Charged particles in strong magnetic fields

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PLAN OF THE LECTURES

- Lecture 1: Order of magnitude of B in different contexts. Units of B in different unit systems. Charged particles in a magnetic field: Classical case and non- relativistic quantum case.

- Lecture 2: Charged particles in a magnetic field: Relativistic quantum case. Charged fields and propagators for spin 0, ½ and 1

- Lecture 3: Schwinger phase (SP). SP and charged particle propagators. Connection between different representations of charged particle propagators. An example: Leading Order Correction to the charged pion propagator.

Notation and general definitions

In what follows we use the Minkowski metric $g^{\mu\nu}$ = diag(1,-1,-1,-1) while for a space-time coordinate four-vector x^{μ} we adopt the notation $x^{\mu} = (t, \vec{x})$ with $\vec{x} = (x^1, x^2, x^3)$. Moreover, the symbol e denotes the proton electric charge. Recall $\partial^{\mu} = (\frac{\partial}{\partial x^0}, -\vec{\nabla})$

The electromagnetic field strength $F^{\mu\nu}$ associated with a general electromagnetic field $\mathcal{A}^{\mu}$ is

$$F^{\mu\nu} = \partial^{\mu}\mathcal{A}^{\nu} - \partial^{\nu}\mathcal{A}^{\mu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

And its dual is

$$F_*^{\mu\nu} = \frac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\,F_{\alpha\beta} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & -B_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}$$

Next, we introduce the covariant derivative $\mathcal{D}^\mu$ appearing in the field equation associated with particles with electric charge Q . It is given by

$$\mathcal{D}^\mu = \partial^\mu + i\, Q\, \mathcal{A}^\mu(x)$$

Under a gauge transformation $\Lambda(x)$ the electromagnetic field transforms as

$$\mathcal{A}^\mu(x) \rightarrow \tilde{\mathcal{A}}^\mu(x) = \mathcal{A}^\mu(x) + \partial^\mu \Lambda(x)$$

While $F^{\mu\nu}$ is invariant under this transformation, $\mathcal{D}^\mu$ transforms in a covariant way. Namely,

$$\mathcal{D}^\mu \rightarrow \tilde{\mathcal{D}}^\mu = e^{-i\, Q\, \Lambda(x)}\; \mathcal{D}^\mu\; e^{i\, Q\, \Lambda(x)}$$

So far we have considered a general external electromagnetic field $\mathcal{A}^\mu$. Now we concentrate on the case associated with a constant and uniform magnetic field $\vec{B}$

The corresponding electromagnetic field strength $F^{\mu\nu}$ is

$$F^{ij} = F_{ij} = -\epsilon_{ijk}\, B^k \qquad ; \qquad F^{0j} = 0$$

where $i, j = 1,2,3$. The associated electromagnetic field in an arbitrary gauge is given by

$$\mathcal{A}^\mu(x) = \frac{1}{2} x_\nu F^{\nu\mu} + \partial^\mu \Psi(x) \ ,$$

where $\Psi(x)$ is, in principle, an arbitrary function.

For a given form of $\Psi(x)$ one obtains a particular gauge. Without loosing generality one can take $\vec{B}$ along the 3-axis. In addition, $\Psi(x)$ can be taken to depend only on x^1, x^2 since only the components F^{12} = - $F^{21} \neq 0$, which implies that only ∂^1A^2 and ∂^2A^1 are relevant. Some commonly used ("standard") gauges are

Symmetric gauge (SG),	$\Psi(x) = 0\ ,$	$\mathcal{A}^\mu(x) = \left(0, -\frac{B}{2}\, x^2, \frac{B}{2}\, x^1, 0\right)$
Landau gauge 1 (LG1),	$\Psi(x) = \frac{B}{2}\, x^1 x^2\ ,$	$\mathcal{A}^\mu(x) = (0, -B\, x^2, 0, 0)$
Landau gauge 2 (LG2),	$\Psi(x) = -\frac{B}{2}\, x^1 x^2\ ,$	$\mathcal{A}^\mu(x) = (0, 0, B\, x^1, 0)$

A basic set of functions

We introduce the scalar functions $\mathcal{F}_Q\,(x,\bar{q})$ which are solutions of the eigenvalue equation

$$\mathcal{D}^\mu\,\mathcal{D}_\mu\,\mathcal{F}_Q(x,\bar{q}) \;=\; f_{\bar{q}}\,\mathcal{F}_Q(x,\bar{q})$$

$$\mathcal{D}^\mu = \partial^\mu + i\;Q\;\mathcal{A}^\mu(x)$$

$$\mathcal{A}^\mu(x) = \frac{1}{2}x_\nu F^{\nu\mu} + \partial^\mu\Psi(x)\;,$$

$\bar{q}$ stands for a set of 4 labels that are needed to completely specify each eigenfunction

One can be more explicit and write the eigenvalue equation in the form

$$\left[\partial^\mu\,\partial_\mu \;-\; Q\,\vec{B}\cdot\vec{L} \;+\; \frac{Q^2}{4}\left(\vec{x}\times\vec{B}\right)^2\right] e^{iQ\Psi(x)}\,\mathcal{F}_Q(x,\bar{q}) \;=\; f_{\bar{q}}\,e^{iQ\Psi(x)}\,\mathcal{F}_Q(x,\bar{q})$$

where $L^k = i\,\varepsilon_{klm}\,x_l\partial_m$. We see that while the eigenfunctions $\mathcal{F}_Q\,(x,\bar{q})$ are gauge dependent, the eigenvalues $f_{\bar{q}}$ are not.

As mentioned, we can always take B along the 3-axis, and $\Psi(x^1,x^2)$. Thus, as in the case of a free particle, the eigenvalues of the components of the four-momentum along the time direction, q^0, and the magnetic field direction, q^3, can be taken as two of the labels required to specify $\mathcal{F}_Q\,(x,\bar{q})$

From what was shown in previous lecture we have

$$f_{\bar{q}} = -\left[(q^0)^2 - (2k+1)B_Q - (q^3)^2\right]$$

where $B_Q = |QB|$ and k is a non-negative integer, to be related with the Landau level. The eigenvalues depend only on three of the labels included in $\bar{q}$. As we know there is a degeneracy, which arises as a consequence of gauge invariance; to fully specify the eigenfunctions, a fourth quantum number χ is required. One has $\bar{q} = (q^0, k, \chi, q^3)$

The quantum number χ can be conveniently chosen according to the gauge in which the eigenvalue problem is analyzed. In particular, since for the standard gauges SG, LG1 and LG2 one has unbroken continuous symmetries, it is natural to consider quantum numbers χ associated with the corresponding group generators.

SG:	$\chi = \imath$,	nonnegative integer, associated to L^3 (eigenvalue of L^3: $m = \mathrm{sign}(QB)(\imath - k)$)
LG1:	$\chi = q^1$,	real number, eigenvalue of $-i\dfrac{\partial}{\partial x^1}$
LG2:	$\chi = q^2$,	real number, eigenvalue of $-i\dfrac{\partial}{\partial x^2}$

The explicit form of $\mathcal{F}_Q\,(x,\bar{q})$ in the standard gauges are $[B_Q = |QB|,\ s = \text{sign(QB)}]$

Sym Gauge (SG)

$$\mathcal{F}_Q(x,\bar{q})^{(\mathrm{SG})} = \sqrt{2\pi}\; e^{-i(q^0 x^0 - q^3 x^3)}\, e^{-is(k-\imath)\phi}\, R_{k,\imath}(\rho)\ ,$$

$$\bar{q} = (q^0, k, \imath, q^3)$$

where

$$R_{k,\imath}(\rho) = N_{k,\imath}\, \xi^{(k-\imath)/2}\, e^{-\xi/2}\, L_{\imath}^{k-\imath}(\xi)$$

with $\xi = B_Q \rho^2/2$, $N_{k\imath} = \left(B_Q\, \imath!/k!\right)^{1/2}$. Here, $L_j^m(x)$ are generalized Laguerre polynomials.

Landau Gauge (LG1, LG2)

$$\mathcal{F}_Q(x,\bar{q})^{(\mathrm{LG1})} = (-is)^k\, N_k\, e^{-i(q^0 x^0 - q^1 x^1 - q^3 x^3)}\, D_k(\rho_s^{(1)}) \qquad \bar{q} = (q^0, k, q^1, q^3)$$

$$\mathcal{F}_Q(x,\bar{q})^{(\mathrm{LG2})} = N_k\, e^{-i(q^0 x^0 - q^2 x^2 - q^3 x^3)}\, D_k(\rho_s^{(2)}) \qquad \bar{q} = (q^0, k, q^2, q^3)$$

where $\rho_s^{(1)} = \sqrt{2B_Q}\left(x^2 + \frac{sq^1}{B_Q}\right), \rho_s^{(2)} = \sqrt{2B_Q}\left(x^1 - \frac{sq^2}{B_Q}\right)$ and $N_k = \left(4\pi B_Q\right)^{1/4}/\sqrt{k!}$

The cylindrical parabolic functions $D_k(x)$ are defined as

$$D_k(x) = 2^{-k/2}\, e^{-x^2/4}\, H_k(x/\sqrt{2})$$

$H_k(x)$ are Hermite polynomials, with the standard convention $H_{-1}(x) = 0$

$\mathcal{F}_Q(x,\bar{q})$ satisfy the completeness and orthogonality relations

$$\sum\!\!\!\!\!\!\int_{\bar{q}} \mathcal{F}_Q(x,\bar{q})^* \, \mathcal{F}_Q(y,\bar{q}) = \delta^{(4)}(x-y) \, ,$$

$$\int d^4x \, \mathcal{F}_Q(x,\bar{q}')^* \, \mathcal{F}_Q(x,\bar{q}) = \hat{\delta}_{\bar{q}\bar{q}'} \, .$$

We have introduced some shorthand notation whose explicit form depends on the chosen gauge. For SG we have

$$\sum\!\!\!\!\!\!\int_{\bar{q}} \equiv \frac{1}{(2\pi)^2} \sum_{k,\imath=0}^{\infty} \int \frac{dq^0}{2\pi}\frac{dq^3}{2\pi} \, , \quad \hat{\delta}_{\bar{q}\bar{q}'} \equiv (2\pi)^4 \, \delta_{kk'} \, \delta_{\imath\imath'} \, \delta(q^0-q'^0) \, \delta(q^3-q'^3)$$

While for LGi (with i=1,2)

$$\sum\!\!\!\!\!\!\int_{\bar{q}} \equiv \frac{1}{2\pi} \sum_{k=0}^{\infty} \int \frac{dq^0}{2\pi}\frac{dq^i}{2\pi}\frac{dq^3}{2\pi} , \quad \hat{\delta}_{\bar{q}\bar{q}'} \equiv (2\pi)^4 \, \delta_{kk'} \, \delta(q^0-q'^0) \, \delta(q^i-q'^i) \, \delta(q^3-q'^3)$$

For later use we define $\breve{q} = (k, \chi, q^3)$ and

$$\sum\!\!\!\!\!\!\int_{\{\bar{q}_E\}} = \sum\!\!\!\!\!\!\int_{\bar{q}} 2\pi\delta(q^0-E)$$

It can be shown that $\mathcal{F}_Q(x,\bar{q})$ satisfy the useful relations

$$\begin{aligned} \mathcal{D}^0\,\mathcal{F}_Q(x,\bar{q}) &= -iq^0\,\mathcal{F}_Q(x,\bar{q}) \\ \left(\mathcal{D}^1 \pm i\mathcal{D}^2\right)\mathcal{F}_Q(x,\bar{q}) &= (\mp s)\left[(2k+1\mp s)B_Q\right]^{1/2}\mathcal{F}_Q(x,\bar{q}_{k\mp s}) \\ \mathcal{D}^3\,\mathcal{F}_Q(x,\bar{q}) &= -iq^3\,\mathcal{F}_Q(x,\bar{q}) \end{aligned}$$

where $\bar{q}_{k\pm s} = (q^0, k\pm s, \chi, q^3)$

Under a gauge transformation $\Lambda(x)$ the functions $\mathcal{F}_Q(x,\bar{q})$ transform as

$$\mathcal{F}_Q(x,\bar{q}) \;\rightarrow\; \tilde{\mathcal{F}}_Q(x,\bar{q}) = e^{-iQ\Lambda(x)}\,\mathcal{F}_Q(x,\bar{q})$$

Spin 0 charged particles: the charged pions

The gauged Klein-Gordon action for a point-like charged pion in the presence of a static and homogeneous magnetic field is

$$\mathcal{S}_{KG} = -\int d^4x \; \pi^{\mathcal{Q}}(x)^* \left(\mathcal{D}^\mu \mathcal{D}_\mu + m_\pi^2\right) \pi^{\mathcal{Q}}(x)$$

$$Q_\pi = \mathcal{Q}\, e, \text{with } \mathcal{Q} = \pm 1$$

The associated gauged Klein-Gordon equation is

$$\left(\mathcal{D}^\mu \mathcal{D}_\mu + m_\pi^2\right) \pi^{\mathcal{Q}}(x) = 0$$

Gauge invariance of the gauged Klein-Gordon action requires that under a gauge transformation $\Lambda(x)$ the $\pi^Q(x)$ field transforms as

$$\pi^{\mathcal{Q}}(x) \;\rightarrow\; \tilde{\pi}^{\mathcal{Q}}(x) = e^{-iQ_\pi \Lambda(x)}\, \pi^{\mathcal{Q}}(x)$$

The quantized charged pion field can be written as

$$\pi^{\mathcal{Q}}(x) \;=\; \pi^{-\mathcal{Q}}(x)^{\dagger} \;=\; \sum\!\!\!\!\!\!\!\int_{\{\bar{q}_{E_\pi}\}} \frac{1}{2E_\pi}\Big\{ a_\pi^{\mathcal{Q}}(\breve{q})\, \mathbb{F}^{\mathcal{Q}}(x,\bar{q}) + a_\pi^{-\mathcal{Q}}(\breve{q})^{\dagger}\, \mathbb{F}^{-\mathcal{Q}}(x,\bar{q})^{*} \Big\}$$

Here the pion energy is given by $E_\pi = \sqrt{m_\pi^2 + (2k+1)B_\pi + (q^3)}$, $k \geq 0$, while the functions $\mathbb{F}^{\mathcal{Q}}(x,\bar{q})$ are given by

$$\mathbb{F}^{\mathcal{Q}}(x,\bar{q}) = \mathcal{F}_{Q_\pi}(x,\bar{q})$$

They satisfy the completeness and orthogonality relations obtained for $\mathcal{F}_{\mathrm{Q}}\,(x,\bar{q})$

The creation and annihilation operators satisfy the commutation relations

$$\big[\, a_\pi^{\mathcal{Q}}(\breve{q})\,,\, a_\pi^{\pm\mathcal{Q}}(\breve{q}\,')\,\big] \;=\; \big[\, a_\pi^{\mathcal{Q}}(\breve{q})^{\dagger}\,,\, a_\pi^{\pm\mathcal{Q}}(\breve{q}\,')^{\dagger}\,\big] \;=\; \big[\, a_\pi^{\mathcal{Q}}(\breve{q})\,,\, a_\pi^{-\mathcal{Q}}(\breve{q}\,')^{\dagger}\,\big] \;=\; 0\;,$$

$$\big[\, a_\pi^{\mathcal{Q}}(\breve{q})\,,\, a_\pi^{\mathcal{Q}}(\breve{q}\,')^{\dagger}\,\big] \;=\; 2E_\pi\,(2\pi)^3\,\delta_{kk'}\,\delta_{\chi\chi'}\,\delta(q^3 - q'^{\,3})\;.$$

The equation that defines the charged pion meson propagator $\triangle_{\pi^Q}(x, y)$ is

$$\left(\mathcal{D}^{\mu}\,\mathcal{D}_{\mu} + m_{\pi}^{2}\right)\Delta_{\pi^{\mathcal{Q}}}(x, y) = -\,\delta^{(4)}(x - y)\ .$$

It can be shown that

$$\Delta_{\pi^{\mathcal{Q}}}(x, y) \quad = \quad \sum\!\!\!\!\!\!\int_{\bar{q}} \mathbb{F}^{\mathcal{Q}}(x, \bar{q})\,\hat{\Delta}_{\pi^{\mathcal{Q}}}(k, q_{\|})\,\mathbb{F}^{\mathcal{Q}}(y, \bar{q})^{*}$$

Here $\hat{\Delta}_{\pi^Q}(k, q_{\|}) = 1/\left[q_{\|}^{2} - m_{\pi}^{2} - (2k+1)B_{\pi} + i\epsilon\right]$ with $q_{||} = (q^0, q^3)$ and $q_{||}^2 = (q^0)^2 - (q^3)^2$

This can be proven using

$$\left(\mathcal{D}^{\mu}\,\mathcal{D}_{\mu} + m_{\pi}^{2}\right)\mathbb{F}^{\mathcal{Q}}(x, \bar{q}) = -\left[(q^{0})^{2} - m_{\pi}^{2} - (2k+1)B_{\pi} - (q^{3})^{2}\right]\mathbb{F}^{\mathcal{Q}}(x, \bar{q})$$

and

$$\sum\!\!\!\!\!\!\int_{\bar{q}} \mathbb{F}^{\mathcal{Q}}(x, \bar{q})\ \mathbb{F}^{\mathcal{Q}}(y, \bar{q})^{*} = \delta^{(4)}(x - y)$$

We say that the charged pion propagator is diagonal in $\bar{q}$-space (or “Ritus space”)

Under a gauge transformation $\triangle_{\pi^Q}(x, y)$ transforms as

$$\Delta_{\pi^{\mathcal{Q}}}(x, y) \ \rightarrow\ \tilde{\Delta}_{\pi^{\mathcal{Q}}}(x, y) = e^{-iQ_{\pi}\Lambda(x)}\ \Delta_{\pi^{\mathcal{Q}}}(x, y)\ e^{iQ_{\pi}\Lambda(y)}$$

Spin 1/2 charged particles: the quarks

We consider the gauged Dirac action for a point-like quark of flavor f in the presence of a static and homogeneous magnetic field. We express the quark charge as $Q_f = \mathcal{Q}\, e$ with $\mathcal{Q}_u = 2/3$, $\mathcal{Q}_d = -1/3$ for $f = u, d$

$$\mathcal{S}_D = \int d^4x\, \bar{\psi}_f(x)\,(i\not{\mathcal{D}} - m_f)\,\psi_f(x)$$

$$\bar{\psi}_f = \psi_f^\dagger \gamma^0$$

$$\not{\mathcal{D}} = \gamma_\mu \mathcal{D}^\mu$$

The associated gauged Dirac equation reads

$$(i\not{\mathcal{D}} - m_f)\,\psi_f(x) = 0$$

Gauge invariance of the gauged Dirac action requires that under a gauge transformation $\Lambda(x)$ the $\psi_f(x)$ field transforms as

$$\psi_f(x) \;\to\; \tilde{\psi}_f(x) = e^{-iQ_f\Lambda(x)}\,\psi_f(x)\;.$$

The quantized quark fields are given by

$$\psi_f(x) = \sum\!\!\!\!\!\!\int_{\{\bar{q}_{E_f}\}} \sum_{a=1,2} \frac{1}{2E_f} \left\{ b_f(\breve{q},a)\, U_f(x,\bar{q},a) + d_f(\breve{q},a)^\dagger\, V_f(x,\bar{q},a) \right\}$$

where the quark energy is given by $E_f = \sqrt{m_f^2 + 2kB_\pi + (q^3)}$ with $k > 0$. For $k = 0$ only the value $a = 1$ in the sum over a is allowed.

The spinors U_f and V_f can be written as

$$U_f\,(x,\bar{q},a) = \mathbb{E}^{\mathcal{Q}_f}(x,\bar{q})\; u_{\mathcal{Q}_f}(k,q^3,a)\ ,$$
$$V_f\,(x,\bar{q},a) = \tilde{\mathbb{E}}^{-\mathcal{Q}_f}(x,\bar{q})\; v_{-\mathcal{Q}_f}(k,q^3,a)\ ,$$

$\mathbb{E}_{Qf}\,(x,\bar{q})$ and $\tilde{\mathbb{E}}_{Qf}\,(x,\bar{q})$ are Ritus functions. Their explicit forms are

$$\mathbb{E}^{\mathcal{Q}}(x,\bar{q}) = \sum_{\lambda=\pm} \Gamma^\lambda\, \mathcal{F}_Q(x,\bar{q}_\lambda)$$
$$\tilde{\mathbb{E}}^{-\mathcal{Q}}(x,\bar{q}) = \sum_{\lambda=\pm} \Gamma^\lambda\, \mathcal{F}_{-Q}(x,\bar{q}_{-\lambda})^*$$

$$\bar{q}_\lambda = (q^0, k_{s\lambda}, \chi, q^3)$$
$$k_{s\pm} = k - (1 \mp s)/2$$

Here, $\Gamma^\pm = (1 \pm S_3)/2$ where $S_3 = i\,\gamma^1\gamma^2$ is the 3-component of the spin operator in the spin one-half representation

The functions $\mathbb{E}^{\mathcal{Q}_f}$ satisfy the orthogonality and completeness relations

$$\int d^4x\, \bar{\mathbb{E}}^{\mathcal{Q}_f}(x,\bar{q})\,\mathbb{E}^{\mathcal{Q}_f}(x,\bar{q}') = \hat{\delta}_{\bar{q}\bar{q}'}\left[\mathcal{I}+\delta_{k0}\left(\Gamma^s-\mathcal{I}\right)\right]$$

$$\sum\!\!\!\!\!\!\int_{\bar{q}} \mathbb{E}^{\mathcal{Q}_f}(x,\bar{q})\,\bar{\mathbb{E}}^{\mathcal{Q}_f}(x',\bar{q}) = \delta^{(4)}(x-x')\,\mathcal{I}$$

$$\bar{\mathbb{E}}^{Q_f}(x,\bar{q}') = \gamma^0\left[\mathbb{E}^{Q_f}(x,\bar{q})\right]^\dagger\gamma^0$$

$$\mathcal{Q}_f = Q_f/e$$

$$s = \mathrm{sign}(Q_f B)$$

$$\mathcal{I} = \mathrm{diag}(1,1,1,1)$$

The spinors are [in Weyl representation of γ matrices]

$$u_{\mathcal{Q}_f}(k,q^3,a) = \frac{1}{\sqrt{2(E_f+m_f)}}\left[\hat{\not{\Pi}}_s(E_f,k,q^3)+m_f\,\mathcal{I}\right]\begin{pmatrix}\phi^{(a)}\\ \phi^{(a)}\end{pmatrix}$$

$$v_{-\mathcal{Q}_f}(k,q^3,a) = \frac{1}{\sqrt{2(E_f+m_f)}}\left[-\hat{\not{\Pi}}_{-s}(E_f,k,q^3)+m_f\,\mathcal{I}\right]\begin{pmatrix}\tilde{\phi}^{(a)}\\ -\tilde{\phi}^{(a)}\end{pmatrix}$$

$$\phi^{(1)} = -\tilde{\phi}^{(2)} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\phi^{(2)} = \tilde{\phi}^{(1)} = \begin{pmatrix}0\\1\end{pmatrix}$$

where $\Pi_s^\mu = \left(q^0, 0, -s\sqrt{2kB_f}, q^3\right)$

The anticommutation relations between creation and annihilation operators are

$$\begin{aligned}
\{b_f(\breve{q},a), b_f(\breve{q}',a')\} &= \{d_f(\breve{q},a), d_f(\breve{q}',a')\} = 0\ ,\\
\{b_f(\breve{q},a), d_f(\breve{q}',a')\} &= \{b_f(\breve{q},a), d_f(\breve{q}',a')^\dagger\} = 0\ ,\\
\{b_f(\breve{q},a), b_f(\breve{q}',a')^\dagger\} &= \{d_f(\breve{q},a), d_f(\breve{q}',a')^\dagger\} = 2E_f\,\delta_{aa'}\,(2\pi)^3\,\delta_{kk'}\,\delta_{\chi\chi'}\,\delta(q^3-q'^3)
\end{aligned}$$

To understand how the expressions for the quark energy and for the spinors are obtained it is useful to consider the relation,

$$i\not{\mathcal{D}}\,\mathbb{E}^{\mathcal{Q}_f}(x,\bar{q}) = \mathbb{E}^{\mathcal{Q}_f}(x,\bar{q})\,\hat{\not{\Pi}}_s(q^0,k,q^3)$$

which follows from the properties of $\mathcal{F}_Q\,(x,\bar{q})$.

We consider the Dirac equation

$$(i\not{\mathcal{D}} - m_f)\,U_f(x,\bar{q},a) = (i\not{\mathcal{D}} - m_f)\,\mathbb{E}^{\mathcal{Q}_f}(x,\bar{q})\;u_{\mathcal{Q}_f}(k,q^3,a)$$

From the property above we have

$$\left[\hat{\not{\Pi}}_s(q^0,k,q^3) - m_f\right] u_{\mathcal{Q}_f}(k,q^3,a) = 0$$

To satisfy the equation in general we must have

$$\det\left(\hat{\not{\Pi}}_s(q^0,k,q^3) - m_f\right) = \left((q^0)^2 - m_f^2 - 2kB_f - (q^3)^2\right)^2 = 0 \quad \rightarrow \quad E_f^2 = (q^0)^2 = m_f^2 + 2kB_f + (q^3)^2$$

The explicit forms of $u_{Q_f}(k,q^3,a)$ are obtained by solving the above equation under this condition and properly normalization.

Something similar holds for $V_f(x,\bar{q},a)$ and $v_{Q_f}(k,q^3,a)$

Note that for S=0,1/2 the energies are

$$S=0 \qquad E^2 = m_\pi^2 + (2k+1)B_\pi + (q^3)^2$$

$$S=1/2 \qquad E^2 = m_f^2 + 2kB_f + (q^3)^2$$

Then, the S=1/2 energies can be written as

$$E_f^{\,2} = m_f^2 + \underbrace{\begin{pmatrix} \text{"orbital"} & \text{spin} \\ 2n+1 & +\ 2s_3 \\ & \end{pmatrix}}_{2k} B_f + (q^3)^2$$

with $s_3 = \pm 1/2$

n	s_3	k
0	-1/2	0
0	+1/2	1
1	-1/2	1
1	+1/2	2
2	-1/2	2

→ Lowest Landau Level (LLL) non-degenerated

All the other LL's appear twice

For LLL, $E_f^2 = m_f^2 + (q^3)^2$ like particle in 1-dim (dimensional reduction)

The equation that defines the quark propagator $S_f(x,y)$ is

$$(i\not{D} - m_f)\, S_f(x,y) = \delta^{(4)}(x-y)$$

In terms of the Ritus eigenfunctions it can be written as

$$S_f(x,y) \;=\; \sum\!\!\!\!\!\!\!\int_{\bar{q}} \; \mathbb{E}^{Q_f}(x,\bar{q})\; \hat{S}_f(k,q_\parallel)\; \bar{\mathbb{E}}^{Q_f}(y,\bar{q}) \;,$$

where

$$\hat{S}_f(k,q_\parallel) = \frac{\hat{\not{\Pi}}_s + m_f}{q_\parallel^2 - m_f^2 - 2kB_f + i\epsilon}$$

This can be proven noting that, according to the property of $\mathbb{E}^{Q_f}$ given before, we have

$$(i\not{D} - m_f)\, \mathbb{E}^{Q_f}(x,\bar{q}) \left(\hat{\not{\Pi}}_s + m_f\right) = \mathbb{E}^{Q_f}(x,\bar{q}) \left(\hat{\not{\Pi}}_s - m_f\right) \left(\hat{\not{\Pi}}_s + m_f\right)$$
$$= \mathbb{E}^{Q_f}(x,\bar{q}) \left(\hat{\not{\Pi}}_s^2 - m_f^2\right) = \mathbb{E}^{Q_f}(x,\bar{q}) \left(q_\parallel^2 - m_f^2 - 2kB_f\right)$$

and using the completeness relations of the $\mathbb{E}^{Q_f}$

Under a gauge transformation the quark propagator transforms as

$$S_f(x,y) \;\to\; \tilde{S}_f(x,y) = e^{-iQ_f\Lambda(x)}\, S_f(x,y)\, e^{iQ_f\Lambda(y)}$$

Spin 1 charged particles: the charged ρ meson

The gauged Proca action for a charged point-like ρ meson in the presence of a static and homogeneous B is

$$\mathcal{S}_P = \int d^4x \left\{ -\frac{1}{2}\rho^{\mathcal{Q},\mu\nu}(x)^\dagger \rho^{\mathcal{Q}}_{\mu\nu}(x) + m_\rho^2 \rho^{\mathcal{Q},\mu}(x)^\dagger \rho^{\mathcal{Q}}_\mu(x) + \frac{i}{2} Q_\rho F^{\mu\nu}\left[\rho^{\mathcal{Q}}_\mu(x)^\dagger \rho^{\mathcal{Q}}_\nu(x) - \rho^{\mathcal{Q}}_\nu(x)^\dagger \rho^{\mathcal{Q}}_\mu(x)\right]\right\},$$

with $\rho^Q_{\mu\nu} = \mathcal{D}_\mu \rho^Q_\nu - \mathcal{D}_\nu \rho^Q_\mu$.

The associated gauged Proca equation reads

$$\mathcal{D}^\mu \mathcal{D}_\mu \rho^{\mathcal{Q}}_\nu(x) + m_\rho^2 \rho^{\mathcal{Q}}_\nu(x) - 2i\, Q_\rho F_\nu{}^\alpha \rho^{\mathcal{Q}}_\alpha(x) = 0 \qquad \text{with } \mathcal{D}^\mu \rho^{\mathcal{Q}}_\mu(x) = 0,$$

The rho field reads

$$\rho^{\mathcal{Q},\mu}(x) = \sum\!\!\!\!\!\!\int_{\{\bar{q}_{E_\rho}\}} \sum_c \frac{1}{2E_\rho}\left[a^{\mathcal{Q}}_\rho(\breve{q},c)\, W^\mu_{\mathcal{Q}}(x,\bar{q},c) + a^{-\mathcal{Q}}_\rho(\breve{q},c)^\dagger\, W^\mu_{-\mathcal{Q}}(x,\bar{q},c)^*\right]$$

The rho energy is $E_\rho = \sqrt{m_\rho^2 + (2k+1)B_\rho + (q^3)^2}$, with $k \geq -1$

The functions $W_Q^\mu(x,\bar{q},c)$ are given by

$$W_{\mathcal{Q}}^{\mu}(x,\bar{q},c) = \mathbb{R}^{\mathcal{Q},\mu\nu}(x,\bar{q})\ \epsilon_{\mathcal{Q},\nu}(k,q^3,c)$$

We have separated the wavefunction into a function $\mathbb{R}^{Q,\mu\nu}$ that depends on x and $\bar{q}$ and a polarization vector $\epsilon_{Q,\nu}(k,q^3,c)$, the index c denoting the polarization state.

For explicit expressions for polarization vectors see Dumm et al, PRD108(2023)016012

For $k = -1$ there is only one possible polarization vector, i.e. $c = 1$ in this case.
For $k = 0$ two polarization vectors can be constructed, i.e. $c = 1,2$.
For $k \geq 1$ the sum over c runs over the full set of values $c = 1,2,3$.

The functions $\mathbb{R}^{Q,\mu\nu}$ are given by

$$\mathbb{R}^{\mathcal{Q},\mu\nu}(x,\bar{q}) = \sum_{\lambda=-1,0,1} \mathcal{F}_{Q_\rho}(x,\bar{q}_\lambda)\ \Upsilon_\lambda^{\mu\nu}\ ,$$

where $\bar{q}_\lambda = (q^0, k - s\,\lambda,\ \chi, q^3)$. There are different possible choices for $\Upsilon_\lambda^{\mu\nu}$; we use

$$\Upsilon_0^{\mu\nu} = g_{\parallel}^{\mu\nu}\ , \qquad \Upsilon_{\pm 1}^{\mu\nu} = \frac{1}{2}(g_{\perp}^{\mu\nu} \mp S_3^{\mu\nu})$$

$S_3 = i(\delta_1^\mu \delta_2^\nu - \delta_2^\mu \delta_1^\nu)$ is the 3-component of spin operator in the $S = 1$ representation.

The equation that defines the rho propagator $D^{\mu\nu}_{\rho^Q}(x,y)$ is

$$\left[\left(\mathcal{D}^{\alpha}\mathcal{D}_{\alpha}+m_{\rho}^{2}\right)\,g_{\mu\nu}-\mathcal{D}_{\mu}\mathcal{D}_{\nu}+2i\,Q_{\rho}\,F_{\mu\nu}\right]D^{\nu\gamma}_{\rho^{\mathcal{Q}}}(x,y)=\delta_{\mu}{}^{\gamma}\,\delta^{(4)}(x-y)$$

In terms of the Ritus eigenfunctions it can be written as

$$D^{\nu\gamma}_{\rho^{\mathcal{Q}}}(x,y)\quad=\quad\sum\!\!\!\!\!\!\!\int_{\bar{q}}\ \mathbb{R}^{\mathcal{Q},\nu\alpha}(x,\bar{q})\ \hat{D}_{\rho^{\mathcal{Q}},\alpha\beta}(k,q_{\|})\ \mathbb{R}^{\mathcal{Q},\gamma\beta}(y,\bar{q})^{*}\ ,$$

where

$$\hat{D}_{\rho^{\mathcal{Q}},\alpha\beta}(k,q_{\|})=\frac{-\,g_{\alpha\beta}+(1-\delta_{k,-1})\ \Pi_{\alpha}(k,q_{\|})\,\Pi_{\beta}(k,q_{\|})^{*}/m_{\rho}^{2}}{q_{\|}^{2}-m^{2}-(2k+1)B_{\rho}+i\epsilon}$$

with

$$\Pi^{\mu}(k,q_{\|})=\left(q^{0}\,,\,i\sqrt{B_{\rho}/2}\left(\sqrt{k+1}-\sqrt{k}\right),\,-s\sqrt{B_{\rho}/2}\left(\sqrt{k+1}+\sqrt{k}\right),\,q^{3}\right)\qquad\text{for } k\geq 0$$

Under a gauge transformation the rho propagator transforms as

$$D^{\nu\gamma}_{\rho^{Q}}(x,y)\to\tilde{D}^{\nu\gamma}_{\rho^{Q}}(x,y)=e^{-iQ_{f}\Lambda(x)}\ D^{\nu\gamma}_{\rho^{Q}}(x,y)\ e^{iQ_{f}\Lambda(y)}$$

For S=1 the energy is $E_\rho^2 = m_\rho^2 + (2k+1) + (q^3)^2$ with $k \geq -1$

This can be written as

$$E_\rho{}^2 = m_\rho^2 + \underbrace{\begin{pmatrix} \text{"orbital"} & \text{spin} \\ 2n+1 & +\ 2s_3 \end{pmatrix}}_{2k+1} B_\rho + (q^3)^2$$

with $s_3 = -1, 0, 1$

n	s_3	k
0	-1	-1
0	0	0
0	1	1
1	-1	0
1	0	1
1	1	2
2	-1	1

Lowest Landau Level, only 1 polarization

1st Landau Level, only 2 polarizations

All the other LL's, 3 polarizations

Summary II

We have introduced a relativistic notation to express gauge transformations and fields.

We have introduced a set of basic functions and show their explicit form in the standard gauges.

We have shown how to express the relativistic fields for particles of S=0, 1/2, 1 in terms of the basis functions.

We have shown how to write the corresponding propagators in terms of the associated (Ritus) eigenfunctions.

We have shown that the propagators transform in a covariant way under gauge transformations.

Of course, our results can be used for other charged particles: e^{-},$W^{\pm}$, etc

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