)^{2n}\Gamma\left(n+\frac{1}{2}\right).$$

On the other hand, since the integrand in $I_2(s)$ is free of singularities for $y \in [1,\infty)$, we can assume that $\Re z_k < 1$ or $\Im z_k \neq 0$. Taking into account that, for $u > 0$, $-\pi < \alpha < \pi$ and $\frac{1}{2} < s < 1$, we have

$$\int_1^\infty dy \frac{\left(y^2-1\right)^{-s}}{y+e^{i\alpha}u} = \frac{\pi}{\sin(2\pi s)}\left[\left(e^{i\alpha}u\right)^2-1\right]^{-s} +$$
$$\frac{1}{2\sqrt{\pi}e^{i\alpha}u}\Gamma(1-s)\Gamma\left(s-\frac{1}{2}\right){}_2F_1\left(\frac{1}{2},1;\frac{3}{2}-s;\frac{1}{\left(e^{i\alpha}u\right)^2}\right)$$

Zeta function X

Hypergeometric function ${}_2F_1(1/2, 1; 3/2 - s; x)$ is analytic in s. The meromorphic extension of the contribution of $I_2(s)$ to $\zeta_A(s)$ on the open half-plane $\Re s < \frac{1}{2}$ presents simple poles at $s = \frac{1}{2} - n$, with $n = 0, 1, 2, \cdots$, with residues

$$\text{Res}\left[\left(\frac{m}{\mu}\right)^{-2s} I_2(s)\right]_{s=\frac{1}{2}-n} = -\frac{1}{\pi}\left(\frac{m}{\mu}\right)^{2n-1} \times$$

$$\sum_{k=1}^{3}\left\{\frac{(-1)^n}{2}\left({z_k}^2 - 1\right)^{n-\frac{1}{2}} + \frac{\Gamma\left(n+\frac{1}{2}\right)}{2\sqrt{\pi}n!z_k}\,{}_2F_1\left(\frac{1}{2}, 1; n+1; \frac{1}{{z_k}^2}\right)\right\}.$$

Zeta function XI

However, since the integrand in the definition of $I_2(s)$

$$(y^2-1)^{-s}\frac{d}{dy}\log P(y) = (y^2-1)^{-s}\times\begin{cases}\frac{3}{y}+\frac{\beta_1'}{\beta_2' m y^2}+O\left(y^{-3}\right), & \beta_2'\neq 0,\\ \frac{2}{y}-\frac{\beta_2}{\beta_1' m y^2}+O\left(y^{-3}\right), & \beta_2'=0,\end{cases}$$

$$\operatorname{Res}\left[\left(\frac{m}{\mu}\right)^{-2s} I_2(s)\right]_{s=-\frac{1}{2}} = \begin{cases}-\frac{\beta_1'}{2\pi\mu\beta_2'}, & \beta_2'\neq 0,\\ \frac{\beta_2}{2\pi\mu\beta_1'}, & \beta_2'=0,\end{cases} \tag{1}$$

Determinant I

$\zeta_A(s)$ is analytic in a neighborhood of the origin, we get for $s \approx 0$

$$\zeta_A(s) = -Ms + \left[\frac{3}{2} - \left(3\log\left(\frac{m}{\mu}\right) + \sum_{k=1}^{3}\log\left(z_k - 1\right)\right)s\right] +$$

$$+ \left[1 - 2s\log\left(\frac{m}{\mu}\right)\right]\left\{-\frac{1}{2} + sF'(0)\right\} + O\left(s^2\right)$$

and, from the usual definition of the functional determinant,

$$\log \mathrm{Det}\left(A/\mu^2\right) := -{\zeta_A}'(0) =$$

$$ml + 2\log\left(\frac{m}{\mu}\right) + \sum_{k=1}^{3}\log\left(z_k - 1\right) - F'(0),$$

where

$$\sum_{k=1}^{3}\log\left(z_k - 1\right) = \log\left\{\frac{-P(1)}{\beta_2' l^2 m^3}\right\}$$

Determinant II

and

$$F'(0) = -\log\left[1 - e^{-2ml}\frac{P(-1)}{P(1)}\right].$$

Notice that $\zeta_A(0) = 1$ and $\mathrm{Det}\,(A/\mu^2)$ do depend on the external scale μ.

Casimir energy I

$\zeta_A(s)$ has a simple pole at $s = -1/2$.

$$\zeta_A(s) = \left\{ \frac{lm^2}{4\pi\mu} - \frac{\beta_1'}{2\pi\mu\beta_2'} \right\} \frac{1}{\left(s + \frac{1}{2}\right)} + O\left((s + 1/2)^0\right).$$

Around $s = -1/2$ we have

$$\left(\frac{m}{\mu}\right)^{-2s} I_1(s) = \frac{Mm}{4\pi\mu} \left\{ \frac{1}{\left(s+\frac{1}{2}\right)} - \left[2\log\left(\frac{m}{2\mu}\right) + 1\right] + O\left(s + \frac{1}{2}\right) \right\}$$

Casimir energy II

And

$$\left(\frac{m}{\mu}\right)^{-2s} I_2(s) = \frac{3m}{2\mu} - \frac{dm}{2\pi b\mu M}\left\{\frac{1}{s+\frac{1}{2}} - 2\left[\log\left(\frac{m}{2\mu}\right)+1\right]\right\} +$$
$$-\frac{m}{\pi\mu}\int_1^\infty dy\sqrt{y^2-1}\left\{\frac{P'(y)}{P(y)} - \frac{3}{y} - \frac{d}{bMy^2}\right\} + O\left(s+\frac{1}{2}\right)$$

where the integral in the last line can also be written interms of the zeroes of $P(y)$ as

$$-\frac{m}{\pi\mu}\lim_{\Lambda\to\infty}\int_1^\Lambda dy\sqrt{y^2-1}\left\{\sum_{k=1}^{3}\frac{1}{y-z_k} - \frac{3}{y} - \frac{d}{bMy^2}\right\}$$

Casimir energy III

Taking into account that the zeroes of $P(y)$ are l-independent, after appropriate subtractions of the divergent linear in l and constant terms, thus renormalizing the linear energy density and the zero energy level, (and assuming that $P(y)$ has no zeroes in $[1,\infty)$) we can define de Casimir energy as

$$E_{\mathsf{Cas}}\,(l) := \mathcal{E}_0 + \mathcal{E}_1 l + \frac{\hbar m}{2} F(-1/2) =$$

$$= \mathcal{E}_0 + \mathcal{E}_1 l - \frac{\hbar m}{2\pi} \int_1^\infty dy \sqrt{y^2-1} \frac{d}{dy} \log\left(1 - e^{-2My} \frac{P(-y)}{P(y)}\right) =$$

$$= \mathcal{E}_0 + \mathcal{E}_1 l + \Delta E(l),$$

$$\Delta E(l) := \frac{\hbar m}{2\pi} \int_1^\infty dy \frac{y}{\sqrt{y^2-1}} \log\left(1 - e^{-2My} Q(y)\right).$$

Casimir energy IV

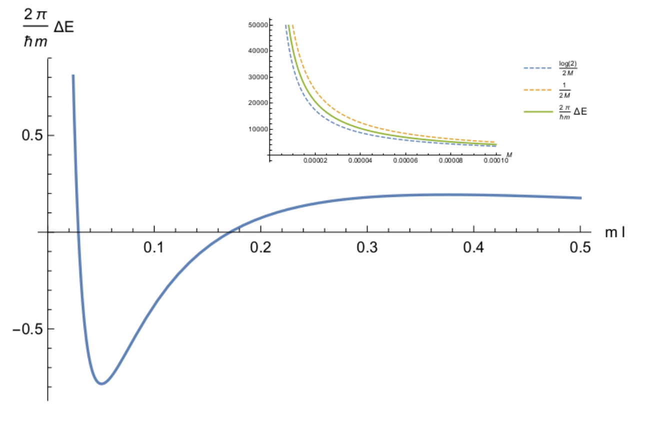


Figure: Figure. ΔE as a function of M for $m = 1, \beta_1 = -1, \beta_2 = -22, \beta_1' = 0, \beta_2' = 1$