Charged particles in strong magnetic fields

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PLAN OF THE LECTURES

- Lecture 1: Order of magnitude of B in different contexts. Units of B in different unit systems. Charged particles in a magnetic field: Classical case and non- relativistic quantum case.

- Lecture 2: Charged particles in a magnetic field: Relativistic quantum case. Charged fields and propagators for particles of spin 0, ½ and 1.

- Lecture 3: Schwinger phase (SP). SP and charged particle propagators. Connection between different representations of charged particle propagators. An example: Leading Order Correction to the charged pion propagator.

Order of magnitude of magnetic field strength

B [Gauss = 10^{-4} Tesla]	Example
1-100 x 10^{-9} G	Intergalatic magnetic field
1-100 x 10^{-6} G	Heliosphere
10^{-3} G	Coffeemaker (30 cm away)
0.3 G	Earth magnetic (on equator)
10 G	Refrigerator magnet
15-70 x 10^{3} G	Medical magnetic resonance
0.45 10^{6} G	Strongest continuous B produced in lab
12 – 28 x 10^{6} G	Record for human produced pulsed B
10^{15} G	Typical Magnetar (at surface)
10^{18} G	Heavy Ion Collision at RHIC

Units

SI (aka MKS) : Unit for B is Tesla [T]   Unit for Q is Coulomb[C]

$$\vec{F}_c = \frac{1}{4\pi\epsilon_0}\frac{Q_1\,Q_2}{r^2}\,\hat{r} \quad \text{Coulomb force} \qquad ; \qquad \vec{F}_L = Q\left(\vec{E} + \vec{v}\times\vec{B}\right) \quad \text{Lorentz force}$$

Gaussian cgs : Unit for B is Gauss [G]   Unit for Q is statCoulomb [statC (esu)]

$$\vec{F}_c = \frac{Q_1\,Q_2}{r^2}\,\hat{r} \quad \text{Coulomb force} \qquad ; \qquad \vec{F}_L = Q\left(\vec{E} + \frac{1}{c}\,\vec{v}\times\vec{B}\right) \quad \text{Lorentz force}$$

1 statC $\cong 3.336 \times 10^{-10}$ C   1 G = 10^{-4} T

Natural units: Basic quantities are c , $\hbar$, eV.

The relation with the basic cgs units is

1 s = 1.52×10^{15} $\hbar$ eV^{-1} 1 cm = 5.07×10^{4} $\hbar$ c eV^{-1} 1 g = 5.61×10^{32} eV c^{-2}

Useful conversion factors are (e in Gaussian units)

$\hbar$ c = 197.33 MeV fm

$e^2 = \alpha \hbar c$ where $\alpha = 1 / 137.036$

The unit of B is $eV^2 / (\hbar c)^{3/2}$ and that of eB is $eV^2 / \hbar c$

We have that $eB = 1\,\mathrm{GeV}^2/\hbar c = 8.1194 \times 10^{10}$ esu G $= 2.70834 \times 10^{-3}$ C T

corresponds to $B = 1.69 \times 10^{20}$ G $= 1.69 \times 10^{16}$ T

By convention when quantities are written in natural units *all factors of $\hbar$ and c are suppressed*. Then

$eB = m_e^2 \rightarrow$ B = 4.41×10^{13} G $eB = m_\pi^2 = (0.14\ \mathrm{MeV})^2 \rightarrow$ B = 3.31×10^{18} G

and

$$\vec{F}_c = \frac{Q_1 Q_2}{r^2}\,\hat{r} \quad \text{Coulomb force} \qquad ; \qquad \vec{F}_L = Q\left(\vec{E} + \vec{v} \times \vec{B}\right) \quad \text{Lorentz force}$$

Charged particle in a constant and homogeneous B: Classical case

<u>Elementary physics approach</u>

Lorentz force $\vec{F_L} = Q\,\vec{v} \times \vec{B}$ is perpendicular to $\vec{v}$, therefore, energy is conserved

We first take $\vec{v}$ perpendicular to $\vec{B}$

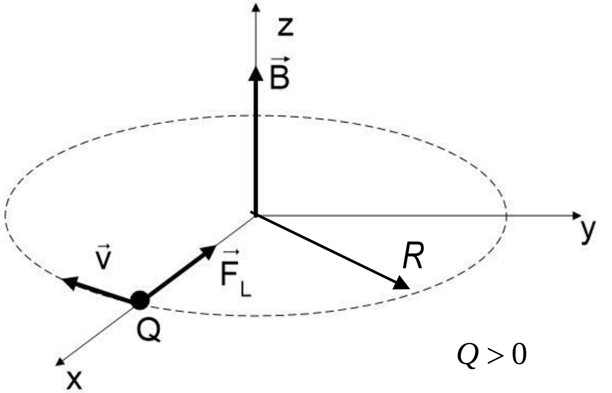
Lorentz force supplies the centripetal force

$$Q\,v\,B = M\,\frac{v^2}{R}$$

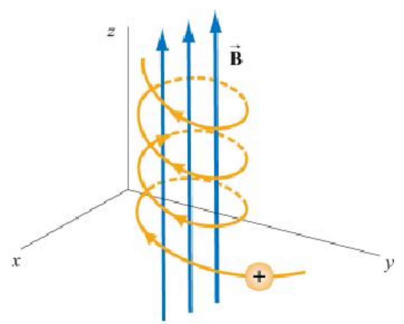
Then, $R = \frac{M\,v}{QB}$, thus $T = \frac{2\pi R}{v} = \frac{2\pi M}{QB}$

and $\omega_c = \frac{2\pi}{T} = \frac{QB}{M}$ Cyclotron frequency

Without lost of generality we assume $\vec{B} = (0,0,\mathrm{B})$ with $B > 0$



If $\vec{v}$ has component along direction of $\vec{B}$ such component is conserved and trajectory is helicoidal



A more sophisticated approach

According to Newton's law we have $M\ \ddot{\vec{x}} = Q\ \dot{\vec{x}} \times \vec{B}$

Taking $\vec{B} = (0,0,\mathrm{B})$ this implies

$$\begin{cases} M\ \ddot{x}_1(t) = Q\ B\ \dot{x}_2(t) \\ M\ \ddot{x}_2(x) = -Q\ B\ \dot{x}_1(t) \\ M\ \ddot{x}_3(t) = 0 \end{cases}$$

equations of motion (eom)

The general solution of this set of equations for constant and homogenous B is

$$\begin{cases} x_1(t) = X_1 + R\ \cos\omega_c(t-t_0) \\ x_2(t) = X_2 + R\ \sin\omega_c(t-t_0) \\ x_3(t) = X_3 + A\ t \end{cases}$$

where $\omega_c = \frac{|QB|}{M}$

Here, X_3, A as well as X_1, X_2, R and t_0 are arbitrary constants fixed by the initial conditions (2 per each 2nd order equation). Note that X_1, X_2 are the coordinates of the center and R the radius of the circle in the 1-2 plane

Note also that that eom are invariant under translations in any direction and rotations around the 3-axis.

An even more sophisticated approach

Let us recall that the electric field $\vec{E}$ and the magnetic field $\vec{B}$ can be obtained from the scalar potential $\phi(\vec{x}, t)$ and the vector potential $\vec{A}(\vec{x}, t)$ according to

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \quad ; \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

In terms of them, the Lagrangian of a charged particle in an electromagnetic field is

$$L = \frac{1}{2} M \, \dot{\vec{x}}^{\,2} + Q \, \dot{\vec{x}} \cdot \vec{A} - Q \, \phi$$

For the case in which only a constant magnetic field is present (i.e. $\phi = 0$ and $\vec{A}$ time independent)

$$L = \frac{1}{2} M \, \dot{\vec{x}}^{\,2} + Q \, \dot{\vec{x}} \cdot \vec{A}$$

Then the Euler-Lagrange equations given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

coincide with those shown before when $\vec{B} = (0,0,\mathrm{B})$

It is very important to note that, given the fields $\vec{E}$ and $\vec{B}$, the potentials ϕ and $\vec{A}$ are not unique. In fact, we can perform the "gauge" transformations

$$\phi \to \phi' = \phi + \frac{\partial \Lambda}{\partial t} \quad ; \quad \vec{A} \to \vec{A}' = \vec{A} - \vec{\nabla}\Lambda \qquad \text{where } \Lambda(\vec{x}, t)$$

without changing the fields $\vec{E}$ and $\vec{B}$.

The Lagrangian of a particle in an electromagnetic field changes by a total time derivative (it is said it is *quasi-invariant*). Therefore the eom remain unchanged.

The existence of gauge transformations is a redundancy in our description of the system: fields which differ by the above transformation describe physically identical configurations. Nothing that we can physically measure can depend on our gauge choice.

Turning back to the problem of particle in a constant and homogenous $\vec{B}$, if we start from the Lagrangian formulation we have to fix a gauge. Usual gauges for $\vec{B} = (0,0,B)$ are

Symmetric gauge (SG),	$\vec{A} = B\left(-x^2, x^1, 0\right)/2$
Landau gauge 1 (LG1),	$\vec{A} = B\left(-x^2, 0, 0\right)$
Landau gauge 2 (LG2),	$\vec{A} = B\left(0, x^1, 0\right)$

Let us consider our problem in the LG2. The Lagrangian reads

$$L = \frac{1}{2} M \, \dot{\vec{x}}^{\,2} + Q \, B \, x_1 \, \dot{x}_2$$

It is easy to see that using the Euler-Lagrange equations we recover the eom given before

$$\begin{cases} M \, \ddot{x}_1 - Q \, B \, \dot{x}_2 = 0 \\ M \, \ddot{x}_2 + Q \, B \, \dot{x}_1 = 0 \\ M \, \ddot{x}_3 = 0 \end{cases}$$

We have already mentioned that the system is expected to be invariant under translations and rotations around the 3-axis. However, the Lagrangian in the LG2 does not seem to be invariant under translations in x_1 or rotations around the 3-axis.

Let's consider the translation $x_1 \to x_1' = x_1 + b$ acting on the LG2 Lagrangian. We have

$$L \to L' = \frac{1}{2} M \, \dot{\vec{x}}^{\,2} + Q \, B \, x_1 \, \dot{x}_2 + Q \, B \, b \, \dot{x}_2$$

which implies

$$\Delta L = Q \, B \, \dot{x}_2 = \frac{d(Q \, b \, x_2)}{dt}$$

Thus, L is quasi-invariant and leads to the same eom.

Even more interestingly, it can be seen that this translation is equivalent to the gauge transformation

$$\vec{A} = B(0, x_1, 0) \to \vec{A}' = B\,(0, x_1, 0) + \vec{\nabla}(B \, b \, x_2)$$

Similarly for rotations around 3-axis. Equivalent situation in other gauges.

Coming back to the Lagrangian for a charged particle in a general electromagnetic field

$$L = \frac{1}{2} M \, \dot{\vec{x}}^{\,2} + Q \, \dot{\vec{x}} \cdot \vec{A} - Q \, \phi$$

we define the *canonical* momenta

$$p_i = \frac{dL}{d\dot{x}_i} = M \, \dot{x}_i + Q \, A_i$$

which satisfy the Poisson bracket relations $\{x_i, p_j\} = \delta_{ij} \quad ; \quad \{x_i, x_j\} = \{p_i, p_j\} = 0$

Clearly, p_i is different from $\Pi_i = M \, \dot{x}_i$ which is the quantity usually called momentum. In the present context Π_i is called *mechanical* momentum

We note that p_i is not gauge invariant and, consequently, is not an observable quantity. On the hand, Π_i is gauge invariant and, thus, it is an observable quantity.

We can now define the Hamiltonian of the system

$$H = \dot{\vec{x}} \cdot \vec{p} - L = \frac{1}{2M} \left(\vec{p} - Q \, \vec{A} \right)^2 + Q \, \phi = \frac{\vec{\Pi}^2}{2M} + Q \, \phi$$

Charged particle in a constant and homogeneous B: Non-relativistic quantum case

We turn now to the non-relativistic quantum theory. Following the usual quantization procedure, we replace the canonical momentum with

$$\vec{p} \to \hat{\vec{p}} = -i\,\vec{\nabla}$$

Then time-dependent Schrödinger equation for a particle in an electric and magnetic field takes the form

$$i\frac{\partial \psi}{\partial t} = H\,\psi = \left[\frac{1}{2M}\left(-i\vec{\nabla} - Q\,\vec{A}\right)^2 + Q\,\phi\right]\psi$$

Before returning to the particular case in which only a constant and homogeneous B is present, it is important to note two important points:

1) Differently from the classical case, it is not possible to formulate the quantum mechanics of particles moving in electric and magnetic fields in terms of $\vec{E}$ and $\vec{B}$ alone. We're obliged to introduce the gauge fields $\vec{A}$ and ϕ

2) Under gauge transformations the Schrödinger equation transforms covariantly (i.e. in a nice way) only if the wavefunction itself also transforms with a position-dependent phase

$$\psi(\vec{x},t) \rightarrow e^{-i\,Q\,\Lambda(\vec{x},t)}\;\psi(\vec{x},t)$$

This transformation does not affect physical probabilities, which are given by $|\psi|^2$

The simplest way to see that the Schrödinger equation transforms “nicely” under the gauge transformations is to define the covariant derivatives

$$\begin{aligned} \mathcal{D}_t &= \frac{\partial}{\partial t} + i\,Q\,\phi \\ \mathcal{D}_i &= \frac{\partial}{\partial x_i} - i\,Q\,A_i \end{aligned}$$

that transform as

$$\mathcal{D}_{t,i} \rightarrow e^{-i\,Q\,\Lambda(\vec{x},t)}\;\mathcal{D}_{t,i}\;e^{i\,Q\,\Lambda(\vec{x},t)}$$

In terms of these covariant derivatives, the Schrödinger equation becomes

$$i\,\mathcal{D}_t\psi = \frac{1}{2M}\;\vec{\mathcal{D}}^2\psi$$

Noting that under gauge transformations (assuming ψ transforms as mentioned above)

$$\mathcal{D}_t\psi(\vec{x},t) \rightarrow e^{-i\,Q\,\Lambda(\vec{x},t)}\;\mathcal{D}_t\psi(\vec{x},t) \qquad ; \qquad \mathcal{D}_i\psi(\vec{x},t) \rightarrow e^{-i\,Q\,\Lambda(\vec{x},t)}\;\mathcal{D}_i\psi(\vec{x},t)$$

We have that the Schrödinger equation transforms, in fact, covariantly

Turning back to the problem of a particle in a constant and homogenous $\vec{B}$ we have that

$$i\frac{\partial\psi}{\partial t} = H\psi = \left[\frac{1}{2M}\left(\hat{\vec{p}} - Q\,\vec{A}\right)^2\right] = \frac{\hat{\vec{\Pi}}^{\,2}}{2M}\,\psi$$

$$\hat{\vec{p}} = -i\vec{\nabla}$$

$$\hat{\vec{\Pi}} = -i\vec{\nabla} - Q\vec{A}$$

Canonical commutation relations are (also valid for general EM field)

$$[x_i, \hat{p}_j] = i\hbar\;\delta_{ij} \qquad ; \qquad [x_i, x_j] = [\hat{p}_i, \hat{p}_j] = 0$$

On the other hand, for the mechanical momenta we have

$$\left[x_i, \hat{\Pi}_j\right] = i\hbar\;\delta_{ij} \qquad ; \qquad \left[\hat{\Pi}_i, \hat{\Pi}_j\right] = i\hbar\;\epsilon_{ijk}\;B_k$$

If we take $\vec{B} = (0,0,\mathrm{B})$ then $[\hat{\Pi}_1, \hat{\Pi}_2] = i\,\hbar\,Q\,B \qquad ; \qquad [\hat{\Pi}_1, \hat{\Pi}_3] = [\hat{\Pi}_2, \hat{\Pi}_3] = 0$

The components of $\hat{\vec{\Pi}}$ (or velocities) perpendicular to $\vec{B}$ are incompatible physical quantities

$$\Delta\Pi_1\;\Delta\Pi_2 \geq \frac{\hbar}{2}\;|QB|$$

We are now in position of solving the Schrödinger equation to find the eigenvalues and eigenfunctions associated to a charged particle moving in a constant and homogeneous B. To do so we have to fix a gauge.

We consider first the LG2 which implies $\vec{A} = B\,(0, x_1, 0)$. Then

$$H = \frac{1}{2M}\left[\hat{p}_1^2 + (\hat{p}_2 - Q\,B\,x_1)^2 + \hat{p}_3^2\right]$$

Because we have manifest translational invariance in the x_2 and x_3 directions, we have $[\hat{p}_2, H] = [\hat{p}_3, H] = 0$ and can look for energy eigenstates that are also eigenstates of $\hat{p}_2$ and $\hat{p}_3$. This motivates the ansatz

$$\psi(\vec{x}) = e^{i(p_2 x_2 + p_3 x_3)}\ \phi(x_1)$$

The time-independent Schrödinger equation is $H\psi = E\psi$. Substituting our ansatz we simply replace $\hat{p}_2$ and $\hat{p}_3$ with their eigenvalues, and we have

$$H\,\psi(\vec{x}) = \frac{1}{2M}\left[\hat{p}_1^2 + (p_2 - Q\,B\,x_1)^2 + p_3^2\right]\psi(\vec{x}) = E\ \psi(\vec{x})$$

We can write the previous equation as an eigenvalue equation for the $\phi(x_1)$. We have

$$\tilde{H}\ \phi(x_1) = \left(E - \frac{p_3^2}{2M}\right)\ \phi(x_1)$$

where $\tilde{H}$ is something very familiar: it's the Hamiltonian for a harmonic oscillator in the x_1 direction, with the centre displaced from the origin,

$$\tilde{H} = -\frac{1}{2M}\frac{\partial^2}{\partial x_1^2} + \frac{M\,\omega_c^2}{2}\left(x_1 - s\frac{p_2}{B_Q}\right)^2$$

where $\begin{cases} B_Q = |\mathrm{QB}| \\ s = \mathrm{sign}(QB) \\ \omega_c = B_Q / M \end{cases}$

Something rather strange has happened in this Hamiltonian: the momentum in the x_2 direction, p_2, has turned into the position of the harmonic oscillator in the x_1 direction, that is now centered at $x_1 = p_2 / |\mathrm{QB}|$

We can immediately write down the energy eigenvalues E ; they are simply those of the harmonic oscillator in the perp plane plus a free kinetic energy in the parallel direction

$$E = \omega_c\left(n + \frac{1}{2}\right) + \frac{p_3^2}{2M}$$

$n = 0, 1, 2\ldots$

The wavefunctions depend on three quantum numbers: n, p_2, p_3 . They are

$$\psi_{n,p_2,p_3}(\vec{x}) = N\ e^{i(p_2 x_2 + p_3 x_3)}\ e^{-B_Q (x_1 - s p_2/B_Q)^2/2}\ H_n(x_1 - s\, p_2/B_Q)$$

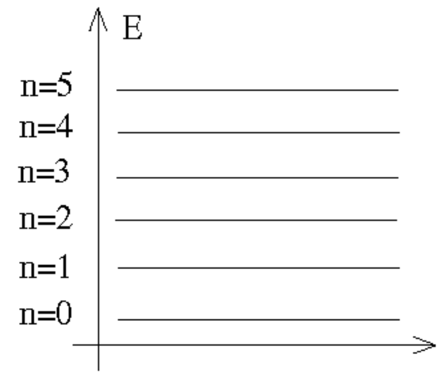
with H_n is the usual Hermite polynomial wavefunctions of the harmonic oscillator and N is some normalization factor.

The wavefunctions look like strips, extended in the x_2 direction but exponentially localized around $s\, p_2/B_Q$ in the x_1 direction.

However, there is large degeneracy of wavefunctions and by taking linear combinations of these states we can cook up wavefunctions that have pretty much any shape you like.

The dynamics of the particle in the x_3-direction is unaffected by the magnetic field B = (0, 0,B). Thus, we restrict to particles with p_3 = 0. The energy spectrum then coincides with that of a harmonic oscillator,

$$E = \omega_c \left(n + \frac{1}{2} \right)$$



In the present context, these are called Landau levels. We see that, in the presence of a magnetic field, the energy levels of a particle become equally spaced, with the gap between each level proportional to the magnetic field B.

The states in a given Landau level are not unique. Instead, there is a huge degeneracy, with many states having the same energy. We can see this in the form of the wavefunctions which, when p_3 = 0, depend on two quantum numbers, n and p_2. Yet, the energy is independent of p_2.

We can now determine how large this degeneracy of states is. To do so, we need to restrict ourselves to a finite region of the perpendicular (i.e. $x_1 - x_2$) plane. We pick a rectangle of area A with sides of lengths L_1 and L_2. We want to know how many states fit inside this rectangle.

Having a finite size L_2 is like putting the system in a box in the x_2-direction. The wavefunctions must obey

$$\psi(x_1, x_2 + L_2, x_3) = \psi(x_1, x_2, x_3)$$

which implies

$$e^{i\, p_2\, L_2} = 1$$

This means that the momentum p_2 is quantized in units of $2\pi/L_2$.

Having a finite size L_1 is somewhat more subtle since the LG2 does not have manifest translational invariance in the x_1-direction. Thus, our argument will be a little heuristic. Because the wf's are exponentially localized around $x_1 = sp_2/B_Q$ for a finite sample restricted to $0 \leq x_1 \leq L_1$ we would expect the allowed p_2 values to range between $0 \leq p_2 \leq B_Q L_1$.

Then number of states in each Landau level is given by

$$\mathcal{N} = \frac{L_2}{2\pi} \int_0^{B_Q L_1} dp_2 = \frac{B_Q L_1 L_2}{2\pi}$$

which implies

$$\frac{\mathcal{N}}{A} = \frac{B_Q}{2\pi}$$

We now consider the same problem but using the SG, $\vec{A} = \frac{B}{2}(-x_2, x_1, 0)$

This choice of gauge breaks translational symmetry in both the x_1 and the x_2 directions. However, it does preserve rotational symmetry around 3-axis. This means that eigenvalue of L_3 is now a good quantum number to label states.

In this gauge, the Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2M}\left[\left(\hat{p}_1 + \frac{Q\,B\,x_2}{2}\right)^2 + \left(\hat{p}_2 - \frac{Q\,B\,x_1}{2}\right)^2 + \hat{p}_3^2\right] \\ &= -\frac{1}{2M}\nabla^2 - s\frac{B_Q}{2M}\,\hat{L}^3 + \frac{B_Q^2}{8M}\left(x_1^2 + x_2^2\right) \end{aligned}$$

where $\begin{cases} B_Q = |QB| \\ s = \mathrm{sign}(QB) \\ \hat{L}_3 = x_1\hat{p}_2 - x_2\hat{p}_1 \end{cases}$

Introducing polar ρ and φ coordinates in the $x_1 - x_2$ plane

$$H\psi = -\frac{1}{2M\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) - \frac{1}{2M\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} - \frac{1}{2M}\frac{\partial^2\psi}{\partial x_3^2} + is\frac{B_Q}{2M}\frac{\partial\psi}{\partial\varphi} + \frac{B_Q^2}{8M}\rho^2\,\psi = E\psi$$

We now propose

$$\psi(\vec{x}) = e^{i\,p_3\,x_3} e^{-i\,s\,m\,\varphi}\ R(\rho)$$

We get

$$-\frac{1}{2M\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{2M}\left(\frac{m^2}{\rho^2} + p_3^2 + m\,B_Q + \frac{B_Q^2\,\rho^2}{4}\right) R = E\,R$$

The solution of this eigenvalue equation (which is well behaved at $\rho = 0, \infty$) is

$$R_{nl}(\rho) = N\,\xi^{m/2}\,e^{-\xi/2}\,L_l^m(\xi)$$

$\xi = B_Q \rho^2 / 2$

$m = n - l$

$n, l \geq 0$

where $L_l^m(x)$ are the generalized Laguerre polynomials.

The eigenvalues are

$$E = \omega_c \left(n + \frac{1}{2}\right) + \frac{p_3^2}{2M}$$

We see that the eigenvalues in the SG coincide (as expected) with those previously obtained in the LG2.

The eigenfunctions are different. They are connected by a gauge transformation. In fact, it can be seen that, given a Landau level (LL) one can be written as a linear combination of the other.

Of course, it can be shown that degeneracy of the LL's, given in this case by the fact that E is independent of the eigenvalue of L_3 (i.e. m), is the same as the one obtained in the LG2.

Summary I

Classically, a charged particle in a constant and homogeneous B describes a circular (or helicoidal) trajectory of fixed radius with frequency given by ω_c=|QB|/M

The classical Lagrangian is quasi-invariant and, thus, the eom are gauge invariant. They are also invariant under translations and rotations around the B-axis.

The canonical momenta are not gauge invariant but the mechanical momenta (and velocities) are. Thus, the latter are observable.

To solve the quantum problem one has to fix a gauge. The eigenvalues are gauge invariant are given by

$$E = \omega_c \left(n + \frac{1}{2} \right) + \frac{p_3^2}{2M}$$

where n indicates the corresponding Landau level (LL). Each level has a large degeneracy.

The eigenfunctions are gauge dependent. They are connected by a gauge transformation.

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