Nambu-Jona—Lasinio model in the presence of intense magnetic fields

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Outline for the lectures

- (i) NJL model at finite T and B - basics

- (ii) Issues related to regularizing thermo and magnetic contributions within nonrenormalizable theories and **applications**

- (iii) hot quark matter and hot bosonic matter with a strong electric field

Outline

- Motivation
- Schrödinger Equation and Dirac Equation
- Particle in the Presence of an Electromagnetic Field
- NJL model in MFA
- NJL model at finite eB
- Magnetic Catalysis
- Thermodynamical Quantities
- Magnetic Field Independent Regularization - MFIR

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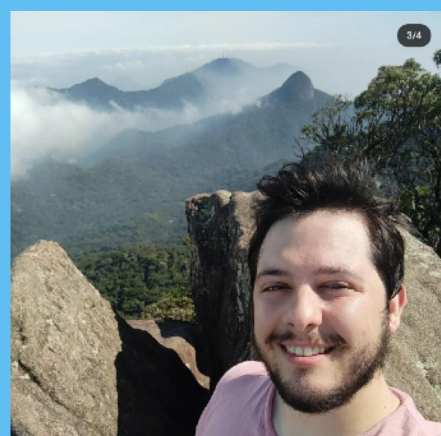

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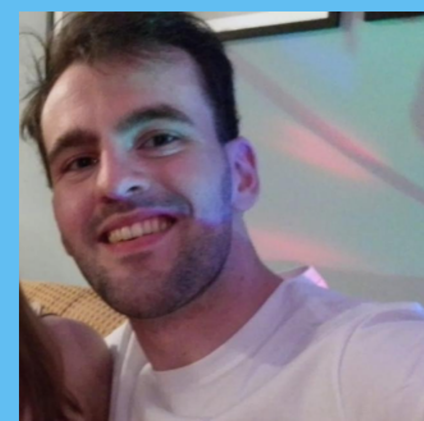

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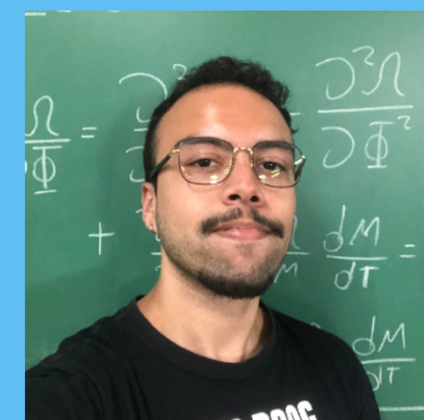

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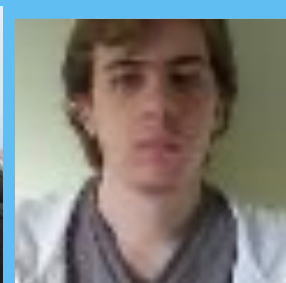
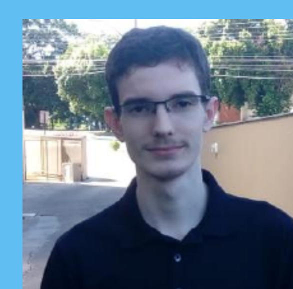
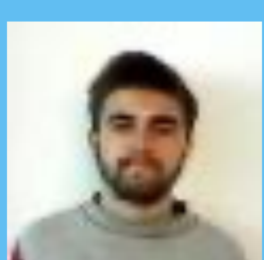
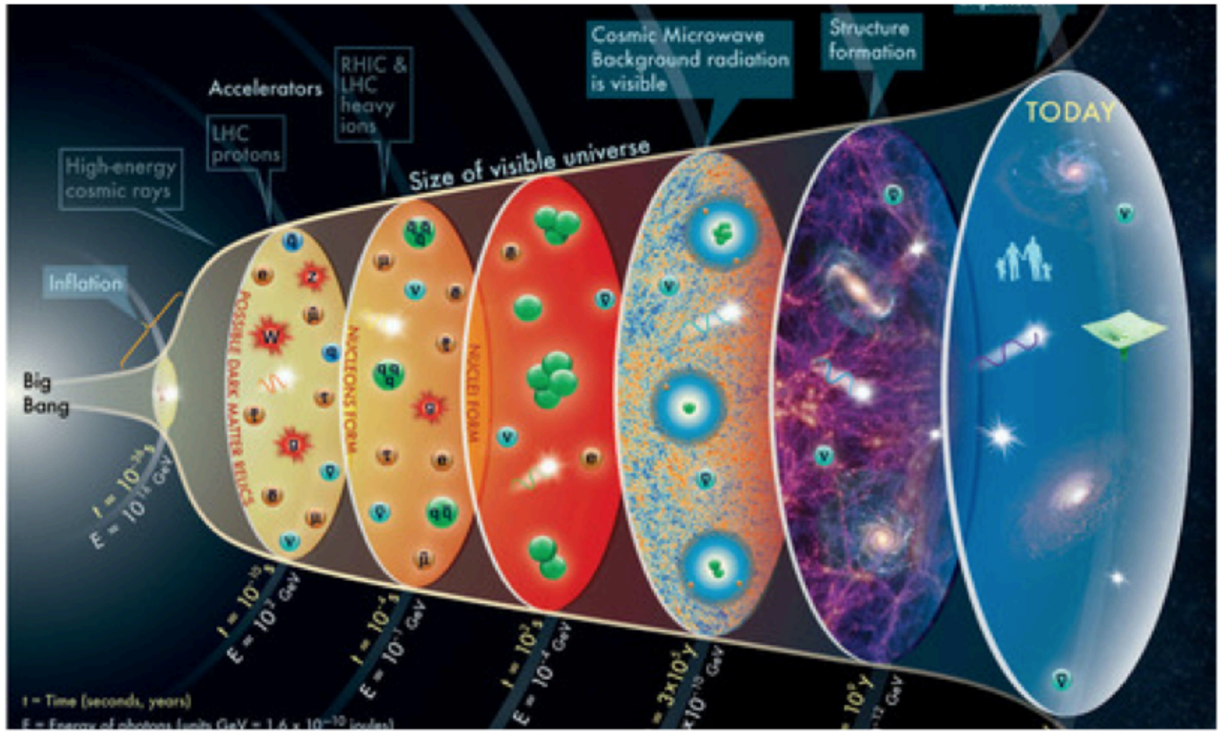




Quarks and gluons in extreme conditions

- heavy ion collisions $T \lesssim 10^{12}\,^\circ C = 200$ MeV, $n \lesssim 0.12$ fm^{-3}
  $B \lesssim 10^{19}$ G $= 0.3$ GeV$^2/e$
- neutron stars $T \lesssim 1$ keV, $n \lesssim 2$ fm^{-3}
  magnetars $B \lesssim 10^{15}$ G
- neutron star mergers $T \lesssim 50$ MeV
- eary universe, QCD epoch $T \lesssim 200$ MeV
  standard scenario: $n \approx 0$ also allowed: $n_Q = 0,\; n_\ell/s \lesssim 0.01$



* G. Endrodi slide - SQM 2024

Strengths of magnetic fields

- **Strong magnetic fields are also present in magnetars:** C. Kouveliotou et al., Nature 393, 235 (1998).

magnetars:

at surface $B \lesssim 10^{15}\,\mathrm{G}$

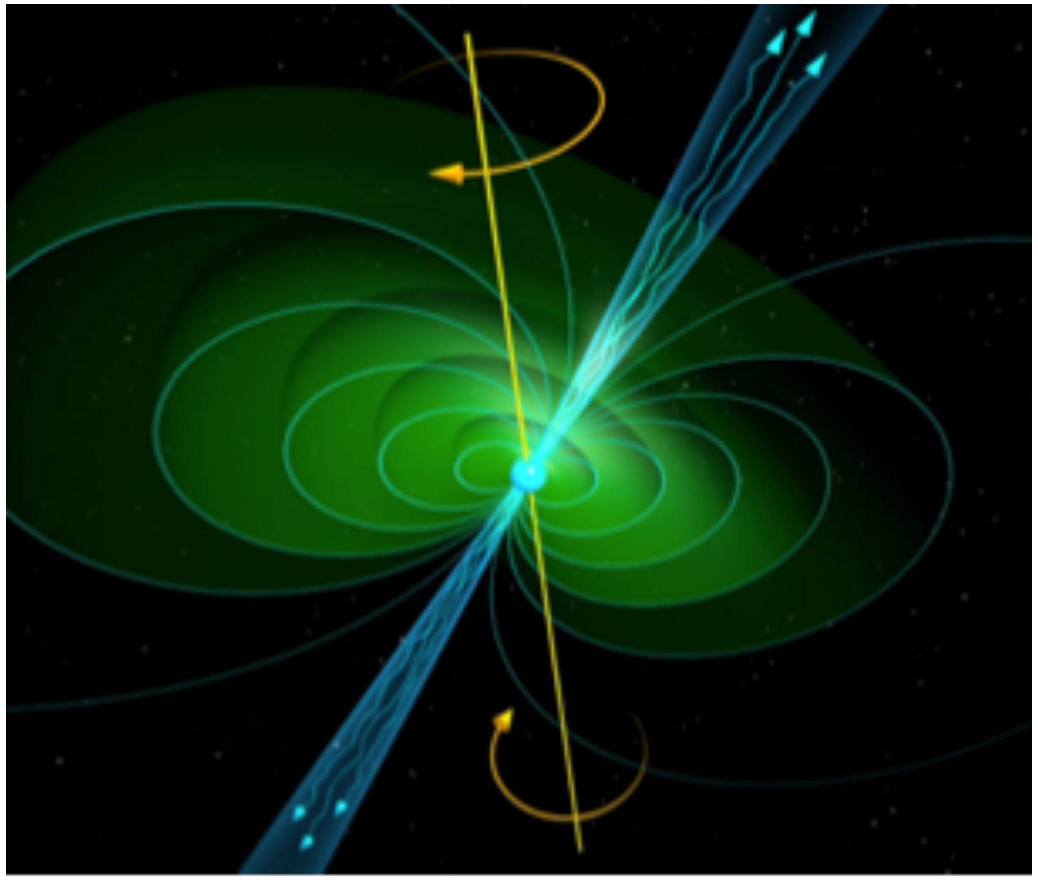
Duncan, Thompson, Astrophys.J. 392, L9 (1992)

larger in the interior,

$B \sim 10^{18-20}\,\mathrm{G}$?

Lai, Shapiro, Astrophys.J. 383, 745 (1991)

E. J. Ferrer *et al.*, PRC 82, 065802 (2010)



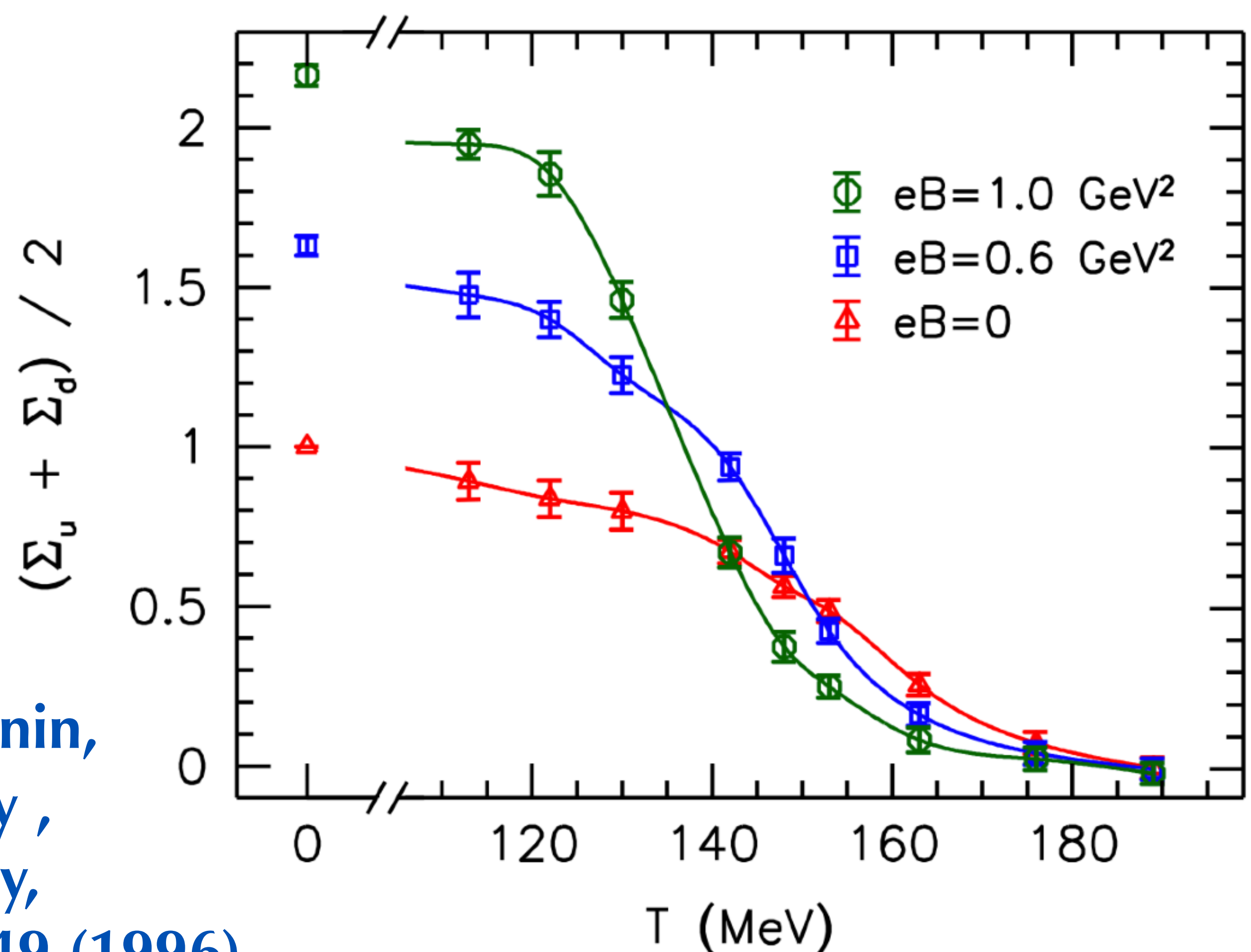
A. K. Harding, D. Lai, Rept. Prog. Phys. 69, 2631 (2006)

- **and might have played an important role in the physics of the early universe.** T. Vaschapati, Phys. Lett. B 265, 258 (1991).

D. Grasso and H.R. Rubinstein, Phys. Rep. 348, 163 (2001).

B Effects on QCD phase transitions?

$$\Lambda_{\rm QCD}^2 \sim (200\,{\rm MeV})^2 \sim 2 \times 10^{18}\,{\rm G}$$

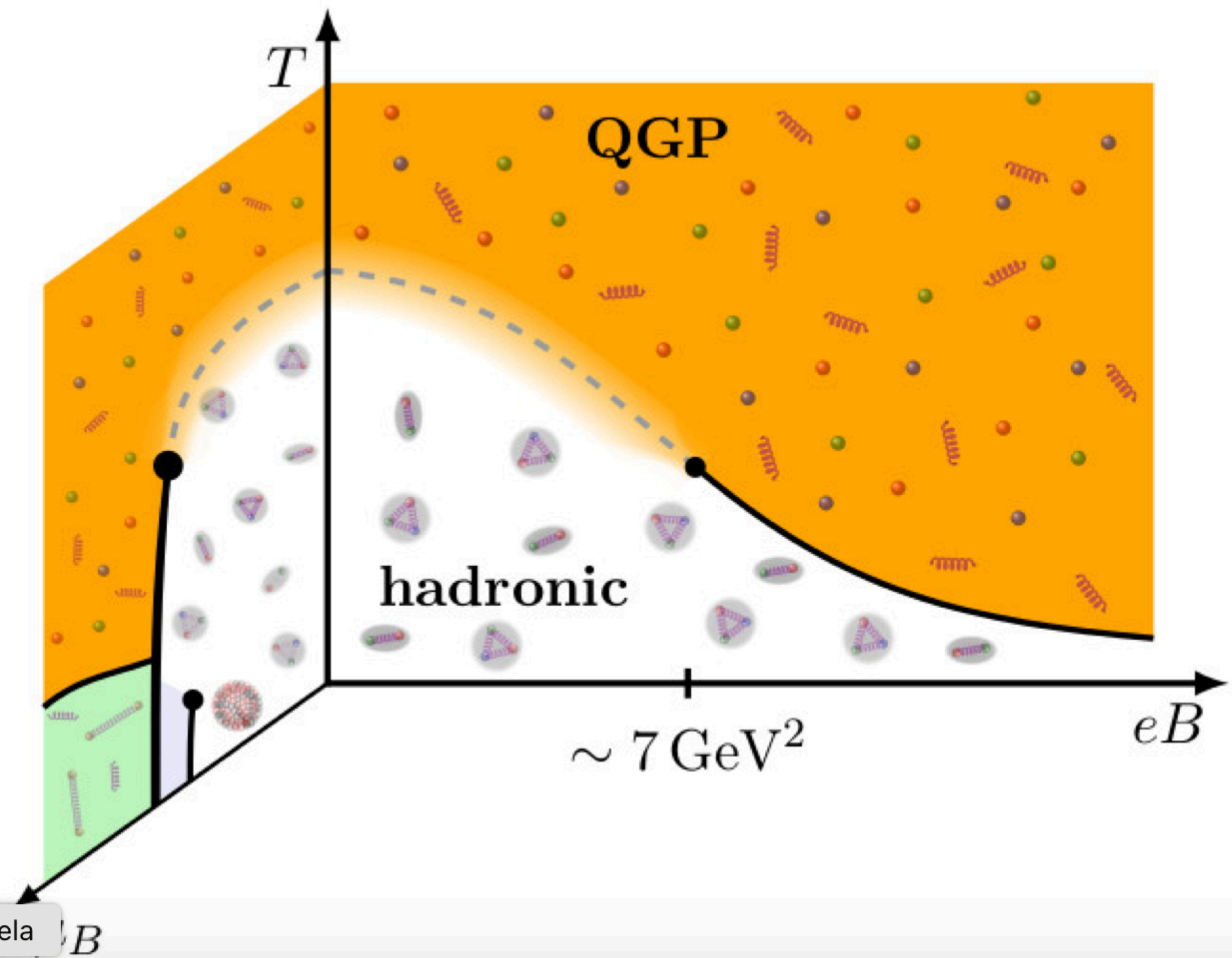


IMC: Bali, Bruckmann, Endrodi, Fodor, Katz et al. *JHEP* 02 (2012) 044 *Phys.Rev.D* 86 (2012) 071502

MC: V.P. Gusynin, V.A. Miransky , I.A. Shovkovy, *Nucl. Phys. B* 462 249 (1996)

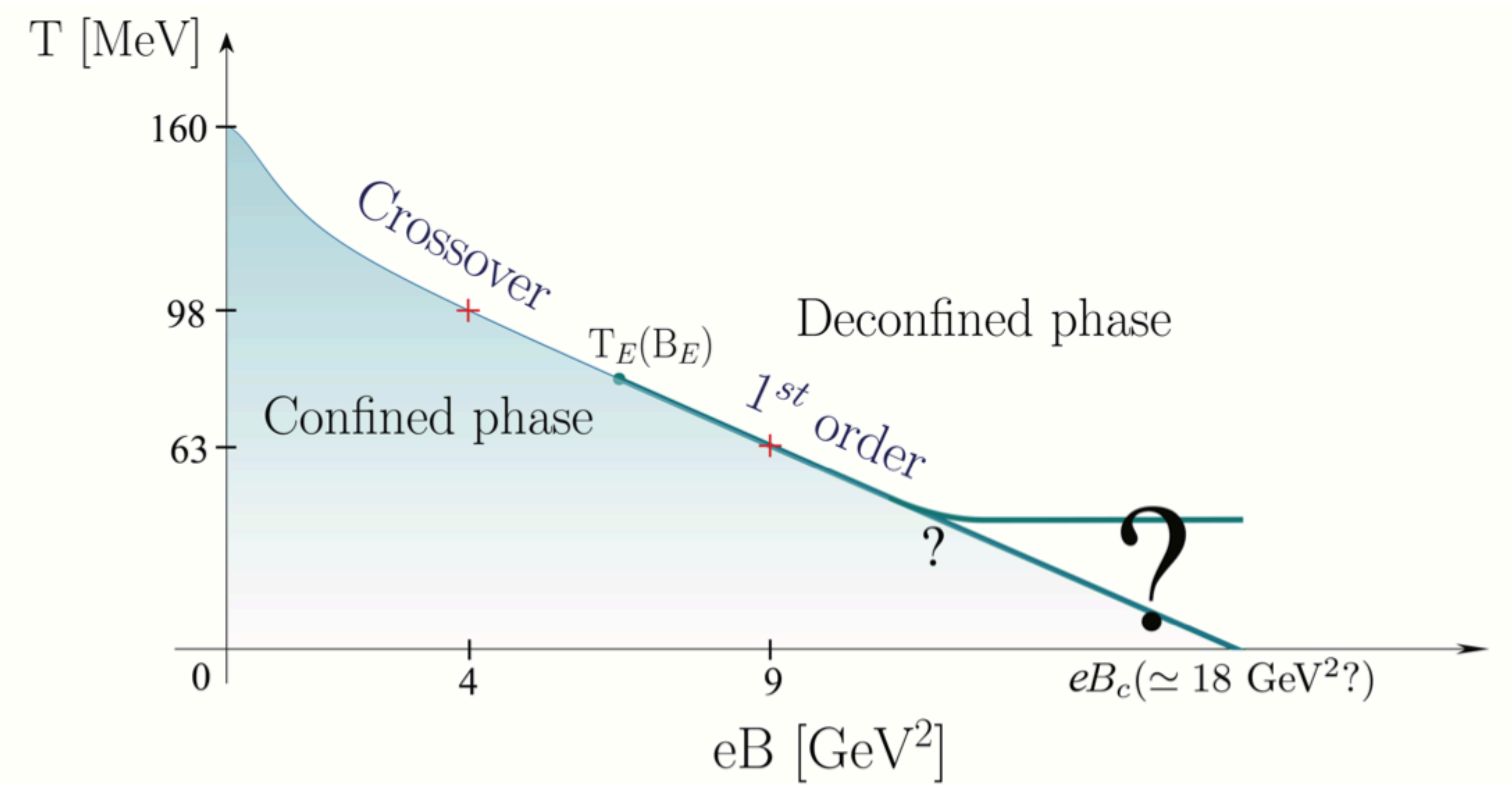
Phase diagram

- control parameters: T, $n \leftrightarrow \mu$, B $\mu_{\{u,d,s\}}$ / $\mu_{\{B,Q,S\}}$ / $\mu_{\{B,I,S\}}$
- ~~well-known~~ famous phase diagram
- well-known, less famous phase diagram: $T - B$



* G. Endrodi slide - SQM 2024

B Effects on QCD phase transitions?



M. D'Elia , L. Maio, F. Sanfilippo, A. Stanzione, *Phys. Rev. D* **105** , 034511 (2022).

Strength of the magnetic fields

	B [Gauss]	eB [MeV^2]
Earth surface	0.5	$(0.05\text{x}10^{-6}\text{MeV})^2$
Magnetic Ressonance	$1.5\text{x}10^4$	$(8.6\text{x}10^{-6}\text{MeV})^2$
magnet - CERN	$8.4\text{x}10^4$	$(20.5\text{x}10^{-6}\text{MeV})^2$
frog levitation *	10^5	$(25\text{x}10^{-6}\text{MeV})^2$
Critical quantum field of the electron	$4.4 \text{ x } 10^{13}$	$(0.5 \text{ MeV})^2 = \mathbf{m}_e{}^2$
Magnetares (field on the surface)	$5.0\text{x}10^{15}$	$(5 \text{ MeV})^2 = (10\mathbf{m}_e)^2$
(Au+Au) Heavy ion collisions	10^{19}	$(400 \text{ MeV})^2 = (3\mathbf{m}_\pi)^2$

(1 Tesla = 10^4 Gauss)

* Andre Geim - Ig Nobel-2000 and Nobel-2010 (graphene)

Schrödinger Equation: free particle

Let's consider relativistic particles and, therefore, we will start by discussing the appropriate equation of motion for this case, namely the Dirac equation. In the non-relativistic case, we heuristically obtain the Schrödinger equation from the energy

$$E = \frac{\vec{p}^{\,2}}{2m} = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

using the prescription:

$$E \to i\frac{\partial}{\partial t} \;\; , \;\; p_x \to \frac{\hbar}{i}\frac{\partial}{\partial x} \;\; , \;\; p_y \to \frac{\hbar}{i}\frac{\partial}{\partial y}, \quad p_z \to \frac{\hbar}{i}\frac{\partial}{\partial z}$$

we obtain

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}, t)$$

Schrödinger Equation: free particle

In relativistic case the energy is given by:

$$E = \sqrt{\vec{p}^{\,2} c^2 + m^2 c^4}$$

using the prescription:

$$E \rightarrow i\frac{\partial}{\partial t} \quad , \quad p \rightarrow \frac{\hbar}{i}\vec{\nabla}$$

we obtain

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = \sqrt{-c^2\hbar^2\vec{\nabla}^2 + m^2 c^4}\ \ \psi(\vec{r},t)$$

Extremely complex equation (nature is simpler!)

Schrödinger Equation: free particle

Dirac's idea was to "take the square root" of:

$$E = \sqrt{\vec{p}^{\,2} c^2 + m^2 c^4} \;\;, \;\; E \to i \frac{\partial}{\partial t} \;\;, \;\; p \to \frac{\hbar}{i} \vec{\nabla}$$

$$E = i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = H_D \, \psi(\vec{r}, t) = \left(c\vec{\alpha} \cdot \vec{p} + \beta m c^2 \right) \, \psi(\vec{r}, t)$$

Requiring that in the operatorial form $E^2 = H_D^2 = p^2 c^2 + m^2 c^4$
($\to$ relativistic dispersion relation)
we can determine $\vec{\alpha}$ and β.

$$\left(c\vec{\alpha} \cdot \vec{p} + \beta m c^2 \right) \, \left(c\vec{\alpha} \cdot \vec{p} + \beta m c^2 \right) \psi(\vec{r}, t) = (c^2 \vec{p}^{\,2} + m^2 c^4) \psi(\vec{r}, t)$$

For the last equation to have a solution $\vec{\alpha}$ and β must be matrices.

Schrödinger Equation: free particle

The minimum dimension of the matrices α_i, $i = x, y, z$ e β that satisfy the desired conditions is 4. A standard representation is the following:

$$\alpha_i = \begin{pmatrix} 0_{2\times 2} & \sigma_i \\ \sigma_i & 0_{2\times 2} \end{pmatrix}, \; i = x, y, z, \;, \; \beta = \begin{pmatrix} 1_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & -1_{2\times 2} \end{pmatrix},$$

where σ_i are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

as an example:

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; \psi(\vec{r}, t) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Schrödinger Equation: free particle

The Dirac equation is given by:

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = H_D\,\psi(\vec{r},t) = \left(c\vec{\alpha}\cdot\vec{p} + \beta mc^2\right)\ \psi(\vec{r},t)$$

Let's rewrite the Dirac equation in a more compact form using the γ Dirac matrices:

$$\gamma_0 = \beta\ \ ,\ \ ,\gamma_i = \beta\alpha_i\ \ ,\ i = x,y,z$$

multiplying the Dirac equation by β

$$i\hbar\beta\frac{\partial}{\partial t}\psi(\vec{r},t) = \left(c\beta\vec{\alpha}\cdot\vec{p} + (\beta)^2 mc^2\right)\ \psi(\vec{r},t)$$

or even

$$\left(i\hbar\gamma_0\frac{\partial}{\partial ct} + \vec{\gamma}\cdot i\hbar\vec{\nabla}\right)\psi(\vec{r},t) = mc\,\psi(\vec{r},t)$$

Solution of Dirac equation for free particle

Using natural units: $\hbar$=1 e c=1

$$(\not{p} - m)\,\Psi(t,\vec{r}) \;=\; 0 \;\;,\;\; \not{p} = p^{\mu}\gamma_{\mu} \;\;,$$
$$i\partial_t\Psi(t,\vec{r}) \;=\; H_D\Psi(t,\vec{r}) \;=\; \left(\vec{\alpha}\cdot\hat{\vec{p}} + \beta m\right)\Psi(t,\vec{r})$$

$$i\frac{\partial}{\partial t}\psi(\vec{r},t) = \left[\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}\right]\psi(\vec{r},t)$$

Ansatz to find the positive energy solution:

$$\Psi(\vec{r},t) \;=\; \Psi(\vec{p})e^{-ip^{\mu}x_{\mu}} \;=\; \begin{bmatrix} \chi \\ \phi \end{bmatrix} e^{-i(Et-\vec{p}\cdot\vec{r})}$$

Substituting into the Dirac Equation:

$$E\begin{bmatrix} \chi \\ \phi \end{bmatrix} = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{bmatrix} \chi \\ \phi \end{bmatrix}$$

Solution of Dirac equation for free particle

Which results in the following 2×2 matrix equations

$$E\,\chi = \vec{\sigma}\cdot\vec{p}\,\phi + m\,\chi \;\;,\;\; E\,\phi = \vec{\sigma}\cdot\vec{p}\,\chi - m\,\phi$$

Which, isolating ϕ on the right-hand side, results in :

$$\phi = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\chi$$

The positive and negative energy solutions are:

$$\Psi^{(+)}(\vec{r},t) = N\begin{bmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{bmatrix} e^{-ip^{\mu}x_{\mu}} = u_s e^{-ip^{\mu}x_{\mu}}$$
$$\Psi^{(-)}(\vec{r},t) = N\begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \\ \chi_s \end{bmatrix} e^{ip^{\mu}x_{\mu}} = v_s e^{ip^{\mu}x_{\mu}}$$

N corresponds to the normalization constant, and χ to the Pauli spinor:

$$N = \sqrt{\frac{E+m}{2E}} \;,\; \chi_{+} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;,\; \chi_{-}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution of Dirac equation for free particle

Considering the case where the particle's momentum is zero:

$$\vec{p} = 0 \rightarrow i\partial_t \Psi(t,\vec{r}) = H_D \Psi(t,\vec{r}) = \left(\vec{\alpha} \cdot \hat{\vec{p}} + \beta m\right) \Psi(t,\vec{r}) \ = \beta m \Psi(t,\vec{r})$$

And the ansatz for the positive and negative energy solutions:

$$\Psi^{(+)}(\vec{r},t) = u_s e^{-ip^\mu x_\mu} = u_s e^{-iEt} \ , \ \Psi^{(-)}(\vec{r},t) = v_s e^{ip^\mu x_\mu} = v_s e^{iEt}$$

$$i\partial_t \Psi^{(+)}(t,\vec{r}) = \beta m \Psi^{(+)}(t,\vec{r}) \implies$$

$$E \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} e^{-iEt} = m \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} e^{-iEt}$$

Solution of Dirac equation for free particle

$$i\partial_t \Psi^{(-)}(t,\vec{r}) = \beta m \Psi^{(-)}(t,\vec{r}) \implies$$

$$-E\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} e^{iEt} = m\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} e^{iEt}$$

The four independent solutions are:

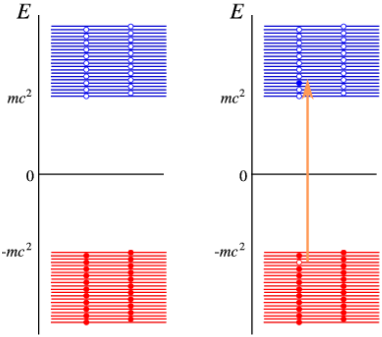
$$\Psi^{(+)}_{\uparrow}(\vec{r},t) = \begin{bmatrix} \chi_+ \\ 0 \end{bmatrix} e^{-iEt} \ , \ \Psi^{(+)}_{\downarrow}(\vec{r},t) = \begin{bmatrix} \chi_- \\ 0 \end{bmatrix} e^{-iEt}$$

$$\Psi^{(-)}_{\uparrow}(\vec{r},t) = \begin{bmatrix} 0 \\ \chi_+ \end{bmatrix} e^{iEt} \ , \ \Psi^{(-)}_{\downarrow}(\vec{r},t) = \begin{bmatrix} 0 \\ \chi_- \end{bmatrix} e^{iEt}$$

$$H_D \Psi^{(+)}_s(t,\vec{r}) = E\Psi^{(+)}_s(t,\vec{r}) = m\Psi^{(+)}_s(t,\vec{r}) \ , \ s = \{\uparrow,\downarrow\} \text{ positive energy}$$

$$H_D \Psi^{(-)}_s(t,\vec{r}) = E\Psi^{(-)}_s(t,\vec{r}) = -m\Psi^{(-)}_s(t,\vec{r}) \ , \ s = \{\uparrow,\downarrow\} \text{ negative energy}$$

Solution of Dirac equation for free particle



Dirac Sea $\rightarrow$ set of negative energy states

$$E = \pm\sqrt{p^2c^2 + m^2c^4}$$

Particle-Hole Pair Creation ($e^- e^+$) (Electron-Positron)
Dirac Sea Hole $\rightarrow$ Positron

Particle in the Presence of an Electromagnetic Field

The free Dirac equation:

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = H_D\, \psi(\vec{r},t) = \left(\vec{\alpha}\cdot\hat{\vec{p}} + \beta m\right)\, \psi(\vec{r},t)$$

Transforms in the Presence of an Electromagnetic Field $A^\mu(\vec{x},t)$=($\phi(\vec{x},t)$, $\vec{A}(\vec{x},t)$) in:

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = H_D\, \psi(\vec{r},t) = \left(\vec{\alpha}\cdot(\hat{\vec{p}} - q\vec{A}) + \beta m\right)\, \psi(\vec{r},t) + q\phi\psi(\vec{r},t)$$

Rearranging and multiplying by $\beta = \gamma_0$ to rewrite the equation in terms of matrices, $\gamma^\mu = (\gamma_0, \vec{\gamma}) = (\beta, \beta\alpha)$

$$\left(\gamma_0(\hat{p}_0 - q\phi) - \vec{\gamma}\cdot(\hat{\vec{p}} - q\vec{A}) - m\right)\, \psi(\vec{r},t) = 0$$

Particle in the Presence of an Electromagnetic Field

Recalling the 4-vector notation, which the prescription corresponds to

$$x^\mu = (t, \vec{x}) \ , \ x_\mu = (t, -\vec{x}) \ , \ \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = (\frac{\partial}{\partial t}, -\nabla) \ ,$$

$$p^\mu = (\hat{p}_0, \hat{\vec{p}}) = i\hbar(\frac{\partial}{\partial t}, -\nabla) \ , \ A^\mu = (\phi, \vec{A})$$

$$\left(\gamma_0(\hat{p}_0 - q\phi) - \vec{\gamma} \cdot (\hat{\vec{p}} - q\vec{A}) - m\right) \ \psi(\vec{r}, t) = \left(\gamma_\mu(p^\mu - qA^\mu) - m\right) \ \psi(\vec{r}, t) = 0$$

Therefore, to describe a particle in the presence of an external electromagnetic field, we use the prescription: (Miminal coupling) :

$$p^\mu \rightarrow p^\mu - qA^\mu \Rightarrow i\hbar\frac{\partial}{\partial t} \rightarrow i\hbar\frac{\partial}{\partial t} - q\phi \ , \ - i\hbar\nabla \rightarrow -i\hbar\nabla - q\vec{A}$$

Particle in a Magnetic Field

Let's Introduce the External Magnetic Field $\vec{B}$ via Minimal Coupling:

$$\not{p} \equiv \hat{p}^{\mu}\gamma_{\mu} \to (\hat{p}^{\mu} - qA^{\mu})\gamma_{\mu} \ ,$$

$q =$ particle charge
$A^{\mu} \ = \ (\phi, \vec{A}) \ = \ (0, 0, Bx, 0)$ (Landau gauge)

$\Rightarrow \vec{B} = \nabla \times \vec{A} = B\hat{z}$, $\quad \nabla \cdot \vec{A}=0$, $\vec{E}=0$, $\phi=0$

$$\begin{aligned} (\not{p} - q\not{A} - m)\,\Psi(t,\vec{r}) \quad &= \quad 0 \ , \\ i\partial_t\Psi(t,\vec{r}) \ = \ H(A^{\mu}(\vec{r}))\Psi(t,\vec{r}) \quad &= \quad \left(\vec{\alpha}\cdot\left[\hat{\vec{p}} - q\vec{A}(x^{\mu})\right] + \beta m\right)\Psi(t,\vec{r}) \end{aligned}$$

Particle in a Magnetic Field

Let's consider an electron ($q = -e$) , and Landau gauge ($A^\mu = (0, 0, Bx, 0)$), e= proton charge > 0
The Dirac equation assumes the following expression:

$$i\partial_t \Psi(t, \vec{r}) \; = \; H(x)\Psi(t, \vec{r}) \; = \; \left(\vec{\alpha} \cdot \left[\hat{\vec{p}} + eBx\hat{j}\right] + \beta m\right) \Psi(t, \vec{r})$$

$$i\frac{\partial}{\partial t}\psi(\vec{r}, t) = \left[\left(\begin{array}{cc} 0 & \vec{\sigma} \cdot \left[\hat{\vec{p}} + eBx\hat{j}\right] \\ \vec{\sigma} \cdot \left[\hat{\vec{p}} + eBx\hat{j}\right] & 0 \end{array}\right) + \left(\begin{array}{cc} m & 0 \\ 0 & -m \end{array}\right)\right]\psi(\vec{r}, t)$$

Analogously to what we did when B=0, we will use an ansatz for the solution (positive energy)

$$\Psi(t, \vec{r}) \; = \; f(x)e^{-iEt + ip_y y + ip_z z} \quad , \quad f(x) \to 4 - \text{spinor}$$

Johnson-Lippmann solution

$\Rightarrow$ ansatz for the solution (positive energy):

$$\Psi(t,\vec{r}) = \begin{pmatrix} C_1 v_{n-1}(\xi) \\ C_2 v_n(\xi) \\ C_3 v_{n-1}(\xi) \\ C_4 v_n(\xi) \end{pmatrix} e^{-iEt+ip_y y+ip_z z}$$

A given choice of $C_1, C_2, C_3, C_4 \Rightarrow$ four independent solutions:

$$\Psi^\epsilon(\vec{r}) = \left[\frac{1+s}{2} \begin{bmatrix} (\epsilon E_n+m)v_{n-1}(\xi) \\ 0 \\ \epsilon p_z v_{n-1}(\xi) \\ ip_n v_n(\xi) \end{bmatrix} + \frac{1-s}{2} \begin{bmatrix} 0 \\ (\epsilon E_n+m)v_n(\xi) \\ -ip_n v_{n-1}(\xi) \\ -\epsilon p_z v_n(\xi) \end{bmatrix} \right]$$

$$\Psi^\epsilon(t,\vec{r}) = \frac{(eB)^{1/4}}{(2\pi)} \frac{1}{\sqrt{2\epsilon E_n(\epsilon E_n+m)}} \Psi^\epsilon(\vec{r})\, e^{-i\epsilon(Et+p_y y+p_z z)}$$

$\epsilon = +1(-1) \rightarrow$ positive (negative) state of energy
s=+1(-1) $\rightarrow$ spin states up (down)
$p_n = \sqrt{2eBn}$ $\quad \xi = (eB)^{1/2}(x+\epsilon \frac{p_y}{eB})$
Convenient notation:

$$\boxed{\psi^\epsilon(\vec{x},t) = \phi^{(\epsilon)}_{n,s,p_y,p_z}(\vec{x})e^{-i\epsilon Et}}\,.$$

The Positive Energy Solution for an Electron in the Presence of a Magnetic Field $\vec{B}$:

$$\psi^{(+)}(t,\vec{r}) = \begin{pmatrix} C_1 v_{n-1}(\xi) \\ C_2 v_n(\xi) \\ C_3 v_{n-1}(\xi) \\ C_4 v_n(\xi) \end{pmatrix} e^{-iEt+ip_y y+ip_z z}$$

$$\xi = (eB)^{1/2}(x + \frac{p_y}{eB}) \ , \ v_n(\xi) = \frac{1}{(\pi^{1/2}2^n n!)^{1/2}} H_n(\xi) e^{-\frac{1}{2}\xi^2}$$

$\frac{p_y}{eB}$ Determine the position where the oscillator wave functions are centered. If our system is contained in a box of side L:

$$0 \leq \frac{p_y}{eB} \leq L \ ,$$

$$\sum_{p_x} \to \sum_{n=0}^{\infty} g_n \ , \ \sum_{p_y} \to \frac{L}{2\pi} \int dp_y = \frac{L}{2\pi} L\, eB \ , \ \sum_{p_z} \to \frac{L}{2\pi} \int_{-\infty}^{\infty} dp_z$$

$$\boxed{\frac{2}{V} \sum_{p_x,p_y,p_z} \equiv \frac{2}{(2\pi)^3} \int d^3p \to \sum_{n=0}^{\infty} g_n \frac{eB}{(2\pi)^2} \int_{-\infty}^{\infty} dp_z} , \quad g_n = 2 - \delta_{n\,0}$$

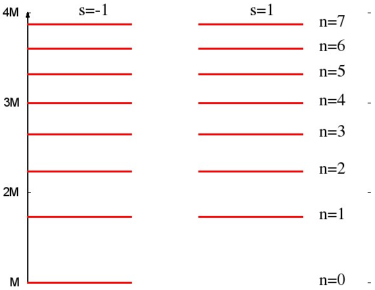
3. IS THE DIMENSIONAL REDUCTION 3+1 $\to$ 1+1 (2+1 $\to$ 0+1) CONSISTENT WITH SPONTANEOUS CHIRAL SYMMETRY BREAKING?

V.P. Gusynin,V.A. Miransky, I.A. Shovkovy, Nucl. Phys. B462, 249 (1996)

In this section we consider the question whether the dimensional reduction 3+1 $\to$ 1+1 (2+1 $\to$ 0+1) in the dynamics of the fermion pairing in a magnetic field is consistent with spontaneous chiral symmetry breaking. This question occurs naturaly since, due to the Mermin-Wagner-Coleman (MWC) theorem [12], there cannot be spontaneous breakdown of continuous symmetries at $D = 1 + 1$ and $D = 0{+}1$. The MWC theorem is based on the fact that gapless Nambu-Goldstone (NG) bosons cannot exist in dimensions less than 2+1. This is in particular reflected in that the $(1 + 1)$–dimensional propagator of would be NG bosons would lead to infrared divergences in perturbation theory (as indeed happens in the $1/N_c$ expansion in the $(1 + 1)$–dimensional Gross–Neveu model with a continuous symmetry [13]).

However, the MWC theorem is not applicable to the present problem. The central point is that the condensate $\langle 0|\bar{\psi}\psi|0\rangle$ and the NG modes are **neutral** in this problem and the dimensional reduction in a magnetic field does not affect the dynamics of the center of mass of **neutral** excitations. Indeed, the dimensional reduction $D \to D{-}2$ in the fermion propagator, in the infrared region, reflects the fact that the motion of **charged** particles is restricted in the directions perpendicular to the magnetic field. Since there is no such restriction for the motion of

Particle in a Magnetic Field



Electron Landau Levels

$$E_n = \sqrt{p_z^2 + M^2 + 2eBn} \ ,$$

$$n = l + \frac{1}{2}(1+s) \ , \quad s = \pm 1 \ ,$$

$$l = 0, 1, 2, \dots$$

$$E_n^2 - M^2 - 2eBn = p_z^2 \geq 0 \Rightarrow$$

$$n \leq \left[\frac{E_n^2 - M^2}{2eB} \right] \ , \ 2eBn \to p_x^2 + p_y^2$$

- Landau Levels with n=1,2,3... are doubly degenerate (spin $s = \pm 1$)
- Ground state, $n = 0$, is not degenerate and has spin s=-1 (for the electron)

(In the figure we take $p_z = 0$, $\frac{eB}{M} = 1$)

DIRAC FERMIONS AT B≠0

- Dirac equation for charged fermions:

$$(i\gamma^\mu D_\mu - m)\psi = 0$$

where $A_\mu = (A_0, -\vec{A})$ and the Landau gauge $\vec{A} = (-By, 0,0)$ is used.

- Look for a solution in the form: $\psi = (i\gamma^\mu D_\mu + m)\phi$. Then,

$$\left[-\partial_0^2 + (\partial_x + ieBy)^2 + \partial_y^2 + \partial_z^2 + i\gamma^1\gamma^2 eB - m^2\right]\phi = 0$$

- Normalized solutions for ϕ have the form

$$\phi_{k,\pm} \propto \frac{1 \pm i\,\mathrm{sgn}(eB)\gamma^1\gamma^2}{2}\varphi_k(y)e^{-i\omega t + ip_x x + ip_z z}$$

where φ_k are harmonic oscillator wave functions, i.e.,

$$\varphi_k \propto H_k(\xi)e^{-\frac{\xi^2}{2}}, \quad \xi = \frac{y}{l} + p_x l\,\mathrm{sgn}(eB) \quad \text{and} \quad l = \frac{1}{\sqrt{|eB|}}$$

- The dispersion relation is given by

$$\omega = E_n^\pm = \pm\sqrt{2n|eB| + p_z^2 + m^2}$$

where $n = \underbrace{k + \frac{1}{2}}_{\text{orbital}} + \underbrace{\mathrm{sgn}(eB)s_z}_{\text{spin}}$ and $s_z = \pm\frac{1}{2}$ is an eigenvalue of $\frac{i}{2}\gamma^1\gamma^2$

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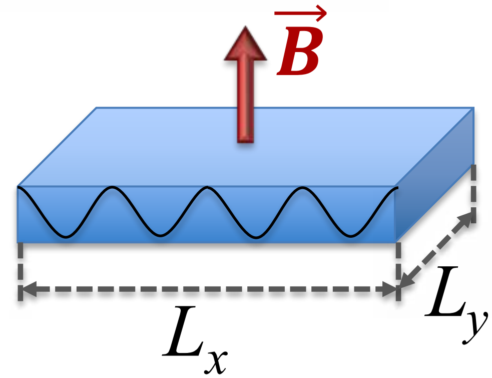
DEGENERACY OF LANDAU LEVELS



- The Landau level energies are independent of p_x

$$E_n^{\pm} = \pm\sqrt{2n|eB| + p_z^2 + m^2}$$

- This means that each level is highly degenerate
- Let's calculate the degeneracy by confining the system in a finite box of size $L_x \times L_y$ with periodic boundary conditions
- The wave function is a plane wave in the *x* direction: $\psi(x) \propto e^{ip_x x}$

$$\psi(0) = \psi(L_x) \quad \Longrightarrow \quad e^{ip_x L_x} = 1 \quad \Longrightarrow \quad p_x = \frac{2\pi n}{L_x}, \quad n = 1,2, \ldots, N_{\max}$$

- The value of p_x sets the center of the Landau orbit in *y*-direction:

$$y_c \approx p_x l^2 \quad \Longrightarrow \quad p_{x,\max} l^2 \lesssim L_y \quad \Longrightarrow \quad \frac{2\pi N_{\max}}{L_x} \frac{1}{|eB|} \approx L_y \quad \Longrightarrow \quad \frac{N_{\max}}{L_x L_y} \approx \frac{|eB|}{2\pi}$$

- The degeneracy is proportional to the field strength and the size (area) of the system in the spatial directions perpendicular to $\vec{B}$

$$N_{\max} \approx \frac{|eB|}{2\pi} L_x L_y$$

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LANDAU ENERGY SPECTRUM

- Landau energy levels at $m = 0$

$$E_n^{\pm} = \pm\sqrt{2n|eB| + p_z^2}$$

where $n = \underbrace{k + \frac{1}{2}}_{\text{orbital}} + \underbrace{\text{sgn}(eB)s_z}_{\text{spin}}$

- Lowest Landau level is *spin polarized*

$$E_0^{\pm} = \pm p_z \qquad (k = 0,\ s_z = -\tfrac{1}{2})$$

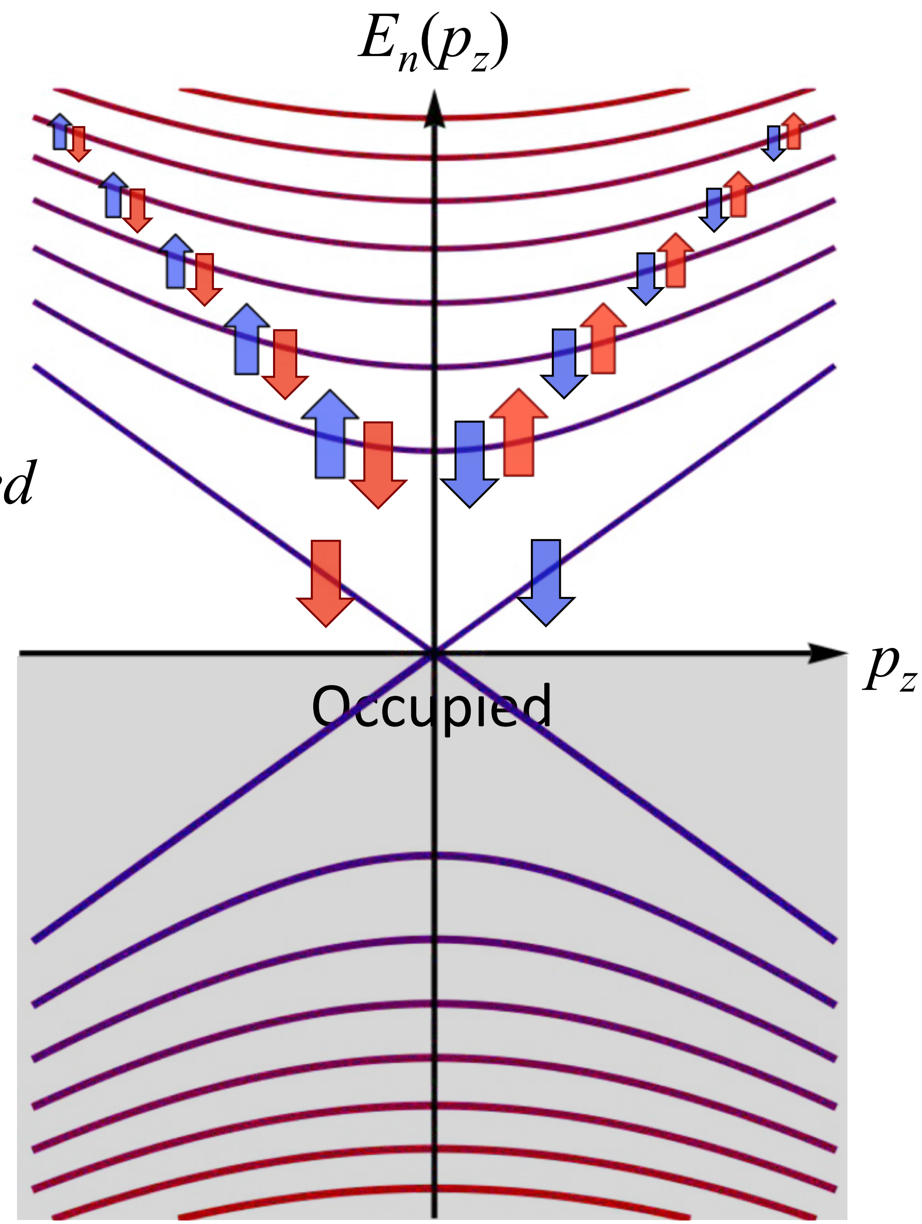
- Density of states at E=0:

$$\left.\frac{dn}{dE}\right|_{E=0} = \frac{|eB|}{2\pi}\frac{1}{2\pi} = \frac{|eB|}{4\pi^2}$$

- Higher Landau levels ($n \geq 1$) are twice as degenerate:

(i) $k = n$ & $s = -\frac{1}{2}$

(ii) $k = n - 1$ & $s = +\frac{1}{2}$



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Example: Dirac Lagrangian

$$\mathcal{L} = \mathcal{L}\left(\bar{\Psi}(t,\vec{r}), \Psi(t,\vec{r}), \partial^\mu \Psi(t,\vec{r})\right) = \bar{\Psi}(t,\vec{r})\left(i\gamma_\mu \partial^\mu - m\right)\Psi(t,\vec{r}) \; , \; \bar{\Psi} \equiv \Psi^\dagger \gamma_0$$

Equation of motion for $\bar{\Psi}$:

$$\frac{\partial \mathcal{L}}{\partial \bar{\Psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\Psi}} = 0 \;\; , \;\; \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = \left(i\gamma_\mu \partial^\mu - m\right)\Psi \;\; , \;\; \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\Psi}} = 0$$

that results in the Dirac equation:

$$\left(i\gamma_\mu \partial^\mu - m\right)\Psi(t,\vec{r}) = 0 \to \left(\not{p} - m\right)\Psi(t,\vec{r}) = 0$$

Equation of motion for Ψ:

$$\frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi} = 0 \;\; , \;\; \frac{\partial \mathcal{L}}{\partial \Psi} = -\bar{\Psi} m \;\; , \;\; \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi} = \bar{\Psi} i \gamma^\mu$$

that results in the Dirac equation:

$$-\bar{\Psi} m - \partial_\mu \bar{\Psi} i \gamma^\mu \; = \; 0 \to \bar{\Psi}\left(\overleftarrow{\partial_\mu} i\gamma^\mu + m\right) = 0 \to \bar{\Psi}\left(\overleftarrow{\not{p}} + m\right) = 0$$

Example: Dirac Hamiltonian

The conjugate momentum to the fields Ψ e $\bar{\Psi}$ are given by:

$$\Pi_\Psi = \frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi}{\partial t}} = \frac{\partial \mathcal{L}}{\partial \partial_0 \Psi} \quad , \quad \Pi_{\bar{\Psi}} = \frac{\partial \mathcal{L}}{\partial \frac{\partial \bar{\Psi}}{\partial t}} = \frac{\partial \mathcal{L}}{\partial \partial_0 \bar{\Psi}}$$

$$\mathcal{L} = \bar{\Psi}(t, \vec{r}) \left(i\gamma_\mu \partial^\mu - m \right) \Psi(t, \vec{r}) \Longrightarrow \Pi_\Psi = \bar{\Psi} i\gamma_0 = i\Psi^\dagger \quad , \quad \Pi_{\bar{\Psi}} = 0$$

Therefore the Hamiltonian density is given by:

$$\mathcal{H} = \Pi_\Psi \dot{\Psi} + \Pi_{\bar{\Psi}} \dot{\bar{\Psi}} - \mathcal{L} \; = \; i\Psi^\dagger \partial_0 \Psi - \bar{\Psi} \left(i\gamma_\mu \partial^\mu - m \right) \Psi$$

simplifying the expression:

$$\mathcal{H} = i\Psi^\dagger \partial_0 \Psi - \Psi^\dagger \gamma_0 \left(i\gamma_\mu \partial^\mu - m \right) \Psi = i\Psi^\dagger \partial_0 \Psi - \Psi^\dagger \gamma_0 \left(i\gamma_0 \partial_0 + i\vec{\gamma} \cdot \nabla - m \right) \Psi$$

we obtain:

$$\mathcal{H} = \Psi^\dagger \left(-i\vec{\alpha} \cdot \nabla + \beta m \right) \Psi \Longrightarrow H = \int d^3 r \mathcal{H} = \int d^3 r \Psi^\dagger \left(-i\vec{\alpha} \cdot \nabla + \beta m \right) \Psi$$

Free Fields Quantization

Let's consider the **canonical quantization** in Quantum Field Theory. As an example we will take the scalar field:

$$\mathrm{QM}\left\{\begin{array}{l}[q_i, p_j] = i\hbar\delta_{i\,j} \\ [q_i, q_j] = [p_i, p_j] = 0\end{array}\right. \quad , \; \mathrm{QFT}\left\{\begin{array}{l}\{\Psi_\alpha(\vec{r}, t), \Pi_\beta(\vec{r}\,', t)\} = i\delta(\vec{r} - \vec{r}\,')\delta_{\alpha\beta} \\ \{\Psi_\alpha(\vec{r}, t), \Psi_\beta(\vec{r}\,', t)\} = \{\Pi_\alpha(\vec{r}, t), \Pi_\beta(\vec{r}\,', t)\} = 0\end{array}\right.$$

we obtain

$$\Pi_\alpha = \frac{\partial \mathcal{L}}{\partial \partial^0 \Psi_\alpha} = i\Psi^\dagger(\vec{r}, t)$$

and, therefore the commutators of the scalar fields need to satisfy the canonical quantization relations:

$$\{\Psi_\alpha(\vec{r}, t), \Psi_\beta(\vec{r}\,', t)\} \;=\; \left\{\Psi^\dagger_\alpha(\vec{r}, t), \Psi^\dagger_\beta(\vec{r}\,', t)\right\} = 0 \;, \; \left\{\Psi_\alpha(\vec{r}, t), \Psi^\dagger_\beta(\vec{r}\,', t)\right\} = i\delta(\vec{r} - \vec{r}\,')\delta_{\alpha\beta}$$

$$\hat{\Psi}(x) = \sum_r \left(\hat{a}_r \phi_r^{(+)}(\vec{x}) e^{-iE_r t} + \hat{b}_r^\dagger \phi_r^{(-)}(\vec{x}) e^{iE_r t}\right)$$

$$\hat{\Psi}^\dagger(x) = \sum_r \left(\hat{a}_r^\dagger \phi_r^{(+)}(\vec{x})^\dagger e^{iE_r t} + \hat{b}_r \phi_r^{(-)}(\vec{x})^\dagger e^{-iE_r t}\right) \;,$$

We need to do the interpretation of $\hat{a}_r$ and $\hat{a}_r^\dagger$ as creation and annihilation operators of fermionic particles
and for $\hat{b}_r$ and $\hat{b}_r^\dagger$ as creation and annihilation operators for fermionic anti-particles (electron-positron or quark-antiquark)

$$\{\hat{a}_r, \hat{a}^\dagger_{r'}\} = \{\hat{b}_r, \hat{b}^\dagger_{r'}\} = \delta_{r\,r'} \quad , \quad \{\hat{a}_r, \hat{a}_{r'}\} = \{\hat{b}_r, \hat{b}_{r'}\} = 0 \;.$$

SU(2) Nambu-Jona-Lasinio model (NJL)

The Lagrangian of the NJL model with two flavors (u and d quarks):

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - \tilde{m}\right)\psi + G\left[(\overline{\psi}\psi)^2 + (\overline{\psi}i\gamma_5\vec{\tau}\psi)^2\right]$$

interaction terms : scalar-isoscalar + pseudoscalar-isovector

$\vec{\tau}$ are the isospin Pauli matrices

ψ is the Dirac fields of quaks u and d,

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}, \ \tilde{m} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}, \ Q = \begin{pmatrix} q_u = \frac{2}{3}e & 0 \\ 0 & q_d = -\frac{1}{3}e \end{pmatrix}.$$

We consider m_u=m_d=m

SU(2) Nambu-Jona-Lasinio model (NJL)

The Lagrangian of the NJL model to be suitable as an effective model for QCD (Quantum Chromodynamics)
$\rightarrow$ It must reflect the symmetries (properties) of the strong interaction!
Positive points:

- Invariant under global phase transformations $\rightarrow$ Baryon number conservation
- The Lagrangian has chiral symmetry(in the limit m_u=m_d=0)
- It has the spontaneous symmetry breaking mechanism (dynamic mass generation)
- The entire QCD phase diagram can be described by a single effective model (a single equation of state)

Negative points:

- The model is non-renormalizable (requires regularization, Λ-cutoff)
- The interaction does not have confinement (there are no gluons or color charge)

NJL model in the mean field approximation (MFA)

$$\mathcal{L} = \overline{\psi}\,(i\partial\!\!\!/ - \tilde{m})\,\psi + G\left[(\overline{\psi}\psi)^2 + (\overline{\psi}i\gamma_5\vec{\tau}\psi)^2\right]$$

MFA $\rightarrow$ Linearization of the interaction terms of $\mathcal{L}$ neglecting quadratic fluctuations:

$$\hat{O} \equiv \langle\hat{O}\rangle + (\hat{O} - \langle\hat{O}\rangle) = \langle\hat{O}\rangle + \Delta\hat{O} \quad , \quad \hat{O} = (\overline{\psi}\psi) \text{ or } (\overline{\psi}\mathrm{i}\gamma_5\vec{\tau}\psi)$$

MFA $\rightarrow$ $(\Delta\hat{O})^2 \cong 0$; $\langle\overline{\psi}i\gamma_5\vec{\tau}\psi\rangle = 0$ (symmetry)

$$\hat{O}_1\hat{O}_2 = (\langle\hat{O}_1\rangle + \Delta\hat{O}_1)(\langle\hat{O}_2\rangle + \Delta\hat{O}_2) \approx \langle\hat{O}_1\rangle\langle\hat{O}_2\rangle + \langle\hat{O}_1\rangle\Delta\hat{O}_2 + \langle\hat{O}_2\rangle\Delta\hat{O}_1$$

$$= \langle\hat{O}_1\rangle\langle\hat{O}_2\rangle + \langle\hat{O}_1\rangle(\hat{O}_2 - \langle\hat{O}_2\rangle) + \langle\hat{O}_2\rangle(\hat{O}_1 - \langle\hat{O}_1\rangle) = \langle\hat{O}_1\rangle\hat{O}_2 + \langle\hat{O}_2\rangle\hat{O}_1 - \langle\hat{O}_1\rangle\langle\hat{O}_2\rangle$$

therefore:

$$(\overline{\psi}\psi)^2 \approx 2\langle\overline{\psi}\psi\rangle\overline{\psi}\psi - \langle\overline{\psi}\psi\rangle^2$$

NJL model in the mean field approximation (MFA)

$$\mathcal{L} \to \mathcal{L}_{MFA} = \overline{\psi}\left(i\partial\!\!\!/ - \tilde{m}\right)\psi + G\left[2\langle\overline{\psi}\psi\rangle\overline{\psi}\psi - \langle\overline{\psi}\psi\rangle^2\right]$$

defining the constituent mass

$$M = m - 2G\left\langle\overline{\psi}\psi\right\rangle$$

we obtain

$$\mathcal{L}_{MFA} = \overline{\psi}\left(i\partial\!\!\!/ - M\right)\psi - G\left\langle\overline{\psi}\psi\right\rangle^2 \ ,$$

As we have seen, the Hamiltonian is easily obtained from the above Lagrangian:

$$\hat{H}_{MFA} = \int d^3r\mathcal{H} = \int d^3r\left[\Psi^\dagger\left(-i\vec{\alpha}\cdot\nabla + \beta M\right)\Psi + G\left\langle\overline{\psi}\psi\right\rangle^2\right]$$

NJL model in the mean field approximation (MFA)

From the Hamiltonian operator, we obtain the energy of the system., E, calculating its statistical average value at T=0:

$$E = \langle \hat{H}_{MFA} \rangle = \int d^3 r \mathcal{H} = \int d^3 r \left[\langle \Psi^\dagger \left(-i\vec{\alpha} \cdot \nabla + \beta M \right) \Psi \rangle + G \left\langle \overline{\psi} \psi \right\rangle^2 \right]$$

noting that $H_{Dirac} = -i\vec{\alpha} \cdot \nabla + \beta M$ and that Dirac field is expanded in a basis of H_{Dirac}:

$$\Psi(\vec{r}, t) = \sum_s \int \frac{d^3 k}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{r}} \left(v_s e^{iE(k)t} b^\dagger_{-\vec{k}, -s} + u_s e^{-iE(k)t} a_{\vec{k}, s} \right)$$

where $a^\dagger_{\vec{k},s}$ is the fermion creation operator(quark) with linear momentum $\vec{k}$ and spin s and $b^\dagger_{\vec{k},s}$ is the anti-fermion creation operator(antiquark) corresponding to the linear momentum $\vec{k}$ and spin s and $E(k) = \sqrt{k^2 + M^2}$. The operators $a_{\vec{k},s}$ and $b_{\vec{k},s}$ are the corresponding annihilation operators.

NJL in MFA - quark gas (fermions)

Substituting the expression for the field into the Dirac Hamiltonian operator, we can show that:

$$\frac{1}{V}\int d^3r\Psi^\dagger\left(-i\vec{\alpha}\cdot\nabla+\beta M\right)\Psi=\sum_\xi\int\frac{d^3p}{2\pi^3}\left(b^\dagger_{\vec{p},\xi}b_{\vec{p},\xi}+a^\dagger_{\vec{p},\xi}a_{\vec{p},\xi}-1\right)$$

The vacuum energy density can be calculated using the expression above for a quark gas at $T=0$:

$$\epsilon=<0|\sum_\xi\int\frac{d^3p}{2\pi^3}\left(b^\dagger_{\vec{p},\xi}b_{\vec{p},\xi}+a^\dagger_{\vec{p},\xi}a_{\vec{p},\xi}-1\right)|0> \quad + \; G\langle\overline{\psi}\psi\rangle^2$$

$$\epsilon=-\frac{2N_cN_f}{(2\pi)^3}\int_{|\vec{p}|<\Lambda}d^3p\,\sqrt{p^2+M^2}\;+\;G\langle\overline{\psi}\psi\rangle^2$$

$$=\;-\frac{N_cN_f}{8\pi^2}\left(2\Lambda E_\Lambda^3-M^2\Lambda E_\Lambda-M^4\ln\left[\frac{\Lambda+E_\Lambda}{M}\right]\right)+G\langle\overline{\psi}\psi\rangle^2$$

where $N_f=2$, $N_c=3$ and $E_\Lambda=\sqrt{\Lambda^2+M^2}$ and we introduce the cutoff Λ to regularize the integral.

NJL in MFA - quark gas (fermions)

usando que

$$M = m - 2G\left\langle\overline{\psi}\psi\right\rangle \rightarrow \left\langle\overline{\psi}\psi\right\rangle = -\frac{M-m}{2G}$$

Therefore, we can rewrite the energy density, ϵ, as:

$$\epsilon = \quad -\frac{N_c N_f}{8\pi^2}\left(2\Lambda E_\Lambda^3 - M^2\Lambda E_\Lambda - M^4 \ln\left[\frac{\Lambda + E_\Lambda}{M}\right]\right) + \frac{(M-m)^2}{4G}$$

Gap Equation

To obtain the Gap equation, we need to calculate

$$\left\langle\overline{\psi}\psi\right\rangle = \left\langle\psi^\dagger\gamma_0\psi\right\rangle$$

where

$$\psi = \sum_s \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}\left(v_s(-\vec{k})e^{iE(k)t}b^\dagger_{-\vec{k},-s} + u_s(\vec{k})e^{-iE(k)t}a_{\vec{k},s}\right)$$

$$\psi^\dagger = \sum_s \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{r}}\left(v_s^\dagger(-\vec{k})e^{-iE(k)t}b_{-\vec{k},-s} + u_s^\dagger(\vec{k})e^{iE(k)t}a^\dagger_{\vec{k},s}\right)$$

NJL in MFA - calculation of the gap equation

$$\left\langle \overline{\psi}\psi \right\rangle = \left\langle \psi^{\dagger}\gamma_0\psi \right\rangle = \sum_s \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{r}} \sum_{s'} \int \frac{d^3k'}{(2\pi)^{3/2}} e^{i\vec{k}'\cdot\vec{r}} \times$$

$$< 0| \left(v_s^{\dagger} e^{-iEt} b_{-\vec{k},-s} + u_s^{\dagger} e^{iEt} a^{\dagger}_{\vec{k},s} \right) \gamma_0 \left(v_{s'} e^{iEt} b^{\dagger}_{-\vec{k}',-s'} + u_{s'} e^{-iEt} a_{\vec{k}',s'} \right) |0>$$

simplifying

$$\left\langle \overline{\psi}\psi \right\rangle = \sum_s \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{r}} \sum_{s'} \int \frac{d^3k'}{(2\pi)^{3/2}} e^{i\vec{k}'\cdot\vec{r}} v_s^{\dagger}\gamma_0 v_{s'} < 0| b_{-\vec{k},-s} b^{\dagger}_{-\vec{k}',-s'} |0>$$

but,

$$< 0| b_{-\vec{k},-s} b^{\dagger}_{-\vec{k}',-s'} |0> = < 0| b_{-\vec{k},-s} b^{\dagger}_{-\vec{k}',-s'} + b^{\dagger}_{-\vec{k}',-s'} b_{-\vec{k},-s} |0> =$$

$$< 0| \{ b_{-\vec{k},-s} \, , \, b^{\dagger}_{-\vec{k}',-s'} \} |0> = \delta(\vec{k} - \vec{k}')\delta_{s\,s'}$$

$$\left\langle \overline{\psi}\psi \right\rangle = \sum_s \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{r}} \sum_{s'} \int \frac{d^3k'}{(2\pi)^{3/2}} e^{i\vec{k}'\cdot\vec{r}} v_s^{\dagger}(-\vec{k})\gamma_0 v_{s'}(-\vec{k}')\delta(\vec{k} - \vec{k}')\delta_{s\,s'}$$

NJL in MFA - calculation of the gap equation

$$\langle\overline{\psi}\psi\rangle=\sum_s\int\frac{d^3k}{(2\pi)^{3/2}}e^{-i\vec{k}\cdot\vec{r}}\sum_{s'}\int\frac{d^3k'}{(2\pi)^{3/2}}e^{i\vec{k}'\cdot\vec{r}}v_s^\dagger\gamma_0 v_{s'}\delta(\vec{k}-\vec{k}')\delta_{s\,s'}$$

$$\langle\overline{\psi}\psi\rangle=\sum_s\int\frac{d^3k}{(2\pi)^3}v_s^\dagger(-\vec{k})\gamma_0 v_s(-\vec{k})=\sum_s\int\frac{d^3k}{(2\pi)^3}\frac{E+M}{2E}$$

$$\times\quad\begin{bmatrix}\chi_s^\dagger & \chi_s^\dagger\frac{-\vec{\sigma}\cdot\vec{k}}{E+M}\end{bmatrix}\quad\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}\quad\begin{bmatrix}\chi_s\\ \frac{-\vec{\sigma}\cdot\vec{k}}{E+M}\chi_s\end{bmatrix}=$$

$$=\sum_s\int\frac{d^3k}{(2\pi)^3}\frac{E+M}{2E}\quad\begin{bmatrix}\chi_s^\dagger & \chi_s^\dagger\frac{-\vec{\sigma}\cdot\vec{k}}{E+M}\end{bmatrix}\quad\begin{bmatrix}\chi_s\\ \frac{\vec{\sigma}\cdot\vec{k}}{E+M}\chi_s\end{bmatrix}$$

$$=-\sum_s\int\frac{d^3k}{(2\pi)^3}\frac{E+M}{2E}\quad\begin{bmatrix}-\chi_s^\dagger\chi_s+ & \chi_s^\dagger\frac{\vec{\sigma}\cdot\vec{k}}{E+M}\frac{\vec{\sigma}\cdot\vec{k}}{E+M}\chi_s\end{bmatrix}$$

using that $\vec{\sigma}\cdot\vec{a}\,\vec{\sigma}\cdot\vec{b}=\vec{a}\cdot\vec{b}+\vec{\sigma}\cdot\vec{a}\times\vec{b}\rightarrow\frac{\vec{\sigma}\cdot\vec{k}}{E+M}\frac{\vec{\sigma}\cdot\vec{k}}{E+M}=\frac{k^2}{(E+M)^2}$

NJL in MFA - calculation of the gap equation

$$\langle \overline{\psi}\psi \rangle = -\sum_s \int \frac{d^3k}{(2\pi)^3} \frac{E+M}{2E} \chi_s^\dagger \chi_s (1 - \frac{k^2}{(E+M)^2})$$

$$\langle \overline{\psi}\psi \rangle = -\sum_s \int \frac{d^3k}{(2\pi)^3} \frac{E+M}{2E} \frac{(E+M)^2 - k^2}{(E+M)^2}$$

$$\langle \overline{\psi}\psi \rangle = -\sum_s \int \frac{d^3k}{(2\pi)^3} \frac{E+M}{2E} \frac{E^2 + 2EM + M^2 - k^2}{(E+M)^2}$$

$$\langle \overline{\psi}\psi \rangle = -\sum_s \int \frac{d^3k}{(2\pi)^3} \frac{E+M}{2E} \frac{k^2 + M^2 + 2EM + M^2 - k^2}{(E+M)^2}$$

$$\langle \overline{\psi}\psi \rangle = -\sum_s \int \frac{d^3k}{(2\pi)^3} \frac{E+M}{2E} \frac{2M(E+M)}{(E+M)^2} = -2 \int \frac{d^3k}{(2\pi)^3} \frac{M}{\sqrt{k^2 + M^2}}$$

therefore, we obtain the Gap Equation:

$$\langle \overline{\psi}\psi \rangle = -\frac{M-m}{2G} \rightarrow \frac{M-m}{2G} = 2N_f N_c \int \frac{d^3k}{(2\pi)^3} \frac{M}{\sqrt{k^2 + M^2}}$$

SU(2)-NJL model in the presence of a B field

NJL Lagrangian with two flavors:

$$\mathcal{L} = \overline{\psi}\left(i\not{D} - \tilde{m}\right)\psi + G\left[(\overline{\psi}\psi)^2 + (\overline{\psi}i\gamma_5\vec{\tau}\psi)^2\right] - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ - electromagnetic field tensor

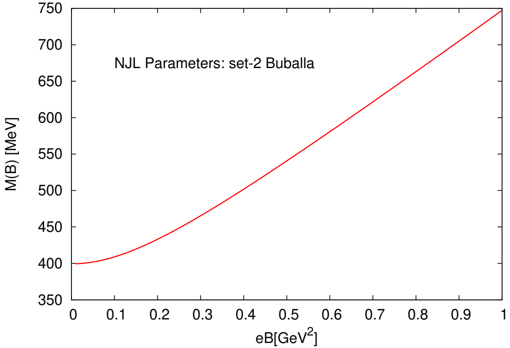
$D^\mu = (i\partial^\mu - QA^\mu)$ - covariant derivative (minimal coupling)
we work in Landau gauge $\rightarrow \vec{B} = B\hat{z}$. Using the prescription:

$$\frac{2}{(2\pi)^3}\int d^3p \; = \; \rightarrow \sum_{n=0}^{\infty} g_n \frac{eB}{(2\pi)^2}\int_{-\infty}^{\infty} dp_z$$

Thus, the Gap equation transforms into:

$$\frac{M-m}{2G} = N_c \sum_{q=u,d}\sum_{n=0}^{\infty} g_n \frac{|e_q|B}{(2\pi)^2}\int_{-\infty}^{\infty} dp_z \frac{M}{\sqrt{p_z^2 + M^2 + 2eBn}}$$

Gap equation - NJL

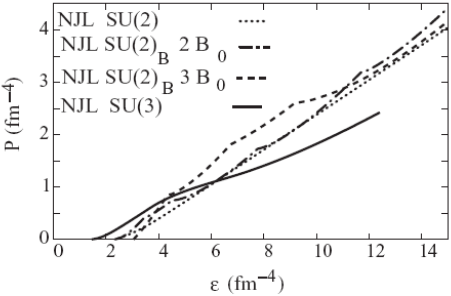


Effective mass increase with B
$\rightarrow$ magnetic catalysis effect (MC)

Refs: parameters NJL : M. Buballa, Physics Reports 407 (2005)205

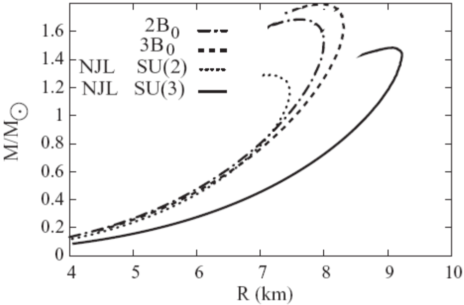
su(2)-NJL EOS: D. P. Menezes, M. Benghi Pinto, S. S. Avancini, A. Pérez Martinez and C. Providência, Phys. Rev. C 79, 035807 (2009).

NJL equation of state with two flavors



Equation of state using NJL model with two flavors.
$B_0 = 1 \times 10^{19}$Gauss

Mass-Radius diagram of a neutron star



Mass-radius diagram of a neutron star using the NJL model with two flavors. B_0 = 1×10^{19}Gauss

β-equilibrium is imposed $\rightarrow$ chemical equilibrium for the reaction: $n \rightleftharpoons p + e^-$

Thermodynamical properties of the model

The mean-field Hamiltonian for the quarks in second quantization is given by:

$$H^{MFA} = \sum_{q=u,d} \sum_{n=0}^{\infty} \sum_{s=\pm 1} \sum_{p_2} \sum_{p_3} \sqrt{M^2 + p_3^2 + 2|Q_q|Bn} \left(\hat{a}^{q\,\dagger}_{nsp_2p_3} \hat{a}^{q}_{nsp_2p_3} + \hat{b}^{q\,\dagger}_{nsp_2p_3} \hat{b}^{q}_{nsp_2p_3} - 1 \right)$$

Grand canonical partition function:

$$Z = Tr[e^{-\beta(H^{MFA} - \sum_q \mu_q \hat{N}_q}] \quad , \quad \Omega = -\frac{1}{\beta} \ln Z \quad ,$$

where β=1/T. Thermodynamic quantities are related to Ω through the following relations:

$$\Omega = \Omega(T, V, \mu_q, \mu_l) = E - TS - \sum_q \mu_q \bar{N}_q - \sum_l \mu_l \bar{N}_l \ ,$$

$$\Omega(T, V, \mu_q) = -PV \ , \ F = \Omega(T, V, \mu_q, \mu_l) + \sum_q \mu_q \bar{N}_q \ , \qquad (1)$$

where $F = E - TS$is the Helmholtz free energy and the average number of particles is obtained from the expression:

$$\bar{N}_\alpha = \frac{1}{\beta} \frac{\partial \ln Z}{\partial \mu_\alpha} = -\frac{\partial \Omega}{\partial \mu_\alpha} \ .$$

Due to the particular form of the mean-field Hamiltonian, we have:

$$Z = Tr[e^{-\beta(H^{MFA} - \sum_q \mu_q \hat{N}_q)}] = e^{-\beta V(G\sigma^2 + \frac{1}{2}B^2)} Tr[e^{-\beta(\bar{H}^{MFA} - \sum_q \mu_q \hat{N}_q)}] ,$$

where $\bar{H}^{MFA}$ It corresponds to the NJL model Hamiltonian without $VG\sigma^2$ and $V\frac{1}{2}B^2$. The representation of the occupation numbers in terms of the quark (n_{q_r}) and antiquark ($\bar{n}_{q_r}$) occupation numbers can be written as:

$$|\tilde{\alpha}\rangle = \left|n_{q_1}, n_{q_2} ...; \bar{n}_{q_1}, \bar{n}_{q_2}, ...\right\rangle \text{ onde } \mathrm{n_{q_r}}, \mathrm{\bar{n}_{q_r}} = 0, 1, ..., \infty \text{ e r} = 1, 2, ..\infty ,$$

We order, for example, the set of independent quark particle states according to the rule:

$$\{n_{q_r}\} = \{n_{q_{nsp_2p_3}}\} = (n_{q_1}, n_{q_2}, ..., n_{l_\infty}) ,$$

$$Tr[e^{-\beta(\bar{H}^{MFA} - \sum_q \mu_q \hat{N}_q)}] = \sum_{\tilde{\alpha}} \langle\tilde{\alpha}|\, e^{-\beta(\bar{H}^{MFA} - \sum_q \mu_q \hat{N}_q)}\, |\tilde{\alpha}\rangle$$

$$= e^{\beta \sum_{q,r} E_r^q} \times e^{-\beta \sum_{q,r,n_{qr}} (E_r^q - \mu_q) n_{qr}} e^{-\beta \sum_{q,r,\bar{n}_{qr}} (E_r^q + \mu_q) \bar{n}_{qr}} .$$

Fermion occupation numbers can only take the values 0 or 1, and therefore we can write it using products:

$$Z = e^{-\beta V(G\sigma^2 + \frac{1}{2}B^2)} e^{\beta \sum_{q,r} E_r^q} \times \prod_{q,r} \left(1 + e^{-\beta(E_r^q - \mu_q)}\right) \prod_{q,r} \left(1 + e^{-\beta(E_r^q + \mu_q)}\right)$$

From the partition function, we can obtain the grand canonical thermodynamic potential:

$$\begin{aligned}\Omega_Q &= -\frac{1}{\beta}\ln Z = V(G\sigma^2+\frac{1}{2}B^2)-\sum_{q,r}E_r^q \\ &- \frac{1}{\beta}\sum_{q,r}\ln\left(1+e^{-\beta(E_r^q-\mu_q)}\right)-\frac{1}{\beta}\sum_{q,r}\frac{1}{\beta}\ln\left(1+e^{-\beta(E_r^q+\mu_q)}\right) .\end{aligned}$$

We have already shown that:

$$\sum_r = \sum_{n,s,p_2,p_3} \Rightarrow V\sum_n g_n\frac{|Q|B}{(2\pi)^2}\int_{-\infty}^{\infty}dp_3 .$$

The grand canonical potential can be written as:

$$\begin{aligned}\omega_Q &= \frac{\Omega_Q}{V}=\omega_Q(0,B)+\frac{1}{2}B^2 \\ &- \frac{1}{\beta}\sum_{q,n}g_n\frac{N_c|Q_q|B}{(2\pi)^2}\int_{-\infty}^{\infty}dp_3\left(\ln\left(1+e^{-\beta(E^q-\mu_q)}\right)+\ln\left(1+e^{-\beta(E^q+\mu_q)}\right)\right) .\end{aligned}$$

$$\begin{aligned}\omega_Q(0,B) &= G\sigma^2-\sum_{q,n}g_n\frac{N_c|Q_q|B}{(2\pi)^2}\int_{-\infty}^{\infty}dp_3\sqrt{M^2+p_3^2+2|Q_q|Bn} \\ &= \frac{(M-m_c)^2}{4G}+N_c\sum_{q=u,d}I_1^q(B)=\frac{(M-m_c)^2}{4G}+\Omega_{T=0}^{(1-Loop)} \quad , \quad N_c=3.\end{aligned}$$

Interesting expression

$$I_1^q(B) \quad = \quad -\sum_{n=0}^{\infty}(2-\delta_{n0})\frac{|Q_q|B}{(2\pi)^2}\int_{-\infty}^{\infty} dp_3\sqrt{M^2+p_3^2+2|Q_q|Bn}\,.$$

This contribution essentially corresponds to the vacuum energy, that is, to the expectation value of the quark Hamiltonian in the vacuum state:

$$I_1^q(B) = \left\langle 0 \left| \frac{H_q}{V} \right| 0 \right\rangle$$

$$= \left\langle 0 \left| \frac{1}{V}\sum_{n=0}^{\infty}\sum_{s=\pm 1}\sum_{p_2}\sum_{p_3} \sqrt{M^2+p_3^2+2|Q_q|Bn}\left(\hat{a}^{q\,\dagger}_{nsp_2p_3}\hat{a}^{q}_{nsp_2p_3}+\hat{b}^{q\,\dagger}_{nsp_2p_3}\hat{b}^{q}_{nsp_2p_3}-1\right)\right| 0 \right\rangle$$

The contributions in I_1^q are clearly divergent and need to be regularized. We will rewrite them in a more convenient form using the generalized Riemann zeta function or the Hurwitz-Riemann zeta function:

$$\zeta(z,x) = \sum_{n=0}^{\infty}\frac{1}{(x+n)^z}\,.$$

We can rewrite I_1^q as the following:

$$
\begin{aligned}
I_1^q(B) &= -\frac{(2|Q_q|B)^{\frac{3}{2}}}{(2\pi)^2}\int_{-\infty}^{\infty} dp_3 \sum_{n=0}^{\infty}\sqrt{\left(\frac{M^2+p_3^2}{2|Q_q|B}+n\right)} + \frac{|Q_q|B}{(2\pi)^2}\int_{-\infty}^{\infty} dp_3\sqrt{M^2+p_3^2} \\
&= -\frac{(2|Q_q|B)^{\frac{3}{2}}}{(2\pi)^2}\int_{-\infty}^{\infty} dp_3\, \zeta(-\frac{1}{2},\frac{M^2+p_3^2}{2|Q_q|B}) + \frac{|Q_q|B}{(2\pi)^2}\int_{-\infty}^{\infty} dp_3\sqrt{M^2+p_3^2}\,.
\end{aligned}
$$

Using the integral representation of the zeta function:

$$
\int_0^{\infty} dyy^{z-1}\exp[-\beta y]\coth(\alpha y) = \Gamma[z]\left\{2^{1-z}\alpha^{-z}\zeta(z,\frac{\beta}{2\alpha}) - \beta^{-z}\right\}\,, \qquad (2)
$$

Making the identification:

$$
\alpha = |Q_q|\,B\,,\ \beta = M^2+p_3^2\,,\ z=-\frac{1}{2}\,,
$$

we obtain:

$$
\begin{aligned}
I_1^q(B) = &-\frac{(2|Q_q|B)^{\frac{3}{2}}}{(2\pi)^2}\int_{-\infty}^{\infty} dp_3\,\frac{1}{2^{3/2}(|Q_q|B)^{1/2}} \\
&\times\left\{\frac{1}{\Gamma(-1/2)}\int_0^{\infty} dyy^{-3/2}\exp[-(M^2+p_3^2)y]\coth(|Q_q|By) + \sqrt{M^2+p_3^2}\right\} \\
+ &\quad \frac{|Q_q|B}{(2\pi)^2}\int_{-\infty}^{\infty} dp_3\sqrt{M^2+p_3^2}\,,
\end{aligned}
$$

using that $\Gamma(-1/2) = -2\pi^{1/2}$ we can rewrite the last expression as the following:

$$I_1^q(B) \quad = \quad \frac{|Q_q|B}{(2\pi)^2 2\pi^{1/2}} \int_{-\infty}^{\infty} dp_3 \int_0^{\infty} dy y^{-3/2} \exp[-(M^2 + p_3^2)y] \coth(|Q_q|By) \, .$$

The p_3 integration can be easily performed:

$$\int_{-\infty}^{\infty} dp_3 \exp[-p_3^2 y] = \frac{1}{y^{1/2}} \int_{-\infty}^{\infty} dp \exp[-p^2] = \frac{\pi^{1/2}}{y^{1/2}} \, .$$

The final results for I_1^q is the following:

$$I_1^q(B) = \frac{|Q_q|B}{8\pi^2} \int_0^{\infty} dy \frac{e^{-M^2 y}}{y^2} \coth(|Q_q|By) = \frac{B_q}{8\pi^2} \int_0^{\infty} dy \frac{e^{-M^2 y}}{y^2} \coth(B_q y), B_q = |Q_q|B \, .$$

The integration $I_1^q(B)$ is clearly divergent and needs to be regularized.

$$\boxed{\Omega_{T=0}^{(1-Loop)} \equiv N_c \sum_{f=u,d} I_1^q(B) = \frac{N_c}{8\pi^2} \sum_{f=u,d} \int_0^{\infty} \frac{dy}{y^3} e^{-yM^{\,2}} \; B_q y \coth(B_q y) \quad \leftarrow \text{(divergent if y} \to 0)} \, .$$

The origin of the divergences can be understood by using the Taylor series expansion of the function:

$$B_q y \coth(B_q y) \sim 1 + \frac{(B_q y)^2}{3} + \frac{(B_q y)^4}{45} + O[(B_q y)^6] \quad ,$$

$\Rightarrow$ **To regularize the effective potential, we need to perform two subtractions.**

1-Loop efective potential - MFIR regularization

$$\Omega^{(1-Loop)}_{T=0} \equiv \frac{N_c}{8\pi^2}\sum_{q=u,d}\left\{\underbrace{\int_0^\infty \frac{dy}{y^3}e^{-yM^2}\left[B_q y\coth(B_q y) - 1 - \frac{(B_q y)^2}{3}\right]}_{finite}\right.$$

$$\left. + \underbrace{\int_0^\infty \frac{dy}{y^3}e^{-yM^2}}_{infinity} + \frac{B_q^2}{3}\underbrace{\int_0^\infty \frac{dy}{y}e^{-yM^2}}_{infinity}\right\}$$

$$\Omega^{(mag)}_{T=0} = \frac{N_c}{8\pi^2}\sum_{q=u,d}\int_0^\infty \frac{dy}{y^3}e^{-yM^2}\left[B_q y\coth(B_q y) - 1 - \frac{(B_q y)^2}{3}\right]$$

$$\Omega^{(vac)}_{T=0} = \frac{N_c}{8\pi^2}\sum_{q=u,d}\int_0^\infty \frac{dy}{y^3}e^{-yM^2} \to -\frac{N_c}{\pi^2}\sum_{q=u,d}\int_0^\Lambda p^2\sqrt{M^2+p^2}\,,$$

$$\Omega^{(field)}_{T=0} = \frac{N_c}{24\pi^2}\sum_{q=u,d}B_q^2\int_0^\infty \frac{dy}{y}e^{-yM^2} \to \frac{N_c}{24\pi^2}\sum_{q=u,d}B_q^2\int_{1/\Lambda^2}^\infty \frac{dy}{y}e^{-yM^2}$$

$$= \frac{N_c}{24\pi^2}\sum_{q=u,d}B_q^2\Gamma\left[0,\frac{M^2}{\Lambda^2}\right] \sim -\frac{N_c}{24\pi^2}\sum_{q=u,d}B_q^2\left[\ln\left(\frac{M^2}{\Lambda^2}\right)+\gamma_E\right]$$

$$\boxed{\Omega^{(1-Loop)}_{T=0} = \Omega^{(mag)}_{T=0} + \Omega^{(vac)}_{T=0} + \Omega^{(field)}_{T=0}}\,.$$

(S. S. Avancini, R. L. S. Farias, M. B. Pinto, T. E. Restrepo and W. Tavares, Phys. Rev. D 103, 056009 (2021).)

Conferences in Brazil

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XXII Escola de Verão Jorge André Swieca de Física Nuclear Teórica

April 28 to May 2 (2025), Niterói, RJ, Brazil

CA1: A modern description of dense matter
Palestrante: Veronica Dexheimer (Kent State University, EUA)

CA2: Hot and dense QCD in colliders
Palestrante: Carlos Alberto Salgado (Universidade de Santiago de Compostela, Espanha)

CA3: Effective Field Theories
Palestrante: Laura Tolos (Institute of Space Sciences, Espanha)

CA4: Nuclear reactions
Palestrante: Chloe Hebborn (Michigan State Uni - EUA)

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Thank you for your attention!